FIVE LECTURES ON LATTICES IN SEMISIMPLE LIE GROUPS

by

Yves Benoist

Abstract. — This text is an introduction to lattices Γ in semisimple Lie groups G, in five independent lectures in which one answers to the following questions: Why do Coxeter groups give lattices in $SO(p,1)$ for $p \leq 9$? Why do arithmetic constructions give lattices in $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$? Why do the unitary representations of G have an influence on the algebraic structure of Γ? Why do the Γ-equivariant factors of the Furstenberg boundary of G also have an influence on the algebraic structure of Γ? Why does one need to study also lattices in semisimple Lie groups over local fields?

Résumé **(Cinq cours sur les réseaux des groupes de Lie semisimples)**

Ce texte est une introduction aux réseaux Γ des groupes de Lie semisimples G, en cinq cours indépendants dans lesquels on répond aux questions suivantes: Pourquoi les groupes de Coxeter donnent-ils des réseaux de $SO(p,1)$ pour $p \leq 9$? Pourquoi les constructions arithmétiques donnent-elles des réseaux de $\mathrm{SL}(d,\mathbb{R})$ et $\mathrm{SO}(p,q)$? Pourquoi les représentations unitaires de G ont-ils une influence sur la structure algébrique de Γ? Pourquoi les facteurs Γ-équivariants de la frontière de Furstenberg de G ont-ils aussi une influence sur la structure algébrique de Γ? Pourquoi doit-on ausi étudier les réseaux des groupes de Lie semisimples sur les corps locaux?

Contents

2000 ***Mathematics Subject Classification.*** — 11F06, 20H10, 22E40, 22E46.
Key words and phrases. — lattices, Coxeter groups, arithmetic groups, unitary representations, mixing, property T, amenability, boundary, local fields.

Introduction

This text is an introduction to lattices in semisimple Lie groups, in five independent lectures. It was given during the first week of the 2004 Summer School at the Fourier Institute in Grenoble. We hope that it will attract young students to this topic and convince them to read some of the many textbooks cited in the references. We illustrate five important methods of this subject: geometry, arithmetics, representations, boundaries, and local fields. One for each lecture.

A lattice Γ in a real semisimple Lie group G is a discrete subgroup for which the quotient G/Γ supports a G-invariant measure of finite volume. One says that Γ is cocompact if this quotient is compact. We will often suppose that the Lie algebra $\mathfrak{g}$ is semisimple. This is the case for $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(p, q)$. The two main sources of lattices are

- the *geometric method*: One constructs a periodic tiling of the symmetric space $X = G/K$, where K is a maximal compact subgroup of G, with a tile P of finite volume. The group of isometries of this tiling is then the required lattice. This very intuitive method, initiated by Poincaré, seems to work only in low dimension: even if one knows by theorical arguments that it does exist, the explicit description of such a tile P in any dimension is still a difficult question. The aim of the first lecture is to construct one for $G = \mathrm{SO}(p, 1)$, where $p \leq 9$.

- the *arithmetic method*: One thinks of G (or better of some product of G by a compact group) as being a group of real matrices defined by polynomial equations with integral coefficients. The subgroup Γ of matrices with integral entries is then a lattice in G. This fact, due to Borel and Harish-Chandra, implies that G always contains a cocompact and a noncocompact lattice. The aim of the second lecture is to construct some of them for the groups $G = \mathrm{SL}(d, \mathbb{R})$ and $G = \mathrm{SO}(p, q)$.

According to theorems of Margulis and Gromov-Schoen, if $\mathfrak{g}$ is simple and different from $\mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$, then all lattices in G can be constructed by the arithmetic method. When $\mathfrak{g} = \mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$, quite a few other methods have been developed in order to construct new lattices. Even though we will not discuss them here, let us quote:

$\star$ for $G = \mathrm{SO}(p, 1)$:
- $p = 2$: gluing trousers (Fenchel-Nielsen); uniformization (Poincaré);
- $p = 3$: gluing ideal tetrahedra and Dehn surgery (Thurston);
- all p: hybridation of arithmetic groups (Gromov, Piatetski-Shapiro).

$\star$ for $G = \mathrm{SU}(p, 1)$:
- $p = 2$: groups generated by pseudoreflections (Mostow); fundamental group of algebraic surfaces (Yau, Mumford);

- $p \leq 3$: moduli spaces of weighted points on the line; holonomy groups of local systems (Deligne, Mostow, Thurston).
- all p: unknown yet.

One of the main successes of the theory of lattices is that it gave in a unified way many new properties of arithmetic groups. One does not use the way in which Γ has been constructed but just the existence of the finite invariant measure. A key tool is the theory of unitary representations, and more precisely the asymptotic behavior of coefficients of vectors in unitary representations. We will explain this in the third lecture.

Another important tool are the boundaries associated to Γ. We will see in the fourth lecture how they are used in the proof of the Margulis normal subgroup theorem, which says that *lattices in real simple Lie groups of real rank at least* 2 *are quasisimple*, i.e. their normal subgroups are either finite or of finite index.

The general theory we described so far gives information on arithmetic groups like $\mathrm{SL}(d,\mathbb{Z})$, $\mathrm{SO}(d,\mathbb{Z}[i])$, or $\mathrm{Sp}(d,\mathbb{Z}[\sqrt{2}])$. It can be extended to S-arithmetic groups like $\mathrm{SL}(d,\mathbb{Z}[i/N])$, $\mathrm{SO}(d,\mathbb{Z}[1/N])$, or $\mathrm{SU}(p,q,\mathbb{Z}[\sqrt{2}/N])$... The only thing one has to do is to replace the real Lie group G by a product of real and p-adic groups. The aim of the last lecture is to explain how to adapt the results of the previous lectures to that setting. For instance, we will construct cocompact lattices in $\mathrm{SL}(d,\mathbb{Q}_p)$ and see that they are quasisimple for $d \geq 3$.

This text is slightly longer than the oral lecture, parce qu'au tableau il est plus facile de remplacer une démonstration technique par un magnifique crobard, un principe général, un exemple insignifiant, un exercice intordable voire une grimace évocatrice. One for each lecture. Nevertheless, there are still many important classical themes in this subject which will not be discussed here. Let us just quote a few: cohomological dimension and cohomology, universal extension and the congruence subgroup property, rigidity and superigidity, counting points and equirepartition, Shimura varieties, quasiisometries...

Un grand merci aux auditeurs de l'École d'été qui par leurs critiques m'ont permis d'améliorer ce texte: Nir Avni, Uri Bader, Pierre Emmanuel Caprace, Yves de Cornulier, Damien Ferté, Francois Guéritaud, Francois Maucourant, Barbara Schapira, et aussi Gaetan Chenevier, Fanny Kassel, Vincent Lafforgue, Bertrand Remy et le referee.

For an undergraduate introduction to tilings and lattices, one can read [2].

1. Lecture on Coxeter Groups

In the first lecture, we construct a few lattices in $SO(p,1)$ by the geometric method, when $p \leq 9$.

1.1. Introduction. — The geometric method of construction of lattices has been initiated by Poincaré in 1880. In his construction, the group G is the group $PO^+(2,1)$ of isometries of the hyperbolic plane $\mathbb{H}^2$. One begins with a polygon $P \subset \mathbb{H}^2$ and with a family of isometries which identify the edges of P two by two. When these isometries satisfy some compability conditions saying that "the first images of P give a tiling around each vertex", the Poincaré theorem says that the group Γ generated by these isometries acts properly on $\mathbb{H}^2$, with P as a fundamental domain. In particular, when P has finite volume, the group Γ is a lattice in G.

There exists a higher-dimensional extension of Poincaré's theorem. One replaces $\mathbb{H}^2$ by the d-dimensional hyperbolic space $\mathbb{H}^d$, the polygon P by a polyhedron, the edges by the $(d-1)$-faces, and the vertices by the $(d-2)$-faces (see [**16**]). In most of the explicitly-known examples, one chooses Γ to be generated by the symmetries with respect to the $(d-1)$-faces of P. The aim of this lecture is to present a proof, due to Vinberg, of this extension of Poincaré's theorem and to describe some of these explicit polyhedra for $d \leq 9$. In this case, the group Γ is a Coxeter group. As a by-product, we will obtain geometric proofs of some of the basic properties of Coxeter groups.

Even though the geometric construction may seem less efficient than the arithmetic one, it is still an important tool.

1.2. Projective transformations. — Let us begin with a few basic definitions and properties. Let $V := \mathbb{R}^{d+1}$, $\mathbb{S}^d = \mathbb{S}(V) := (V-0)/\mathbb{R}_+^\times$ be the projective sphere, and $\mathrm{SL}^\pm(d+1,\mathbb{R})$ be the group of projective transformations of $\mathbb{S}^d$.

Definition 1.1. — *A reflection σ is an element of order 2 of $\mathrm{SL}^\pm(d+1,\mathbb{R})$ which is the identity on an hyperplane. All reflections are of the form $\sigma = \sigma_{\alpha,v} := Id - \alpha \otimes v$ for some $\alpha \in V^*$ and $v \in V$ with $\alpha(v) = 2$.*
- A rotation ρ is an element of $\mathrm{SL}^\pm(d+1,\mathbb{R})$ which is the identity on a subspace of codimension 2 and is given by a matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in a suitable supplementary basis. The real $\theta \in [0,\pi]$ is the angle of the rotation.

Let $\sigma_1 = \sigma_{\alpha_1,v_1}$, $\sigma_2 = \sigma_{\alpha_2,v_2}$ be two distinct reflections, Δ be the group they generate, $a_{12} := \alpha_1(v_2)$, $a_{21} := \alpha_2(v_1)$, and $L := \{x \in \mathbb{S}^d \ / \ \alpha_1|_x \leq 0, \alpha_2|_x \leq 0\}$. The following elementary lemma tells us when the images $\delta(L)$, $\delta \in \Delta$, tile a subset C of $\mathbb{S}^d$, i.e. when the interiors $\delta(\overset{\circ}{L})$, $\delta \in \Delta$, are disjoints. The set C is then the union $C = \bigcup_{\delta\in\Delta} \delta(L)$.

Lemma 1.2. — *a) If $a_{12} > 0$ or $a_{21} > 0$, the $\delta(L)$, $\delta \in \Delta$, do not tile (any subset of $\mathbb{S}^d$.*
b) Suppose now $a_{12} \leq 0$ and $a_{21} \leq 0$. Consider the following four cases :

b1) $a_{12}a_{21} = 0$. *If both* a_{12} *and* a_{21} *are equal to* 0, *then the product* $\sigma_1\sigma_2$ *is of order* 2, *the group* Δ *is* $\mathbb{Z}/2 \times \mathbb{Z}/2$, *and the* $\delta(L)$, $\delta \in \Delta$, *tile* $\mathbb{S}^d$. *Otherwise they do not tile.*
b2) $0 < a_{12}a_{21} < 4$. *The product* $\sigma_1\sigma_2$ *is a rotation of angle* θ *given by* $4\cos(\theta/2)^2 = a_{12}a_{21}$. *If* $\theta = 2\pi/m$ *for some integer* $m \geq 3$ *then* $\sigma_1\sigma_2$ *is of order* m, *the group* Δ *is* $\mathbb{Z}/2 \times \mathbb{Z}/m$, *and the* $\delta(L)$, $\delta \in \Delta$, *tile* $\mathbb{S}^d$. *Otherwise they do not tile.*
b3) $a_{12}a_{21} = 4$. *The product* $\sigma_1\sigma_2$ *is unipotent and the* $\delta(L)$, $\delta \in \Delta$, *tile a subset* C *of* $\mathbb{S}^d$ *whose closure is a half-sphere.*
b4) $a_{12}a_{21} > 4$. *The product* $\sigma_1\sigma_2$ *has two distinct positive eigenvalues and the* $\delta(L)$, $\delta \in \Delta$, *tile a subset* C *of* $\mathbb{S}^d$ *whose closure is the intersection of two distinct half-spheres.*

Proof. — This lemma reduces to a 2-dimensional exercise that is left to the reader. □

Remark The cross-ratio $[\alpha_1, \alpha_2, v_1, v_2] := \frac{\alpha_1(v_2)\alpha_2(v_1)}{\alpha_1(v_1)\alpha_2(v_2)} = \frac{a_{12}a_{21}}{4}$ is a projective invariant.

1.3. Coxeter systems. — A *Coxeter system* (S, M) is the data of a finite set S and a matrix $M = (m_{s,t})_{s,t\in S}$ with diagonal coefficients $m_{s,s} = 1$ and nondiagonal coefficients $m_{s,t} = m_{t,s} \in \{2, 3, \dots, \infty\}$. The cardinal r of S is called the *rank* of the Coxeter system. To such a Coxeter system one associates the corresponding *Coxeter group* $W = W_S$ defined by the set of generators S and the relations $(st)^{m_{s,t}} = 1$, for all $s, t \in S$ such that $m_{s,t} \neq \infty$. For w in W, the length $\ell(w)$ is the smallest integer ℓ such that w is the product of ℓ elements of S.

A Coxeter group has a natural r-dimensional representation σ_S, called the *geometric representation*, which is defined in the following way. Let $(e_s)_{s\in S}$ be the canonical basis of $\mathbb{R}^S$. The *Tits form* on $\mathbb{R}^S$ is the symmetric bilinear form defined by

$$B_S(e_s, e_t) := -\cos(\tfrac{\pi}{m_{s,t}}) \text{ for all } s, t \in S .$$

According to Lemma 1.2, the formula

$$\sigma_S(s)v = v - 2B_S(e_s, v)e_s \ \forall\ s \in S,\ v \in E_S$$

defines a morphism σ_S of W into the orthogonal group of the Tits form. Let P_S be the simplex in the sphere $\mathbb{S}^{r-1}$ of the dual space defined by $P_S := \{f \in \mathbb{S}^{r-1} \ /\ f(e_s) \leq 0 \ /\ \forall s \in S\}$.

As a special case of the Vinberg theorem stated in the next section, we will see the following theorem, due to Tits.

Theorem 1.3. — **(Tits)** *The representation* σ_S *is faithful, its image* Γ_S *is discrete and the translates* ${}^t\gamma(P_S)$, $\gamma \in \Gamma_S$, *tile a convex subset* C_S *of the sphere* $\mathbb{S}^{r-1}$.

Remarks - The convex C_S is called the *Tits convex set.*
- For a few Coxeter groups with $r \leq 10$, called *hyperbolic*, we will prove that the Tits form is Lorentzian of signature $(r-1, 1)$ and that the group Γ_S is a lattice in the corresponding orthogonal group.

Corollary 1.4. — *For every subset $S' \subset S$, the natural morphism $\rho_{S,S'} : W_{S'} \to W_S$ is injective.*

Proof of Corollary 1.4. — The representation $\sigma_{S'}$ is equal to the restriction of $\sigma_S \circ \rho_{S,S'}$ to the vector space $< e_s, s \in S' >$. □

1.4. Groups of projective reflections. —

In this section we study groups generated by projective reflections fixing the faces of some convex polyhedron P of the sphere $\mathbb{S}^d$.

Let $P \subset \mathbb{S}^d$ be a d-dimensional *convex polyhedron*, i.e. the image in $\mathbb{S}^d$ of a convex polyhedral cone of $\mathbb{R}^{d+1}$ with 0 omitted. A *k-face* of P is a k-dimensional convex subset of P obtained as an intersection of P with some hyperspheres which do not meet the interior $\overset{\circ}{P}$. A *face* is a $(d-1)$-face and an edge is a 0-face.

Let S be the set of faces of P and for every s in S, one chooses a projective reflection $\sigma_s = Id - \alpha_s \otimes v_s$ with $\alpha_s(v_s) = 2$ which fixes s. A suitable choice of signs allows us to suppose that P is defined by the inequalities $(\alpha_s \leq 0)_{s \in S}$. Let $a_{s,t} := \alpha_s(v_t)$ for $s, t \in S$. Let Γ be the group generated by the reflections σ_s.

According to Lemma 1.2, if we want the images $\gamma(P)$ to tile some subset of $\mathbb{S}^d$, the following conditions are necessary: for all faces $s, t \in S$ such that the intersection $s \cap t$ is a $(d-2)$-dimensional face of P, one has

$$a_{s,t} \leq 0 \ \text{ and } (\ a_{s,t} = 0 \Longleftrightarrow a_{t,s} = 0 \ ) \tag{1}$$

$$a_{s,t}a_{t,s} \geq 4 \ \text{ or } \ a_{s,t}a_{t,s} = 4\cos^2(\tfrac{\pi}{m_{s,t}}) \ \text{ with } \ m_{s,t} \text{ integer, } m_{s,t} \geq 2 \tag{2}$$

Conversely, the following theorem states that these conditions are also sufficient.

Let (S, M) be the Coxeter system given by these integers $m_{s,t}$ and completed by $m_{s,t} = \infty$ when either $s \cap t = \emptyset$, $\operatorname{codim}(s \cap t) \neq 2$, or $a_{s,t}a_{t,s} \geq 4$. Note that, when the polyhedron is the simplex P_S of the previous section, the Coxeter system is the one we started with.

Theorem 1.5. — **(Vinberg)** *Let P be a convex polyhedron of $\mathbb{S}^d$ and, for each face s of P, let $\sigma_s = Id - \alpha_s \otimes v_s$ be a projective reflection fixing the face s. Suppose that conditions (1) and (2) are satisfied for every s, t such that* $\operatorname{codim}(s \cap t) = 2$. *Let Γ be the group generated by the reflections σ_s. Then*
(a) the polyhedra $\gamma(P)$, for γ in Γ, tile some convex subset C of $\mathbb{S}^d$;
(b) the morphism $\sigma : W_S \to \Gamma$ given by $\sigma(s) = \sigma_s$ is an isomorphism;
(c) the group Γ is discrete in $\mathrm{SL}^{\pm}(d+1, \mathbb{R})$.

In other words, to be sure that a convex polyhedron and its images by a group generated by projective reflections through its faces tile some part of the sphere, it is enough to check local conditions "around each 2-codimensional face".

We will still call C the *Tits convex set*. It may not be open.

Remark The proof of Theorem 1.3 given by Tits in [**8**] can be adapted to get Theorem 1.5 (see [**24**], Lemma 1). In this lecture, we will follow Vinberg's proof, which is more geometric.

1.5. The universal tiling. —

To prove Theorem 1.5, one introduces an abstract space X obtained by gluing copies of P indexed by the Coxeter group $W := W_S$ along their faces and oneproves that this space is convex.

Formally, one defines $X := W \times P/_{\sim}$ where the equivalence relation $\sim$ is generated by

$$(w,p) \sim (w',p') \iff \exists s \in S \;/\; w' = ws \;\text{ and }\; p' = p = \sigma_s(p)\;.$$

One denotes by P^{sing} the union of the 3-codimensional faces of P, and sets $P^{\text{reg}} = P - P^{\text{sing}}$, $X^{\text{sing}} := W \times P^{\text{sing}}/_{\sim}$, $X^{\text{reg}} = X - X^{\text{sing}}$. The Coxeter group W acts naturally on X and on $\mathbb{S}^d$. Let $\pi : X \to \mathbb{S}^d$ be the map defined by $\pi(w,p) := w\,p$.

Lemma 1.6. — *a) For x in P, let $W_x \subset W$ be the subgroup generated by σ_s for $s \ni x$. Then $V_x := W_x \times P/_{\sim}$ is a neighborhood of x in X.*
b) The map π is W-equivariant, i.e. $\forall w \in W\;, \forall x \in X,\; \pi(w\,x) = w\,\pi(x)$.
c) For all x in X^{reg}, there exists a neighborhood V_x of x in X such that $\pi|_{V_x}$ is a homeomorphism onto a convex subset of $\mathbb{S}^d$.

Proof. — a) Let P_x be an open neighborhood of x in P which does not meet the faces of P not containing x. Then, $W_x \times P_x/_{\sim}$ is open in X.
b) Easy.
c) This is a consequence of a), b), lemma 1.2 and of hypotheses (1) and (2). □

A segment on $\mathbb{S}^d$ is a 1-dimensional convex subset which is not a circle. Let us transfer this notion of segment to X.

Definition 1.7. — *For every x, y in X, a segment $[x,y]$ is a compact subset of X such that the restriction of π to $[x,y]$ is a homeomorphism onto some segment of $\mathbb{S}^d$ with end-points $\pi(x)$ and $\pi(y)$.*

We do not know yet that such a segment does exist. It is precisely what we want to show now.

Let us denote by $\partial P = P - \overset{\circ}{P}$ the union of the faces of P and $\partial X := W \times \partial P/_{\sim}$. The following lemma is the key lemma. For each point z in P^{reg} one defines its multiplicity by $m(z) := m_{s,t}$ if $z \in s \cap t$ for some $s \neq t$, and by $m(z) := \mathbf{1}_{\partial P}(z)$ otherwise. We extend this function on X^{reg} by the formula $m(w\,z) := m(z)$.

Lemma 1.8. — *Fix $w \in W$. Let $S = S_w$ be the set of all $(x,y) \in \overset{\circ}{P} \times \overset{\circ}{P} \subset X \times X$ such that $\pi(x) \neq -\pi(y)$, and such that the segment $[x, w\,y]$ exists and is contained in X^{reg}. Suppose $S \neq \emptyset$. Then*

a) the sum $\sum_{z\in[x,wy]} m(z)$ *is a constant* $L(w)$ *on* S *depending only on* w*;*
b) the set S *is dense in* $P \times P$*.*

The above sum counts the number of faces crossed by the segment $[x, wy]$. We will see later that this number $L(w)$ is equal to the length $\ell(w)$.

Proof. — Let $L(x, y, w)$ be the above sum. According to the local analysis given in Lemma 1.2, when the segment $[x, w\,y]$ crosses the interior of a 2-codimensional face $w'(s\cap t)$, one has $m_{s,t} < \infty$. Moreover, this local analysis proves that the function $(x, y) \to L(x, y, w)$ is locally constant (this is the main point in this proof, see the remark below). Choose $L \geq 0$ such that the set $S_L := \{(x, y) \in S \ / \ L(x, y, w) = L\}$ is nonempty. One knows that S_L is open in $\overset{\circ}{P} \times \overset{\circ}{P}$. Notice that, for (x, y) in S_L, the only tiles $w'P \subset X$ crossed by the segment $[x, w\,y]$ satisfy $\ell(w') \leq L$, they belong to a fixed finite set of tiles. So, by a compactness argument, for any (x, y) in the closure $\overline{S}_L$, the segment $[x, w\,y]$ exists and is included in the compact $\bigcup_{\ell(w')\leq L} w'(P)$. Moreover, since $P^{\rm sing}$ is of codimension 3, removing some subset of codimension 2 in S, one can find an open, connected, and dense subset S' of $\overset{\circ}{P} \times \overset{\circ}{P}$ such that $\overline{S}_L \cap S' \subset S_L$. Hence, successively, $S_L \cap S'$ is open and closed in S', S' is included in S_L, S_L is dense in $\overset{\circ}{P} \times \overset{\circ}{P}$, and $S_L = S$. □

The next statement is a corollary of the previous proof.

Lemma 1.9. — *For every* x, x' *in* X*, there exists at least one segment* $[x, x']$ *joining them.*
Moreover, when $\pi(x) \neq -\pi(x')$*, this segment is unique.*

Proof. — Keep notations from the previous lemma with $x' = w\,y$.
We know the implication $S_w \neq \emptyset \Longrightarrow \overline{S}_w = P \times P$. This allows to prove by induction on $\ell(w)$ that $S_w \neq \emptyset$, by letting the point y move continuously through a face. The uniqueness follows from the uniqueness of the segment joining two non-antipodal points on the sphere $\mathbb{S}^d$. □

Lemma 1.10. — *The map* $\pi : X \to C$ *is bijective and* C *is convex.*

Proof. — Let x, x' be two points of X. According to Lemma 1.9, there is a segment $[x, x']$ joining them. Hence if $\pi(x) = \pi(x')$, one must have $x = x'$. This proves that $\pi : X \to C$ is bijective. Two points of C can also be joined by a segment, hence C is convex. □

Proof of Theorem 1.5. — (a), (b) follow from Lemma 1.10, and (c) follows from (a). □

Remark Let us point out how crucial Lemma 1.8 is. Consider the following group Γ generated by two linear transformations g_1 and g_2 of $\mathbb{R}^2$, which identify the opposite faces

of a convex quadrilateral P:
- g_1 is the homothety of ratio 2,
- g_2 is a rotation whose angle α/π irrational, and
- $P := \{(x, y) \in \mathbb{R}^2 \ / \ 1 \leq x \leq 2 \ \text{ and } \ \left|\frac{y}{x}\right| \leq \tan\frac{\alpha}{2}\}$.

The successive images $\gamma(P)$, $\gamma \in \Gamma$, draw a kind of irrational spider web which, instead of tiling an open set in $\mathbb{S}^2$, tile the universal cover of $\mathbb{R}^2 - \{0\}$. The group Γ is not discrete.

1.6. Cocompactness. —

The following corollary tells us when the convex set C is open.

Corollary 1.11. — *With the same notations as Theorem 1.5, the following conditions are equivalent:*
(i) for every x in P, the Coxeter group W_{S_x} is finite, where $S_x := \{s \in S \ / \ x \in s\}$;
(ii) the convex set C is open.

In this case, W acts properly on C with a compact quotient.

To prove this corollary, we will use the following lemma.

Lemma 1.12. — *a) The union of the boundaries of the tiles $\cup_{w \in W} w(\partial P)$ is the intersection of C with a family of hyperspheres. This family is locally finite in $\overset{\circ}{C}$.*
b) One has $L(w) = \ell(w)$ for all w in W.
c) For every x in P, W_{S_x} is the stabilizer of x. Moreover, the union U_x of $w(P)$, for $w \in W_{S_x}$, is a neighborhood of x in C.
d) One has the equivalence: $x \in \overset{\circ}{C} \iff \mathrm{card}(W_{S_x}) < \infty$.
e) The group W acts properly on $\overset{\circ}{C}$.

Remark Point b) is related to the exchange lemma for Coxeter groups ([**8**] ch.IV §1).

Proof. — a) One just has to check that when one walks on a hypersphere containing a face and passes through a face of codimension 2 then one is still on a face. But this is a consequence of the local analysis of Lemma 1.2.b2.

b) $\ell(w)$ is the minimum number of faces a path from $\overset{\circ}{P}$ to $w(\overset{\circ}{P})$ has to cross. According to a), this minimum is achieved when this path is a segment. Hence $\ell(w) = L(w)$.

c) This a consequence of Lemma 1.6 and 1.10.

d) If the union U_x is a neighborhood of x, by local finiteness of the tiling and by compactness of a small sphere centered at x, the index set W_{S_x} must be finite. Conversely, if W_{S_x} is finite, the intersection of U_x with a small sphere is, by induction, simultaneously open and closed.

e) This is a consequence of c) and d). □

Proof of Corollary 1.11. — Use Lemma 1.12 d) and e). □

Let q_0 be a quadratic form of signature $(d,1)$ and $\mathbb{H}^d \subset \mathbb{S}^d$ be the corresponding hyperbolic space: it is one of the two connected components of the set $\{x \in \mathbb{S}^d \ / \ q_0|_x < 0\}$.

Corollary 1.13. — *Keep previous notations.*
a) If $\mathring{P} \subset \mathbb{H}^d$ and if the symmetries σ_s are orthogonal for q_0, then $\mathring{C} = \mathbb{H}^d$.
b) Moreover, if $P \subset \mathbb{H}^d$, then Γ is a cocompact lattice in the orthogonal group $O(d,1)$

In case $a)$ P is called an *hyperbolic Coxeter polyhedron.*

Proof. — a) By contradiction, let x_0 be a point of $\mathring{P}$, y a point of $\mathbb{H}^d - \mathring{C}$ minimizing the distance to x_0 and s a face of P crossed by the segment $[x_0, y]$. Then, one has $d(x_0, \sigma_s(y)) < d(x_0, y)$. Contradiction.
b) Note that $C = \mathbb{H}^d$ is open and use Corollary 1.11. □

1.7. Examples. — a) Consider a convex polygon in $\mathbb{H}^2$ whose angles between the edges are equal to π/m for some $m \leq 2$.
Then the group generated by the orthogonal reflections with respect to the faces is a cocompact lattice in $O(2,1)$.

b) Consider a tetrahedron in $\mathbb{H}^3$ whose group of isometries is S_4 and whose vertices are on the boundary of $\mathbb{H}^3$. The angles between the faces are $\pi/3$.
Then the group generated by the orthogonal reflections with respect to the faces is a noncocompact lattice in $O(3,1)$.

c) Consider a dodecahedron in $\mathbb{H}^3$ whose group of isometries is $\mathcal{A}_5$ such that the angles between the faces are $\pi/2$.
Then the group generated by the orthogonal reflections with respect to the faces is a cocompact lattice in $O(3,1)$.

d) Let $k \geq 5$. Consider a convex k-gon P in $\mathbb{R}^2$, with vertices x_1,..., $x_k = x_0$ and sides $s_1 = [x_1, x_2]$,..., $s_k = s_0 = [x_k, x_1]$. Let ℓ_i be the lines containing s_i. Assume that the points v_i on the intersection $\ell_{i-1} \cap \ell_{i+1}$ are in $\mathbb{R}^2$ and that P is in the convex hull of the points v_i. Denote by $\sigma_i = Id - \alpha_i \otimes v_i$ the projective reflections such that $\mathrm{Ker}(\alpha_i) = \ell_i$.
Then the group generated by σ_i acts cocompactly on some bounded open convex subset of $\mathbb{R}^2$ whose boundary is in general non C^2. This kind of groups has been introduced first in [**14**]. See [**3**] for more information on these examples and their higher-dimensional analogs.

e) Consider the convex polyhedron $P_S \subset \mathbb{R}^{r-1}$ associated to the geometrical representation of a Coxeter group W_S given by some Coxeter system (S, M). Consider also the Tits convex set C_S tiled by the images of P_S and the Tits form B_S, as in Section 1.3.

To each Coxeter system (S, M), one associates its Coxeter diagram. It is a graph whose set of vertices is S and whose edges are weighted by the number $m_{s,t}$, with the convention that an edge is omitted when the weight is is equal to 2 and the weight is not specified when it is equal to 3. The Coxeter system is said to be *irreducible* if the corresponding graph is connected.

The following proposition gives the list of hyperbolic Coxeter simplices which are compact (resp. of finite volume).

Proposition 1.14. — *Let (S, M) be an irreducible Coxeter system.*
a) One has the equivalences: B_S is positive definite $\iff C_S = \mathbb{S}^{r-1} \iff \mathrm{card}(W_S) < \infty$. In this case, (S, M) is said to be elliptic.
(S, M) is said to be parabolic if B_S is positive and degenerate.
b) Suppose B_S is Lorentzian. Then one has the equivalences:
b1) all Coxeter subsystems are elliptic $\iff W_S$ is a cocompact lattice in $O(B_S)$;
b2) all Coxeter subsystems are either elliptic or irreducible parabolic $\iff W_S$ is a lattice in $O(B_S)$.

Proof. — We will just prove the implications $\Rightarrow$ we need for our examples.
a) and b1) are easy consequences of Theorem 1.5 and corollaries 1.11, 1.13.
b2) Use the fact that for $d \geq 2$, for any simplex S with $\mathring{S} \subset \mathbb{H}^d$, the hyperbolic volume of $\mathring{S}$ is finite. □

The lists of Coxeter diagrams satisfying these properties are due to Coxeter in cases a) and to Lanner in cases b). They can be found, for instance, in [**26**] p.202-208. There are only finitely many of them with rank $r \leq 5$ in case b1) and $r \leq 10$ in case b2). Here are two examples.

The Coxeter diagram obtained as a pentagone with one edge of weight 4, gives a cocompact lattice in $0(4, 1)$.

The Coxeter diagram E_{10} (which is a segment with 9 points and a last edge starting from the third point of the segment) gives a noncocompact lattice in $0(9, 1)$.

f) The description of all compact (resp. finite volume) hyperbolic Coxeter polyhedra in $\mathbb{H}^d$ is known only in dimensions 2 and 3. The highest dimension of known examples is $d = 5$ (resp. $d = 21$) and one knows that one must have $d \leq 29$ (resp. $d \leq 995$).

2. Lecture on Arithmetic groups

The aim of this second lecture is to give explicit constructions of lattices in the real Lie groups $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$. These examples are particular cases of a general arithmetic construction of lattices in any semisimple group G, due to Borel and Harish-Chandra. In fact, Margulis showed that all "irreducible" lattices of G are obtained in this way when the real rank of G is at least 2.

2.1. Examples. —

Here are a few explicit examples of lattices.

Write $d = p+q$ with $p \geq q \geq 1$. For any commutative ring A, let $\mathrm{SL}(d,A) := \{g \in \mathcal{M}(d,A) \;/\; \det(A) = 1\}$.

Denote by I_d the $d\times d$ identity matrix, $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $J_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -\sqrt{2}\, I_q \end{pmatrix}$, and let $\mathrm{SO}(p,q) := \{g \in \mathrm{SL}(d,\mathbb{R}) / g\, I_{p,q}\, {}^t g = I_{p,q}\}$.

Example 1 *The group* $\Gamma := \mathrm{SL}(d,\mathbb{Z})$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Example 2 *The group* $\Gamma := \mathrm{SO}(p,q) \cap \mathrm{SL}(d,\mathbb{Z})$ *is a noncocompact lattice in* $\mathrm{SO}(p,q)$.

Example 3 Let σ be the automorphism of order 2 of $\mathbb{Q}[\sqrt{2}]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\sqrt{2}]) \;/\; g\, I_{p,q}\, {}^t g^\sigma = I_{p,q}\}$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Example 4 Let O be a subring in $\mathcal{M}(d,\mathbb{R})$ which is also a lattice in this real vector space. Suppose that $O \subset \mathrm{GL}(d,\mathbb{R}) \cup \{0\}$. We will see that such a subring does exist for every $d \geq 2$: in fact O is an "order in a central division algebra over $\mathbb{Q}$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathcal{M}(d,\mathbb{R})$". *The group* $\Gamma := O \cap \mathrm{SL}(d,\mathbb{R})$ *is a cocompact lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Example 5 Let σ be the automorphism of order 2 of $\mathbb{Q}[\sqrt{2}]$. *The group* $\Gamma := \{(g, g^\sigma) \;/\; g \in \mathrm{SL}(d,\mathbb{Z}[\sqrt{2}])\}$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$.

Example 6 *The group* $\Gamma := \mathrm{SL}(d,\mathbb{Z}[i])$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{C})$.

Example 7 Let τ be the automorphism of order 2 of $\mathbb{Q}[\sqrt[4]{2}]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\sqrt[4]{2}]) \;/\; g\, J_{p,q}\, {}^t g^\tau = J_{p,q}\}$ *is a cocompact lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Example 8 *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\sqrt{2}]) \;/\; g\, J_{p,q}\, {}^t g = J_{p,q}\}$ *is a cocompact lattice in* $\{g \in \mathrm{SL}(d,\mathbb{R}) \;/\; g\, J_{p,q}\, {}^t g = J_{p,q}\} \simeq \mathrm{SO}(p,q)$.

The aim of this lecture is to give a complete proof for Examples 1, 4, 7 and 8, a sketch of a proof for the other examples, and a short survey of the general theory.

2.2. The space of lattices in $\mathbb{R}^d$. —

We study in this section the space X of lattices in $\mathbb{R}^d$ and the subset X_1 of lattices of covolume 1 in $\mathbb{R}^d$. As homogeneous spaces, one has $X = \mathrm{GL}(d,\mathbb{R})/\mathrm{SL}^{\pm}(d,\mathbb{Z})$ and $X_1 = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. We will prove that X_1 has finite volume.

Proposition 2.1. — **(Minkowski)** *The group* $\mathrm{SL}(d,\mathbb{Z})$ *is a lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Let us denote by $g_{i,j}$ the entries of an element g in $G := \mathrm{GL}(d,\mathbb{R})$ and let
$K := O(d) = \{g \in G \;/\; g^t g = 1\}$,
$A := \{g \in G \;/\; g$ is diagonal with positive entries$\}$,
$A_s := \{a \in A \;/\; a_{i,i} \leq s\, a_{i+1,i+1}$, for $i = 1,\dots,d-1$ $\}$ for $s \geq 1$,
$N := \{g \in G \;/\; g - 1$ is strictly upper triangular$\}$, and
$N_t := \{n \in N \;/\; |n_{i,j}| \leq t$, for $1 \leq i < j \leq d$ $\}$ for $t \geq 0$.

According to the Iwasawa decomposition, the multiplication induces a diffeomorphism $K \times A \times N \simeq G$. Let us define the *Siegel domain* $S_{s,t} := K A_s N_t$, and $\Gamma := \mathrm{SL}(d,\mathbb{Z})$.

Lemma 2.2. — *For* $s \geq \frac{2}{\sqrt{3}}$, $t \geq \frac{1}{2}$, *one has* $G = S_{s,t}\Gamma$.

Proof. — Let g be in G and $\Lambda := g(\mathbb{Z}^d)$. One argues by induction on d. A family $(f_1,\dots,f_d)$ of vectors of Λ is said to be *admissible* if
- the vector f_1 is of minimal norm in $\Lambda - \{0\}$;
- the images $\dot{f}_2,\dots,\dot{f}_d$ of $f_2,\dots,f_d$ in the lattice $\dot{\Gamma} := \Gamma/\mathbb{Z}f_1$ of the Euclidean space $\mathbb{R}^d/\mathbb{R}f_1$ form an admissible family of $\dot{\Gamma}$;
- each f_i with $i \geq 2$ is of minimal norm among the vectors of Γ whose image in $\dot{\Gamma}$ is $\dot{f}_i$.

It is clear that Λ contains an admissible family $(f_1,\dots,f_d)$ and that such a family is a basis of Λ. After right multiplication of g by some element of Γ, one may suppose that this family is the image of the standard basis $(e_1,\dots,e_d)$ of $\mathbb{Z}^d$, i.e. for all $i = 1,\dots,d$, one has $g\, e_i = f_i$.

Let us show that $g \in S_{\frac{2}{\sqrt{3}},\frac{1}{2}}$. Write $g = kan$. Since $(k^{-1}f_1,\dots,k^{-1}f_d)$ is an admissible basis of $k^{-1}(\Lambda)$, one may suppose that $k = 1$ i.e. $g = an$. Hence

$$\begin{aligned} f_1 &= a_{1,1}e_1,\\ f_2 &= a_{2,2}\, e_2 + a_{1,1}n_{1,2}\, e_1,\\ &\;\;\dots\\ f_d &= a_{d,d}\, e_d + a_{d-1,d-1}n_{d-1,d}\, e_{d-1} + \cdots + a_{1,1}n_{1,d}\, e_1. \end{aligned}$$

By induction hypothesis, one knows that

$$\begin{aligned} |n_{i,j}| &\leq \frac{1}{2} \quad \text{for } 2 \leq i < j \leq d \quad \text{and}\\ a_{i,i} &\leq \frac{2}{\sqrt{3}}\, a_{i+1,i+1} \quad \text{for } 2 \leq i \leq d-1\,. \end{aligned}$$

It remains to prove these inequalities for $i = 1$.

The first ones are a consequence of the inequalities $\|f_j\| \leq \|f_j + p\, f_1\|$, $\forall p \in \mathbb{Z}$.

The last inequality is a consequence of the inequality $\|f_1\| \leq \|f_2\|$, because this one implies $a_{1,1}^2 \leq a_{2,2}^2 + a_{1,1}^2 n_{1,2}^2 \leq a_{2,2}^2 + \frac{1}{4} a_{1,1}^2$. □

Let $G' := \mathrm{SL}(d, \mathbb{R})$, $K' := K \cap G'$, $A' := A \cap G'$. One still has the Iwasawa decomposition $G' = K'A'N$. One denote $R_{s,t} := S_{s,t} \cap G'$. One still has, thanks to Lemma 2.2, $G' = R_{s,t}\Gamma$.

Proposition 2.1 is now a consequence of the following lemma.

Lemma 2.3. — *The volume of $R_{s,t}$ for the Haar measure is finite.*

Let us first compute the Haar measure in the Iwasawa decomposition. Let dg', dk', da' and dn be right Haar measures on the groups G', K', A', and N respectively. These are also left Haar measure, since these groups are unimodular. The modulus function of the group $A'N$ is

$$a'n \longrightarrow \rho(a'n) = \rho(a') = |\det_{\mathfrak{n}}(\mathrm{Ad}\, a')| = \prod_{i<j} \frac{a'_{i,i}}{a'_{j,j}} \ ,$$

where $\mathfrak{n}$ is the Lie algebra of N.

A left Haar measure on $A'N$ is left A'-invariant and right N-invariant, hence is equal to the product measure $da'dn$, up to a multiplicative constant. Therefore, the measure $\rho(a')da'dn$ is a right Haar measure on $A'N$.

In the same way, the measures dg' and $\rho(a')dk'da'dn$ on G' are both left K-invariant and right $A'N$-invariant. They must be equal, up to a multiplicative constant. Hence

$$dg' = \rho(a')dk'da'dn \ .$$

Proof of Lemma 2.3. — Let $b_i := \dfrac{a'_{i,i}}{a'_{i+1,i+1}}$. The functions $b_1, \ldots, b_{d-1}$ give a coordinate system on A' for which $da' = \frac{db_1}{b_1} \cdots \frac{db_{d-1}}{b_{d-1}}$ and $\rho(a') = \prod_{1 \leq i < d} b_i^{r_i}$ with $r_i \geq 1$. Hence one has

$$\int_{R_{s,t}} dg' = \left(\int_{K'} dk' \right) \left(\prod_{1 \leq i < d} \int_0^s b_i^{r_i - 1} db_i \right) \left(\int_{N_t} dn \right),$$

which is finite because K' and N_t are compact and $r_i \geq 1$. □

2.3. Mahler compactness criterion. —

Let us prove a simple and useful criterion, which tells us when some subset of the set X of lattices in $\mathbb{R}^d$ in compact.

The set X of lattices in $\mathbb{R}^d$ is a manifold as it identifies with the quotient space $\mathrm{GL}(d, \mathbb{R})/\mathrm{SL}^{\pm}(d, \mathbb{Z})$. By definition of the quotient topology, a sequence (Λ_n) of lattices in $\mathbb{R}^d$ converges to some lattice Λ of $\mathbb{R}^d$ if and only if there exists a basis $(f_{n,1}, \ldots, f_{n,d})$ of Λ_n which converges to a basis $(f_1, \ldots, f_d)$ of Λ.

For any lattice Λ in $\mathbb{R}^d$, one denotes by $d(\Lambda)$ the volume of the torus $\mathbb{R}^d/\Lambda$. It is given by the formula $d(\Lambda) = |\det(f_1, \ldots, f_d)|$ where $(f_1, \ldots, f_d)$ is any basis of Λ.

Lemma 2.4. — **(Hermite)** *Any lattice* Λ *in* $\mathbb{R}^d$ *contains a nonzero vector* v *of norm* $\|v\| \leq (\frac{4}{3})^{\frac{d-1}{4}} d(\Lambda)^{\frac{1}{d}}$.

Proof. — This is a consequence of Lemma 2.2, with the inequality $a_{1,1}^d \leq s^{\frac{d(d-1)}{2}} \prod a_{i,i}$. □

Proposition 2.5. — **(Mahler)** *A subset* $Y \subset X$ *is relatively compact in* X *if and only if there exist constants* $\alpha, \beta > 0$ *such that for all* $\Lambda \in Y$*, one has*

$$d(\Lambda) \leq \beta \quad \text{and} \quad \inf_{v \in \Lambda - 0} \|v\| \geq \alpha \ .$$

In other words, a set of lattices is relatively compact if and only if their volumes are bounded and they avoid a small ball.

Proof. — Let us fix $s > \frac{2}{\sqrt{3}}$ and $t > \frac{1}{2}$ and set $\Lambda_0 := \mathbb{Z}^d \in X$. Note that a subset $Y \subset X$ is relatively compact if and only if there exists a compact subset $S \subset S_{s,t}$ such that $Y \subset \{g\,\Lambda_0 \ / \ g \in S\}$.

$\Longrightarrow$ Let us fix $0 < r < R$ such that, for all $g = kan$ in S and all $i = 1, \ldots, d$, one has $r \leq a_{i,i} \leq R$. One has then $|\mathrm{det} g| \leq R^d$ and $\inf_{v_0 \in \Lambda_0 - 0} \|g\, v_0\| \geq r$, because if one writes $v_0 = \sum_{1 \leq i \leq \ell} m_i e_i$ with $m_\ell \neq 0$, one has

$$\|g\,v_0\| \ \geq \ |{<}ke_\ell, g\,v_0{>}| \ = \ |{<}e_\ell, an\,v_0{>}| \ \geq \ a_{\ell,\ell}|m_\ell| \ \geq \ r \ .$$

$\Longleftarrow$ Let $S := \{g \in S_{s,t} \ / \ g\,\Lambda_0 \in \overline{Y}\}$. For all $g = kan$ in S and all $i = 1, \ldots, d$, one has

$$a_{1,1} \geq \alpha \quad , \quad a_{i,i} \leq s\, a_{i+1,i+1} \quad \text{and} \quad \prod_{1 \leq j \leq d} a_{j,j} \leq \beta \ .$$

As a consequence, there exist $0 < r < R$ such that, for all $g = kan$ in S and all $i = 1, \ldots, d$, one has $r \leq a_{i,i} \leq R$. Hence S is compact and $\overline{Y}$ too. □

The same proof can be easily adapted for Examples 5 and 6. The same strategy can also be used for Examples 2 and 3: using the Iwasawa decomposition of G, one introduces the Siegel domains and proves that they are of finite volume and that a finite union of them surjects on G/Γ.

2.4. Algebraic groups. —

In this section we recall a few definitions from the theory of algebraic groups.

Let K be an algebraically closed field of characteristic 0, k a subfield of K, $V_k \simeq k^d$ a k-vector space, $V = K \otimes_k V_k$, and $k[V]$ the ring of k-valued polynomials on V_k.

A *variety* $Z \subset V$ is a subset consisting of the zeros of a family of polynomials on V. Let $I(Z) \subset K[V]$ be the ideal of polynomials on V which are zero on Z. One says that Z is a *k-variety*, or is defined over k, if $I(Z)$ is generated by the intersection $I_k(Z) := I(Z) \cap k[V]$. Let $k[Z] := k[V]/I_k(Z)$ be the ring of *regular functions* of Z. A *k-morphism* of k-varieties $\varphi : Z_1 \to Z_2$ is a map such that, for all f in $k[Z_2]$, $f \circ \varphi$ is in $k[Z_1]$.

A *k-group* is a k-variety $G \subset \mathrm{GL}(V) \subset \mathrm{End}(V)$ which is a group for the composition of endomorphisms. For instance, the k-groups

$$G_a := \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} / \, x \in K\} \quad \text{and} \quad G_m := \{\begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} / \, y, z \in K, \ xy = 1\}$$

are called the additive and the multiplicative k-groups. One has $k[G_a] = k[x]$ and $k[G_m] = k[y, y^{-1}]$. Another example is given by $\mathrm{GL}(V)$ which can be seen as a k-group thanks to the identification

$$\mathrm{GL}(V) \simeq \{(g, \delta) \in \mathrm{End}(V) \times K \ / \ \ \delta \mathrm{det} g = 1\} \ .$$

Note that $G_k := G \cap \mathrm{GL}(d, k)$ is a subgroup of G and, more generally, for any subring A of K, $G_A := G \cap \mathrm{GL}(d, A)$ is a subgroup of G. A *k-morphism* of k-groups $\varphi : G_1 \to G_2$ is a k-morphism of k-varieties which is also a morphism of groups. A *k-isogeny* is a surjective k-morphism with finite kernel. A *k-character* of G is a k-morphism $\chi : G \to G_m$. A *k-cocharacter* of G is a k-morphism $\chi : G_m \to G$. A *k-representation* of G in a k-vector space W_k is a k-morphism $\rho : G \to \mathrm{GL}(W)$. The k-representation is *irreducible* if 0 and W are the only invariant subspaces. It is *semisimple* if it is a direct sum of irreducible representations. A K-group G is *reductive* if all its K-representations are semisimple. A reductive K-group is *semisimple* if all its K-characters are trivial. This definition is well-suited for the groups we are dealing with since we have the following lemma.

This lemma will not be used later on. The reader may skip its proof.

Lemma 2.6. — *The k-groups* $\mathrm{SL}(d)$ *and* $\mathrm{SO}(p, q)$ *are semisimple.*

Proof. — Say for $G = \mathrm{SL}(d, \mathbb{C})$. Since $G = [G, G]$, one only has to prove the semisimplicity of the representations of the group G in a $\mathbb{C}$-vector space V. So one has to prove that any G-invariant subspace W has a G-invariant supplementary subspace. To prove this fact, we will use Weyl's unitarian trick: let $K = \mathrm{SU}(n, \mathbb{C})$ be the maximal compact subgroup of G. By averaging with respect to the Haar measure on K, one can construct a K-invariant hermitian scalar product on V. The orthogonal $W^\perp$ of W is then K-invariant and, since the Lie algebra of G is the complexification of the Lie algebra of K, it is also G-invariant. □

2.5. Arithmetic groups. —

We check that for a $\mathbb{Q}$-group G the subgroup $G_\mathbb{Z} := G \cap \mathrm{GL}(d, \mathbb{Z})$ does not depend, up to commensurability, on the realization of G as a group of matrices.

Lemma 2.7. — *Let* ρ *be a* $\mathbb{Q}$*-representation of a* $\mathbb{Q}$*-group* G *in a* $\mathbb{Q}$*-vector space* $V_\mathbb{Q}$*. Then*
a) the group $G_\mathbb{Z}$ *preserves some lattice* $\Lambda \subset V_\mathbb{Q}$*;*
b) any lattice $\Lambda_0 \subset V_\mathbb{Q}$*, is preserved by some subgroup of finite index of* $G_\mathbb{Z}$*.*

Proof. — a) Choose a basis of $V_{\mathbb{Q}}$. The entries of the matrices $\rho(g)-1$ can be expressed as polynomials with rational coefficients in the entries of the matrices $g-1$. The constant coefficient of these polynomials is zero. Hence there is an integer $m \geq 1$ such that, if g is in the congruence subgroup $\Gamma_m := \{g \in G_{\mathbb{Z}} \ / \ g = 1 \bmod m\}$, then $\rho(g)$ has integral entries. Since Γ_m is of finite index in $G_{\mathbb{Z}}$, the group $G_{\mathbb{Z}}$ also preserves a lattice in $V_{\mathbb{Q}}$.

b) This is a consequence of a), because one can find integers $N, N_0 \geq 1$ such that $N\Lambda \subset N_0\Lambda_0 \subset \Lambda$. □

One easily deduces the following corollary.

Corollary 2.8. — *Let $\varphi : G_1 \to G_2$ be a $\mathbb{Q}$-isomorphism of $\mathbb{Q}$-groups. Then the groups $\varphi(G_{1,\mathbb{Z}})$ and $G_{2,\mathbb{Z}}$ are commensurable.*

2.6. The embedding. —

The following embedding will allow us to reduce the proof of the compactness of G/Γ to Mahler's criterion.

Proposition 2.9. — *Let $G \subset H = \mathrm{GL}(d, \mathbb{C})$ be a $\mathbb{Q}$-group without nontrivial $\mathbb{Q}$-characters. Then the injection $G_{\mathbb{R}}/G_{\mathbb{Z}} \hookrightarrow X = H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a homeomorphism onto a closed subset of X.*

We will need the following proposition.

Proposition 2.10. — **(Chevalley)** *Let G be a k-group and $H \subset G$ a k-subgroup. There exist a k-representation of G on some vector space V_k and a point $x \in \mathbb{P}(V_k)$ whose stabilizer is H, i.e. $H = \{g \in G \ / \ g\,x = x\}$.*

Proof. — We will need the following notations.
- $I(H) := \{P \in K[G] \ / \ P|_H = 0\}$,
- $K^m[G] := \{P \in K[G] \ / \ d^\circ P \leq m\}$, and
- $I^m(H) := I(H) \cap K^m[G]$.

Since $K[G]$ is noetherian, one can choose m such that $I^m(H)$ generates the ideal $I(H)$ of $K[G]$. The action of G on $K^m[G]$ given by $(\pi(g)P)(g') := P(g'g)$ is a k-representation. The k-representation we are looking for is the representation in the p^{th} exterior product $V := \Lambda^p(K^m[G])$, where $p := \dim I^m(H)$ and x is the line in V defined by $x := \Lambda^p(I^m(H))$. By construction, one has the required equality $H = \{g \in G \ / \ g\,x = x\}$. □

Corollary 2.11. — *Let G be a k-group and $H \subset G$ a k-subgroup. Suppose H does not have any nontrivial k-character. Then there exist a k-representation of G on some vector space V_k and a point $v \in V_k$ whose stabilizer is H, i.e. $H = \{g \in G \ / \ g\,v = v\}$.*

Proof. — The action of H on the line x is trivial since all the k-characters of H are trivial. Just choose v on this line. □

Proof of Proposition 2.9. — We have to show that

$$\forall g_n \in G_{\mathbb{R}},\ h \in H_{\mathbb{R}} \text{ such that } \lim_{n\to\infty} g_n H_{\mathbb{Z}} = hH_{\mathbb{Z}} \text{ in } H_{\mathbb{R}}/H_{\mathbb{Z}}$$
$$\exists g \in G_{\mathbb{R}}, \text{ such that } \lim_{n\to\infty} g_n G_{\mathbb{Z}} = gG_{\mathbb{Z}} \text{ in } G_{\mathbb{R}}/G_{\mathbb{Z}}.$$

Since all $\mathbb{Q}$-characters of G are trivial, according to Corollary 2.11 (with H for G and G for H), there exists a $\mathbb{Q}$-representation of H in some $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ and a vector $v \in V_{\mathbb{Q}}$ whose stabilizer in H is G. By Lemma 2.7, the group $H_{\mathbb{Z}}$ stabilizes some lattice Λ in $V_{\mathbb{Q}}$. One can choose Λ containing v. Hence the $H_{\mathbb{Z}}$-orbit of v is discrete in $V_{\mathbb{R}}$.

Let $h_n \in H_{\mathbb{Z}}$ such that $\lim\limits_{n\to\infty} g_n h_n = h$. The sequence $h_n^{-1}v$ converges to $h^{-1}v$ hence is equal to $h^{-1}v$ for n large. Therefore one can write $h_n = \gamma_n g^{-1} h$ with $g \in G_{\mathbb{R}}$, $\gamma_n \in G_{\mathbb{Z}}$, and the sequence $g_n\gamma_n$ converges to g. □

2.7. Construction of cocompact lattices. —

We check that the groups Γ in Examples 4 and 8 of Section 2.1 are cocompact lattices in $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$ respectively.

The following lemma can be applied directly to these examples and enlightens the strategy of the proof in the general case.

Lemma 2.12. — *Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space and $G \subset \mathrm{GL}(V)$ a $\mathbb{Q}$-subgroup without nontrivial $\mathbb{Q}$-character. Suppose that there exists a G-invariant polynomial $P \in \mathbb{Q}[V]$ such that*

$$\forall v \in V_{\mathbb{Q}}\ ,\ P(v) = 0 \Longleftrightarrow v = 0\ .$$

Then the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is compact.

Proof. — Let Λ_0 be a lattice in $V_{\mathbb{Q}}$. One can suppose that $P(\Lambda_0) \subset \mathbb{Z}$.

By Propositions 2.5 and 2.9, we only have to show that no sequence $g_n v_n$ with $g_n \in G_{\mathbb{R}}$ and $v_n \in \Lambda_0 - \{0\}$ can converge to zero.

This is a consequence of the minoration $|P(g_n v_n)| = |P(v_n)| \geq 1$. □

Corollary 2.13. — *a) In Example* 2.1.4, Γ *is cocompact in* $\mathrm{SL}(d,\mathbb{R})$.
b) In Example 2.1.8, Γ *is cocompact in* $\mathrm{SO}(p,q)$.

Proof. — a) Take $V_{\mathbb{Q}} = D$ and $P(v) = det_D(\rho_v)$ where ρ_v is the left multiplication by v.
b) We will apply Weil's recipe called "restriction of scalars". Let us denote by $\mathrm{SO}(J_{p,q},\mathbb{C})$ the special orthogonal group for the quadratic form q_0 whose matrix is $J_{p,q}$. The algebraic group

$$H := \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d,\mathbb{C})\ /\ a + \sqrt{2}\, b \in \mathrm{SO}(J_{p,q},\mathbb{C})\ ,\ a - \sqrt{2}\, b \in \mathrm{SO}(J^{\sigma}_{p,q},\mathbb{C}) \right\}$$

is defined over $\mathbb{Q}$, because the family of equations is σ-invariant.

The map $(a,b) \to a + \sqrt{2}\, b$ gives an isomorphism

$$H_{\mathbb{Z}} \simeq \Gamma$$

and the map $(a, b) \to (a + \sqrt{2}\, b, a - \sqrt{2}\, b)$ gives an isomorphism

$$H_{\mathbb{R}} \simeq \mathrm{SO}(J_{p,q}, \mathbb{R}) \times \mathrm{SO}(J^{\sigma}_{p,q}, \mathbb{R}) \ .$$

One applies Lemma 2.12 with the natural $\mathbb{Q}$-representation in $V_{\mathbb{Q}} = \mathbb{Q}^d \times \mathbb{Q}^d$ and with $P : (u, v) \to q_0(u + \sqrt{2}\, v)\, q_0^{\sigma}(u - \sqrt{2}\, v)$. This proves that $H_{\mathbb{Z}}$ is cocompact in $H_{\mathbb{R}}$. Since $\mathrm{SO}(J^{\sigma}_{p,q}, \mathbb{R})$ is compact, Γ is a cocompact lattice in $\mathrm{SO}(J_{p,q}, \mathbb{R})$. □

Remark To convince the reader that Examples a) do exist in any dimension $d \geq 2$, we will give a construction of

a central division algebra D over $\mathbb{Q}$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathcal{M}(d, \mathbb{R})$,

without using the well-known description of the Brauer group of $\mathbb{Q}$.

Let L be a Galois real extension of $\mathbb{Q}$ with Galois group $\mathrm{Gal}(L/\mathbb{Q}) = \mathbb{Z}/d\mathbb{Z}$ and σ be a generator of the Galois group. One can take $L = \mathbb{Q}[\eta]$ with $\eta = \sum_{1 \leq i \leq q/2d} \cos\left(2\pi g^{id}/q\right)$ where q is a prime number $q \equiv 1 \bmod 2d$ and g is a generator of the cyclic group $(\mathbb{Z}/q\mathbb{Z})^{\times}$ (for $d = 3, 4$ and 5, take $L = \mathbb{Q}[\cos \frac{2\pi}{7}]$, $L = \mathbb{Q}[\cos \frac{\pi}{17} + \cos \frac{2\pi}{17}]$ and $L = \mathbb{Q}[\cos \frac{2\pi}{11}]$).

We will construct D as a d-dimensional left L-vector space $D = L \oplus La \oplus \cdots \oplus La^{d-1}$, with the following multiplication rules: $\forall \ell \in L$, $a\ell a^{-1} = \sigma(\ell)$ and $a^d = p$ where p is another prime number which is inert in L (such a p does exist by Cebotarev theorem). By construction D is an algebra with center $\mathbb{Q}$. It remains to show that every nonzero element $v = \ell_0 + \ell_1 a + \ldots + \ell_{d-1} a^{d-1} \in D$ is invertible. One may suppose that all ℓ_i are in the ring R of integers of L but that $\ell_0 \notin pR$. One computes the determinant Δ_v of the right multiplication by v as an endomorphism of the left L-vector space D. One gets

$$\Delta_v = \det \begin{pmatrix} \ell_0 & p\ell^{\sigma}_{d-1} & \cdots & p\ell_1^{\sigma^{d-1}} \\ \ell_1 & \ell_0^{\sigma} & \cdots & p\ell_2^{\sigma^{d-1}} \\ \vdots & \vdots & & \vdots \\ \ell_{d-1} & \ell^{\sigma}_{d-2} & \cdots & \ell_0^{\sigma^{d-1}} \end{pmatrix} \equiv \ell_0 \ell_0^{\sigma} \cdots \ell_0^{\sigma^{d-1}} \bmod pR \ .$$

Since p is inert, this determinant is nonzero and v is invertible. □

2.8. Godement compactness criterion. —

In this section, we state a general criterion for the cocompactness of an arithmetic subgroup and show how to adapt the previous arguments to prove it.

Let us first recall the definitions of *semisimple* and *unipotent* elements and some of their properties. An element g in $\mathrm{End}(V)$ is *semisimple* if it is diagonalizable over K and *unipotent* if $g - 1$ is nilpotent. The following lemma is the classical Jordan decomposition.

Lemma 2.14. — *Let $g \in \mathrm{GL}(V)$ and $G \subset \mathrm{GL}(V)$ be a k-group.*
i) g can be written in a unique way as $g = su = us$ with s semisimple and u unipotent.
ii) Every subspace $W \subset V$ invariant by g is also invariant by s and u.

iii) $g \in G \Longrightarrow s, u \in G$.
iv) $g \in G_k \Longrightarrow s, u \in G_k$.

Proof. — i) Classical.
ii) s and u can be expressed as polynomials in g.
iii) Consider the action of G on $K^m[\mathrm{End}V] := \{P \in K[\mathrm{End}V] \ / \ d^\circ P \leq m\}$ given by $(\pi(g)P)(x) := P(xg)$. The subspace $I^d[G] := I[G] \cap K^d[\mathrm{End}V]$ is invariant by g. Hence it is also invariant by its semisimple and unipotent part which are nothing else than $\pi(s)$ and $\pi(u)$. So for all $P \in I^d[G]$, one has $P(s) = (\pi(s)P)(1) = 0$ and $P(u) = (\pi(u)P)(1) = 0$. Therefore s and u are in G.
iv) By unicity, s and u are invariant under the Galois group $\mathrm{Gal}(K/k)$. □

Lemma 2.15. — *Let $\rho : G \to H$ be a k-morphism of k-groups and $g \in G$.*
a) g is semisimple $\Longrightarrow \rho(g)$ is semisimple.
b) g is unipotent $\Longrightarrow \rho(g)$ is unipotent.

Proof. — One can suppose that $k = K$ and that G is the smallest K-group containing g. The main point then is to prove that all k-morphisms $\varphi : G_a \to G_m$ and $\psi : G_m \to G_a$ are trivial. But $y \circ \varphi$ is an invertible element of $k[x]$, hence is a constant, and $x \circ \psi$ is an element $F(y) \in k[y, y^{-1}]$ such that $F(y) = F(y^n)/n$ for all $n \geq 1$, hence is a constant. □

Note that the Lie algebra $\mathfrak{g}$ of a $\mathbb{Q}$-group G is defined over $\mathbb{Q}$, because it is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

Theorem 2.16. — **(Godement)** *Let $G \subset \mathrm{GL}(d, \mathbb{C})$ be a semisimple $\mathbb{Q}$-group and $\mathfrak{g}$ its Lie algebra. The following conditions are equivalent:*
(i) $G_\mathbb{R}/G_\mathbb{Z}$ is compact.
(ii) Every element g of $G_\mathbb{Q}$ is semisimple.
(iii) The only unipotent element of $G_\mathbb{Z}$ is 1.
(iv) The only nilpotent element of $\mathfrak{g}_\mathbb{Q}$ is 0.

Remark See Section 2.9 for the general formulation of this theorem.

Sketch of proof of Theorem 2.16. — *(i)* ⇒ *(iii)* Let $u \in G_\mathbb{Z}$ be a unipotent element. According to Jacobson-Morozov, there exists a Lie subgroup S of $G_\mathbb{R}$ containing u whose Lie algebra $\mathfrak{s}$ is iso;orphic to $\mathfrak{sl}(2, \mathbb{R})$. There exists then an element $a \in S$ such that $\lim_{n\to\infty} a^n u a^{-n} = e$. Since $G_\mathbb{R}/G_\mathbb{Z}$ is compact, one can write $a^n = k_n \gamma_n$ with k_n bounded and $\gamma_n \in G_\mathbb{Z}$. But then $\gamma_n u \gamma_n^{-1}$ is a sequence of elements of $G_\mathbb{Z} - \{e\}$ converging to e. Therefore $u = e$.
(ii) ⇔ *(iii)* This follows from Lemma 2.14 and the fact that if $u \in G_\mathbb{Q}$ is unipotent, then u^n is in $G_\mathbb{Z}$ for some positive integer n.
(iii) ⇒ *(iv)* The exponential of a nilpotent element of $\mathfrak{g}_\mathbb{Q}$ is a well-defined unipotent element which is in $G_\mathbb{Q}$. As above, this element has a power in $G_\mathbb{Z}$.

$(iv) \Rightarrow (i)$ The group $\mathrm{Aut}(\mathfrak{g})$ is a $\mathbb{Q}$-group and the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a $\mathbb{Q}$-isogeny, i.e. it is a $\mathbb{Q}$-morphism with finite kernel and cofinite image. Thanks to the following lemma, one can suppose that $G = \mathrm{Aut}(\mathfrak{g})$. We can then apply Lemma 2.12 with P as the G-invariant polynomial on $\mathfrak{g}$ given by $P(X) = (tr\,X)^2 + (tr\,X^2)^2 + \cdots + (tr\,X^d)^2$, where $d := \dim \mathfrak{g}$, since one has the equivalence: $P(X) = 0 \iff X$ is nilpotent. □

In this proof, we have used the following lemma.

Lemma 2.17. — *Let $\varphi : G \to H$ be a $\mathbb{Q}$-isogeny between two semisimple $\mathbb{Q}$-groups. Then $\varphi(G_{\mathbb{Z}})$ and $H_{\mathbb{Z}}$ are commensurable.*

Remark One must be aware that, even though φ is surjective, $\varphi(G_{\mathbb{Q}})$ and $H_{\mathbb{Q}}$ are not commensurable. Take for instance $G = \mathrm{SL}(2)$ and $H = \mathrm{PGL}(2)$, and look at the elements of $H_{\mathbb{Q}}$ given by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. — One may suppose that $H = G/C$ where C is the center of G. Let $G \subset \mathrm{End}(V)$ be this $\mathbb{Q}$-group, $A := \mathrm{End}_C(V)$ the commutator of C in $\mathrm{End}(V)$, $A_{\mathbb{Z}} = A \cap \mathrm{End}(V_{\mathbb{Z}})$, $\Gamma := G_{\mathbb{Z}} = \{g \in G \;/\; gA_{\mathbb{Z}} = A_{\mathbb{Z}}\}$, and $\Delta := \{g \in G \;/\; gA_{\mathbb{Z}}g^{-1} = A_{\mathbb{Z}}\}$. Using the fact that a bijective $\mathbb{Q}$-morphism is a $\mathbb{Q}$-isomorphism, one only has to show that Δ/Γ is finite.

According to Propositions 2.5 and 2.9, we only have to show that no sequence $d_n a_n$ with $d_n \in \Delta$ and $a_n \in A_{\mathbb{Z}} - \{0\}$ can converge to zero. Since the semisimple associative algebra A is the direct sum of its minimal bilateral ideals B, one may suppose that a_n is in some $B_{\mathbb{Z}} - \{0\}$. Let b_i be a basis of B. Since $\det_B d_n = 1$, according to Minkowski's lemma 2.4, one can find a constant C_0 and nonzero elements $c_n \in B_{\mathbb{Z}}$ such that $\|c_n d_n^{-1}\| \leq C_0$. Since the elements $a_n b_i c_n$ are in $B_{\mathbb{Z}}$, the elements $d_n a_n b_i c_n d_n^{-1}$ are also in $B_{\mathbb{Z}}$ and converge to zero. Hence, successively, for $n \gg 0$, one has $a_n b_i c_n = 0$, $a_n B c_n B = 0$, $a_n B = 0$, and $a_n = 0$. Contradiction. □

Corollary 2.18. — *In Example* 2.1.7, Γ *is cocompact in* $\mathrm{SL}(d, \mathbb{R})$.

Proof. — The proof is similar to that of Corollary 2.13, using "restriction of scalar".

The algebraic group

$$G := \left\{ \begin{pmatrix} a & \sqrt{2}\,b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d, \mathbb{C})/(a + \sqrt[4]{2}\,b)\, J_{p,q}\, ({}^t a - \sqrt[4]{2}\,{}^t b) = J_{p,q}\;,\; \det(a + \sqrt[4]{2}\,b) = 1 \right\}$$

is defined over $k_0 = \mathbb{Q}[\sqrt{2}]$, because the family of equations is τ-invariant. The "image" of G by the Galois involution σ of k_0 is the algebraic group

$$G^\sigma := \left\{ \begin{pmatrix} a & -\sqrt{2}\,b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d, \mathbb{C})/(a + i\sqrt[4]{2}\,b) J^\sigma_{p,q} ({}^t a - i\sqrt[4]{2}\,{}^t b) = J^\sigma_{p,q}\;,\; \det(a + i\sqrt[4]{2}\,b) = 1 \right\}$$

which is also defined over $k_0 = \mathbb{Q}[\sqrt{2}]$.

Using the diagonal embedding $x \to (x, x^\sigma)$ of $\mathbb{Q}[\sqrt{2}]$ in $\mathbb{R} \times \mathbb{R}$, one constructs a semisimple $\mathbb{Q}$-group

$$H := \left\{ \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} \in \mathrm{GL}(4d,\mathbb{C}) \;/\; c+\sqrt{2}\,d \in G \;,\; c-\sqrt{2}\,d \in G^\sigma \right\}$$

The maps $(c,d) \to c+\sqrt{2}\,d$ and $(a,b) \to a+\sqrt[4]{2}\,b$ give isomorphisms

$$H_{\mathbb{Z}} \;\simeq\; G_{\mathbb{Z}[\sqrt{2}]} \;\simeq\; \Gamma$$

and the map $(c,d) \to (c+\sqrt{2}\,d, c-\sqrt{2}\,d)$ gives an isomorphism

$$H_{\mathbb{R}} \;\simeq\; G_{\mathbb{R}} \times (G^\sigma)_{\mathbb{R}} \;\simeq\; \mathrm{SL}(d,\mathbb{R}) \times \mathrm{SU}(d,\mathbb{R})\ .$$

Since $\sqrt[4]{2} \in \mathbb{R}$, the group $G_{\mathbb{R}}$ is isomorphic to $\mathrm{SL}(d,\mathbb{R})$. Since the hermitian form h_0 on $\mathbb{C}^d$ whose matrix is $J^\sigma_{p,q}$ is positive definite, the group $(G^\sigma)_{\mathbb{R}}$ is compact. To apply Theorem 2.16, one uses Lemma 2.15 and notices that $H_{\mathbb{Q}}$ does not contain any unipotent element, since its image by $(c,d) \to c-\sqrt{2}\,d$ lies in the compact group $(G^\sigma)_{\mathbb{R}}$. This proves that $H_{\mathbb{Z}}$ is cocompact in $H_{\mathbb{R}}$. Since $(G^\sigma)_{\mathbb{R}}$ is compact, Γ is a cocompact lattice in $\mathrm{SL}(d,\mathbb{R})$. □

2.9. A general overview. —

Let us now describe, without proof, the general theory that these examples illustrate. Roughly speaking, this theory says that for $d \geq 3$ and $q \geq 2$ all lattices in $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$ are constructed in a similar way.

More precisely, let $H \subset \mathrm{GL}(d,\mathbb{C})$ be a $\mathbb{Q}$-group. Then one has the equivalences:

$$\mathrm{vol}(H_{\mathbb{R}}/H_{\mathbb{Z}}) < \infty \iff H \text{ has no nontrivial } \mathbb{Q}-\text{character;}$$
$$H_{\mathbb{R}}/H_{\mathbb{Z}} \text{ is compact} \iff H \text{ has no nontrivial } \mathbb{Q}-\text{cocharacter.}$$

One says that H is $\mathbb{Q}$-*anisotropic* when it does not have any nontrivial cocharacter, i.e. when it does not contain any $\mathbb{Q}$-subgroups $\mathbb{Q}$-isomorphic to G_m.

These facts, due to Borel and Harish-Chandra, are the main motivations of Borel's book [**4**], and are illustrated by Examples 1 to 4.

There is a very important construction of lattices which is simultaneously an extension and a by-product of the previous construction: let $L \subset \mathrm{GL}(d,\mathbb{C})$ be a semisimple algebraic group defined over a number field k, $\mathcal{O}$ the ring of integers of k, $\sigma_1, .., \sigma_{r_1}$ the real embeddings of k, and $\sigma_{r_1+1}, .., \sigma_{r_1+r_2}$ the complex embeddings of k up to complex conjugation. Recall that the image of the diagonal map $\sigma : \mathcal{O} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is a lattice in this real vector space. Thus the diagonal image of the group $L_{\mathcal{O}} := L \cap \mathrm{SL}(d,\mathcal{O})$ in the product $L^{\sigma_1}_{\mathbb{R}} \times \cdots \times L^{\sigma_{r_1}}_{\mathbb{R}} \times L^{\sigma_{r_1+1}}_{\mathbb{C}} \times \cdots \times L^{\sigma_{r_1+r_2}}_{\mathbb{C}}$ is also a lattice. According to Weil's trick of "restriction of scalars" this construction with $L_{\mathcal{O}}$ can be seen as a special case of the previous construction of $H_{\mathbb{Z}}$ for some suitable algebraic group H defined over $\mathbb{Q}$ for which $H_{\mathbb{Q}} \simeq L_k$ (this is illustrated by Examples 5 and 6).

Suppose that $r_1 + r_2 > 2$ and that, in the above product, all the factors are compact except one. Then $L_{\mathcal{O}}$ is a cocompact lattice in the noncompact factor (this is illustrated by Examples 7 and 8).

These examples are the main motivation for the following definition.

Definition 2.19. — *A subgroup* Γ *of a real linear semisimple Lie group* G *is said to be arithmetic if there exist an algebraic group* H *defined over* $\mathbb{Q}$ *and a surjective morphism* $\pi : H_{\mathbb{R}} \to G$ *with compact kernel such that the groups* Γ *and* $\pi(H_{\mathbb{Z}})$ *are commensurable (i.e. their intersection is of finite index in both of them).*

The classification of all arithmetic groups Γ of a given real linear semisimple Lie group G, up to commensurability, relies on the classification of all algebraic absolutely simple groups defined over a number field k (see [**23**] for a reduction of this classification to the anisotropic case). For groups G of classical type different from D_4, this classification is due to Weil ([**28**]) and is equivalent to the classification of all central simple algebras D with antiinvolution $*$ over k (i.e $(a_1^*)^* = a_1$ and $(a_1 a_2)^* = a_2^* a_1^*\ \forall a_1, a_2 \in D$).

Since k is a number field, this is a classical topic in arithmetic, which contains
- the classification of central division algebras over k;
- the classification of bilinear symmetric or antisymmetric forms over k;
- the classification of hermitian forms over a quadratic extension $k \supset k_0$.

The main tool in this classification is the local-to-global principle (see [**19**] or [**29**]).

According to a theorem of Borel ([**4**]), all linear real semisimple Lie groups contain at least one cocompact and one noncocompact lattice.

For an arithmetic subgroup Γ of $G = \mathrm{SL}(d, \mathbb{R})$, Weil's classification implies that, up to commensurability, either
- Γ is the group Γ_1 of units of an order $\mathcal{O}_D$ in a central simple algebra D of rank d over $\mathbb{Q}$ which splits over $\mathbb{R}$ (this generalizes Examples 1 and 4), or
- Γ is the group Γ_2 of $*$-invariant units of an order $\mathcal{O}_D$ in a central simple algebra D of rank d over a real field k, and $*$ is an antiinvolution of D nontrivial on k such that D is split over $\mathbb{R}$ and all the other embeddings of the fixed field k_0 of $*$ in k are real and extend to a complex embedding of k whose corresponding real unitary group is compact (this generalizes Examples 3 and 7).

Moreover, Γ_1 is cocompact if and only if D is a division algebra and Γ_2 is cocompact if and only if either D is a division algebra or $k_0 \neq \mathbb{Q}$.

Conversely, the Margulis arithmeticity theorem says the following. Let G be a real semisimple Lie group of rank at least 2, with no compact factor (a factor is a group G' which is a quotient of G), then *all irreducible lattices* Γ *in* G *are arithmetic groups* (irreducible means that any projection of Γ in a nontrivial factor of G is nondiscrete). This theorem is the main aim of Zimmer's book [**30**] and of Margulis' book [**15**].

3. Lecture on Representations

The aim of this lecture is to show how the properties of the unitary representations of a Lie group G have an influence on the algebraic structure of any lattice Γ of G.

We will deal here with a property due to Kazhdan. Namely, using the decreasing properties of the coefficients of unitary representations of G, when G is simple of rank at least 2, we will show that the abelianization of Γ is finite. We will also see that these properties imply mixing properties for some non-relatively compact flows on G/Γ.

3.1. Decay of coefficients. —

We will first prove a general decreasing property for coefficients of unitary representations of semisimple real Lie groups.

Definition 3.1. — *A unitary representation π of a locally compact group G in a (separable) Hilbert space $\mathcal{H}_\pi$ is a morphism from G to the group $U(\mathcal{H}_\pi)$ of unitary transfomations of $\mathcal{H}_\pi$, such that $\forall v \in \mathcal{H}_\pi$, the map $G \to \mathcal{H}_\pi; g \mapsto \pi(g)v$ is continuous.*

For any $v, w \in \mathcal{H}_\pi$, the coefficient is the continuous function $c_{v,w} : G \to \mathbb{C}$ given by $c_{v,w}(g) = <\pi(g)v, w>$.

Examples - The trivial representation is the constant representation $\pi(g) = Id$. Its coefficients are constant maps.
- Suppose G acts continuously on a locally compact space X preserving a Radon measure ν. Then the formula $(\pi(g)\varphi)(x) := \varphi(g^{-1}x)$ defines a unitary representation π of G in $L^2(X, \nu)$. Its coefficients are the correlation coefficients $c_{\varphi,\psi} : g \to \int_G \varphi(x)\overline{\psi}(gx)d\nu(x)$.
- When G is compact, any unitary representation is a hilbertian orthogonal sum of irreducible unitary representations. By Peter-Weyl, these are finite dimensional.

For $H \subset G$, let us set

$$\mathcal{H}_\pi^H := \{v \in \mathcal{H}_\pi \ / \ \forall h \in H,\ \pi(h)v = v\}$$

the subspace of H-invariant vectors. Recall that a Lie group G is *semisimple* if its Lie algebra $\mathfrak{g}$ does not have any nonzero solvable ideal (or equivalently, if the group of automorphims of $\mathfrak{g}$ is a semisimple $\mathbb{R}$-group) and that G is *quasisimple* if $\mathfrak{g}$ is simple.

Theorem 3.2. — **(Howe, Moore)** *Let G be a connected semisimple real Lie group with finite center and π be a unitary representation of G. Suppose that $\mathcal{H}_\pi^{G_i} = 0$ for every connected normal subgroup $G_i \neq 1$. Then, for all $v, w \in \mathcal{H}_\pi$, one has*

$$\lim_{g \to \infty} <\pi(g)v, w> \quad = \quad 0. \tag{3}$$

Remarks - The proof of this theorem is postponed to Section 3.4.
- The symbol $g \to \infty$ means that g goes out of any compact of G.
- There are only finitely many G_i. When G is quasisimple, the hypothesis is $\mathcal{H}_\pi^G = 0$.

Corollary 3.3. — *Let G be a connected semisimple real Lie group with finite center and π be a unitary representation of G without nonzero G-invariant vectors. Let H be a closed subgroup of G whose images in the factors $G/G_i \neq 1$ are noncompact. Then $\mathcal{H}_\pi^H = 0$.*

Remark - When $\mathfrak{g}$ is simple, the hypothesis is H *noncompact.*

Proof. — By induction, one can suppose that $\forall i$, $\mathcal{H}_\pi^{G_i} = 0$. Let v be a H-invariant vector. The coefficient $c_{v,v}$ is constant on H. By Theorem 3.2, it has to be zero. Hence $v = 0$. □

3.2. Invariant vectors for SL(2). —

Let us begin with a direct proof of Corollary 3.3 for $\mathrm{SL}(2, \mathbb{R})$.

For $t > 0$ and $s \in \mathbb{R}$, let $a_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $u_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $u_s^- := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$.

Proposition 3.4. — *Let π be a unitary representation of $G = \mathrm{SL}(2, \mathbb{R})$, $t \neq 1$, $s \neq 0$ and $v \in \mathcal{H}_\pi$. If v is either a_t-invariant, u_s-invariant or u_s^--invariant then it is G-invariant.*

The proof uses the following lemma

Lemma 3.5. — **(Mautner)** *Let π be a unitary representation of a locally compact group G. For $v \in \mathcal{H}_\pi$, $\|v\| = 1$, let $S_v \subset G$ be its stabiliser $S_v = \{g \in G \ / \ \pi(g)v = v\}$. Then*
a) $S_v = \{g \in G \ / \ c_{v,v}(g) = 1\}$.
b) Let $g \in G$ such that there exist $g_n \in G$, $s_n \in S_v$, $s'_n \in S_v$ satisfying $\lim_{n\to\infty} g_n = g$, $\lim_{n\to\infty} s_n g_n s'_n = e$. Then g is in S_v.

Proof. — a) Use the equality $\|\pi(g)v - v\|^2 = 2\,\|v\|^2 - 2\,\mathrm{Re}(c_{v,v}(g))$.
b) Let n go to infinity in the equality $c_{v,v}(g_n) = c_{v,v}(s_n g_n s'_n)$ to get $c_{v,v}(g) = 1$. □

Proof of Proposition 3.4. — It is enough to prove that the invariance of v by one among a_t, u_s, u_s^- implies the invariance by the other two. Thanks to symmetries, there are only two cases to deal with:

$\mathbf{a_t}$-invariant $\Longrightarrow$ $\mathbf{u_s}$-invariant. One may suppose $t > 1$. One uses Lemma 3.5.b with $g_n = g = u_s$, $s_n = a_t^{-n}$ and $s'_n = a_t^n$. One easily checks that $\lim_{n\to\infty} s_n g_n s'_n = \lim_{n\to\infty} u_{t^{-2n}s} = e$.

$\mathbf{u_s}$-invariant $\Longrightarrow$ $\mathbf{a_t}$-invariant. One may suppose that t is rational, $t = \frac{p}{q}$. One uses Lemma 3.5.b with $g = a_t$, $g_n = \begin{pmatrix} \frac{p}{q} & 0 \\ \frac{t-1}{snp} & \frac{q}{p} \end{pmatrix}$, $s_n = u_s^{-np}$ and $s'_n = u_s^{nq}$. One easily checks that $\lim\limits_{n\to\infty} s_n g_n s'_n = \lim\limits_{n\to\infty} \begin{pmatrix} 1 & 0 \\ \frac{t-1}{snp} & 1 \end{pmatrix} = e$. □

3.3. Real semisimple Lie groups. —

To prove Theorem 3.2, we recall without proof basic facts on the structure of semisimple Lie groups (see [**12**]). We use the language of root systems and parabolic subgroups which, since E.Cartan, is the only convenient one which allows to deal with all real semisimple Lie groups. At the end we will recall the meaning of these concepts for the important example $G = \mathrm{SL}(d, \mathbb{R})$.

Let G be a connected semisimple Lie group with finite center.

Maximal compact subgroups The group G contains a maximal compact subgroup K and all such subgroups are conjugate. Let $\mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie algebras. There exists an involution θ of $\mathfrak{g}$, called the *Cartan involution*, whose fixed point set is $\mathfrak{k}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ where $\mathfrak{q}$ is the fixed point set of $-\theta$. The Killing form $K(X, Y) = tr(adX \circ adY)$ is positive definite on $\mathfrak{q}$ and negative definite on $\mathfrak{k}$.

Cartan subspaces An element X of $\mathfrak{g}$ is said to be *hyperbolic* if $\mathrm{ad}(X)$ is diagonalizable over $\mathbb{R}$. A *Cartan subspace* of $\mathfrak{g}$ is a commutative subalgebra whose elements are hyperbolic and which is maximal for these properties. All Cartan subspaces are conjugate and a maximal commutative algebra in $\mathfrak{q}$ is a Cartan subspace. Let us choose one of them $\mathfrak{a} \subset \mathfrak{q}$ and set $A := \exp(\mathfrak{a})$. By definition, the *real rank* of G is the dimension of $\mathfrak{a}$. The set of real characters of the Lie group A can be identified with the dual $\mathfrak{a}^\star$. Endowed with the Killing form, this space is Euclidean.

Restricted roots Let us diagonalize $\mathfrak{g}$ under the adjoint action of A. One denotes by Δ the set of *restricted roots*, i.e. the set of nontrivial weights for this action. It is a root system. One has a decomposition $\mathfrak{g} = \mathfrak{l} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$, where $\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} \;/\; \forall g \in A\;,\; \mathrm{Ad}g(Y) = \alpha(g)Y\}$ is the root space associated to α and $\mathfrak{l}$ is the centralizer of $\mathfrak{a}$.

Weyl chambers Let Δ^+ be a choice of positive roots, $\Delta^- = -\Delta^+$, and Π the set of simple roots. Π is a basis of $\mathfrak{a}^\star$. Let $\mathfrak{u}^\pm := \oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha$ and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$ the *minimal parabolic subalgebra* associated to Δ^+. Its normaliser $P := N_G(\mathfrak{p})$ is the *minimal parabolic subgroup* associated to Δ^+. Let $A^+ := \{a \in A \;/\; \forall\alpha \in \Delta^+\;,\; \alpha(a) \geq 1\}$ be the corresponding *Weyl chamber* in A. One has the *Cartan decomposition* $G = KA^+K$. Let L be the centralizer of $\mathfrak{a}$ in G and $U^\pm$ be the connected groups with Lie algebra $\mathfrak{u}^\pm$. One has the equality $P = LU^+$.

Parabolic subgroups For every subset $\theta \subset \Pi$, one denotes by $<\theta>$ the vector space generated by θ, $\Delta_\theta := \Delta \cap <\theta>$, $\Delta_\theta^\pm := \Delta_\theta \cap \Delta^\pm$, $\mathfrak{l}_\theta := \mathfrak{l} \oplus \oplus_{\alpha\in\Delta_\theta} \mathfrak{g}_\alpha$, $\mathfrak{u}_\theta^\pm := \oplus_{\alpha\in\Delta^\pm - \Delta_\theta^\pm} \mathfrak{g}_\alpha$, $U_\theta^\pm$ the associated connected groups, $A_\theta := \{a \in A \,/\, \forall \alpha \in \theta \;,\; \alpha(a) = 1\}$, $A_\theta^+ := A^+ \cap A_\theta$, L_θ the centralizer of A_θ in G. Let $\mathfrak{p}_\theta := \mathfrak{l}_\theta \oplus \mathfrak{u}_\theta^+$ and $P_\theta := L_\theta U_\theta^+$ the *parabolic subalgebra and subgroup* associated to θ. One knows the following.
(1) *The quotient L/A is compact.*
(2) *Every group containing P is equal to some P_θ.*
(3) *P_θ is generated by the subgroups $P_{\{\alpha\}}$ for $\alpha \in \theta$.*
(4) *The multiplication $m : U^- \times P \to G$ is a diffeomorphism onto a open subset of full measure.*
(5) *If $\theta_1 \subset \theta_2$, then $\Delta_{\theta_1} \subset \Delta_{\theta_2}$, $P_{\theta_1} \subset P_{\theta_2}$ and $U_{\theta_1}^+ \supset U_{\theta_2}^+$.*

Example $G = \mathrm{SL}(d, \mathbb{R})$. One can take
$K = \mathrm{SO}(d, \mathbb{R})$,
$A = \{a = \mathrm{diag}(a_1, \ldots, a_d) \,/\, a_i > 0 \;,\; a_1 \cdots a_d = 1\}$,
$A^+ = \{a \in A \,/\, a_1 \geq \cdots \geq a_d\}$,
$\Delta = \{\varepsilon_i - \varepsilon_j \;,\; i \neq j \;,\; 1 \leq i, j \leq d\}$,
$\Delta^+ = \{\varepsilon_i - \varepsilon_j \;,\; 1 \leq i < j \leq d\}$,
$\Pi = \{\varepsilon_{i+1} - \varepsilon_i \;,\; 1 \leq i < d\}$,
where $\varepsilon_i \in \mathfrak{a}^\star$ is the differential of the character of A denoted by the same symbol: $\varepsilon_i(a) = a_i$. The root spaces $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ are 1-dimensional (with basis $E_{i,j} = e_j^\star \otimes e_i$) and one has $\mathfrak{l} = \mathfrak{a}$ [Note that these two properties are satisfied only for *split* semisimple Lie groups. They are not satisfied for $\mathrm{SO}(p, q)$ when $p \geq q + 2 \geq 3$]. One has then,

$$\mathfrak{u}^+ = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\} \quad , \quad \mathfrak{p} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \quad , \quad \mathfrak{u}^- = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\} \quad .$$

Choosing for instance θ^c with only two simple roots, one has, in terms of block matrices,

$$\mathfrak{u}_\theta^+ = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad , \quad \mathfrak{p}_\theta = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \quad , \quad \mathfrak{u}_\theta^- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \right\} \quad ,$$

$$\mathfrak{l}_\theta = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad , \quad A_\theta = \left\{ \begin{pmatrix} b_1 Id & 0 & 0 \\ 0 & b_2 Id & 0 \\ 0 & 0 & b_3 Id \end{pmatrix} \in A \right\} \quad ,$$

and $A_\theta^+ = A_\theta \cap A^+$. Note that another value for θ would give different numbers and sizes of block matrices.

3.4. Decay of coefficients. —

In this section we give the proof of Theorem 3.2.

We will need the following lemma which is a special case of the Corollary 3.3 we have not yet proven.

Lemma 3.6. — *Let π be a unitary representation of a connected quasisimple real Lie group with finite center G, $a \neq 1$ be a hyperbolic element of G, and $u \neq 1$ be a unipotent element of G. If v is either a-invariant or u-invariant then it is G-invariant.*

Proof. — First case: v is a-invariant. One can suppose $a \in A^+$. Let $\theta := \{\alpha \in \Pi \;/\; \alpha(a) = 1\}$. The same argument as in Proposition 3.4 shows that v is invariant by U_θ and U_θ^-. One concludes that v is G-invariant thanks to the following fact that the reader can easily check for $G = \mathrm{SL}(d, \mathbb{R})$: *the two groups U_θ and U_θ^- generate G.*

Second case: v is u-invariant. According to Jacobson-Morozov, there exists a Lie subgroup S of G containing u with Lie algebra $\mathfrak{s} \simeq \mathfrak{sl}(2, \mathbb{R})$. By Proposition 3.4, v is S-invariant. Since S contains hyperbolic elements, we are back to the first case. □

Proof of Theorem 3.2. — If the coefficient $<\pi(g)v, w>$ does not decrease to 0, one can find sequences $g_n = k_{1,n} a_n k_{2,n} \in G = KA^+K$ such that

$$\lim_n <\pi(g_n)v, w> = \ell \neq 0 \quad , \quad \lim_n k_{1,n} = k_1 \quad , \quad \lim_n k_{2,n} = k_2 \quad ,$$

and for some $\alpha \in \Pi$, $\lim_n \alpha(a_n) = \infty$. One can suppose that $k_1 = k_2 = e$.

Using the weak compactness of the unit ball of $\mathcal{H}_\pi$, one can suppose that the sequence $\pi(a_n)v$ has a weak limit $v_0 \in \mathcal{H}_\pi$. This vector v_0 is nonzero since

$$<v_0, w> = \lim_n <\pi(a_n)\pi(k_{2,n})v, \pi(k_{1,n}^{-1})w> = \lim_n <\pi(g_n)v, w> \neq 0$$

Moreover, this vector is u-invariant for all $u \in U_{\{\alpha\}^c}$, because, since $a_n^{-1} u a_n \to e$,

$$\|\pi(u)v_0 - v_0\| \leq \overline{\lim_n} \|\pi(a_n)(\pi(a_n^{-1} u a_n)v - v)\| = 0 \; .$$

This contradicts Lemma 3.6. □

3.5. Uniform decay of coefficients. —

In this section, we prove that, for a higher rank semisimple Lie group G and for K-finite vectors, the decay of coefficients is uniform.

For every vector v in a unitary representation $\mathcal{H}_\pi$ of G, set

$$\delta(v) = \delta_K(v) = (\dim <Kv>)^{1/2} \in \mathbb{N}^{1/2} \cup \{\infty\}.$$

Theorem 3.7. — **(Howe, Oh)** *Let G be a connected semisimple real Lie group with finite center, such that, for all normal subgroup $G_i \neq 1$ of G, one has* $\mathrm{rank}_{\mathbb{R}}(G_i) \geq 2$. *Then there exists a K-biinvariant function $\eta_G \in \mathrm{C}(G)$ satisfying $\lim_{g\to\infty} \eta_G(g) = 0$ and such that, for all unitary representation π of G with $\mathcal{H}_\pi^{G_i} = 0$, $\forall\, i$, for any $v, w \in \mathcal{H}_\pi$, with $\|v\| = \|w\| = 1$, one has, for $g \in G$,*

$$| <\pi(g)v, w> | \leq \eta_G(g)\delta(v)\delta(w).$$

Remark The (most often) best function η_G has been computed by H.Oh ([**17**]), thanks to Harish-Chandra's function

$$\xi(t) := (2\pi)^{-1}\int_0^{2\pi}(t\cos^2 s + t^{-1}\sin^2 s)^{-1/2}ds \tag{4}$$

$$\asymp t^{-1/2}\log t \text{ for } t \gg 1\ . \tag{5}$$

For instance, for $G = \mathrm{SL}(d,\mathbb{R})$ where $d \geq 3$ and $a = \mathrm{diag}(t_1,\dots,t_d) \in A^+$, one can take

$$\eta_G(a) = \prod_{1\leq i\leq [n/2]} \xi(\frac{t_i}{t_{n+1-i}})\ . \tag{6}$$

The proof is based on the following two propositions.

Definition 3.8. — *Let σ, τ be unitary representations of G. One says that σ is weakly contained in τ, and one writes $\sigma \prec \tau$, if*

$$\forall\,\varepsilon > 0,\ \forall C \text{ compact in } G,\ \forall\, v_1,\dots,v_n \in \mathcal{H}_\sigma\ ,\ \exists\, w_1,\dots,w_n \in \mathcal{H}_\tau\ /$$
$$|<\sigma(g)v_i, v_j> - <\tau(g)w_i, w_j>| \leq \varepsilon\ ,\quad \forall\, g \in C\ ,\ \forall\, i,j \leq n\ .$$

For $g = kan \in G$ let us set $H(g) := a$, let us introduce the *Harish-Chandra spherical function* ξ_G

$$\xi_G(g) := \int_K \rho(H(gk))^{-1/2}dk \quad \text{where} \quad \rho(a) = \det_{\mathfrak{n}}(\mathrm{Ad}(a))\ .$$

The following proposition will be applied not directly to G but to a subgroup of G isomorphic to $\mathrm{SL}(2,\mathbb{R})^e$.

Proposition 3.9. — *Let G be a connected real semisimple Lie group with finite center, and π be a unitary representation of G which is weakly contained in the left regular representation λ_G. Then for every $v, w \in \mathcal{H}_\pi$ with $\|v\| = \|w\| = 1$, and every $g \in G$, one has*

$$|<\pi(g)v, w>| \leq \xi_G(g)\,\delta_K(v)\,\delta_K(w) \tag{7}$$

Proof. — Let us first prove these inequalities for the left regular representation λ_G.

First note that

for every v in $L^2(G)$, left K-finite, with $\|v\| = 1$, there exists a positive left K-invariant function $\varphi \in L^2(G)$ with $\|\varphi\| = 1$ such that, for all $x \in G$, one has $|v(x)| \leq \delta(v)\varphi(x)$.
One can take $\varphi(x) := \delta(v)^{-1}(\sum_i |v_i(x)|^2)^{1/2}$ where v_i is an orthonormal basis of $<Kv>$.

If ψ is the positive K-invariant function associated in the same way to w, one gets

$$\begin{aligned} |<\pi(g)v, w>| &\leq \int_G |v(g^{-1}x)w(x)|d\mu(x) \\ &\leq \delta(v)\delta(w)\int_G \varphi(g^{-1}x)\psi(x)d\mu(x) \leq \delta(v)\,\delta(w)\ <\pi(g)\varphi, \psi>\ . \end{aligned}$$

These functions $\varphi, \psi \in L^2(G)$ are left K-invariant, positive and of norm 1. We want to prove the majoration

$$|<\pi(g)\varphi, \psi>| \leq \xi_G(g)\ . \tag{8}$$

Using the formula for the Haar measure as in Lemma 2.3, one computes

$$\begin{aligned} |<\pi(g)\varphi, \psi>| &= \int_K \left(\int_{AN} \varphi(an)\psi(gkan)\rho(a)dadn\right) dk \\ &\leq \|\varphi\|_{L^2} \int_K \left(\int_{AN} \psi(H(gk)an)^2\rho(a)dadn\right)^{1/2} dk \\ &= \|\varphi\|_{L^2}\|\psi\|_{L^2} \int_K \rho(H(gk))^{-1/2}dk \end{aligned}$$

using the Cauchy-Schwarz inequality in $L^2(AN)$ and the K-invariance of φ and ψ.

Let us now deduce these inequalities for π. The main point is to show that, starting from a finite family v_i of vectors in $\mathcal{H}_\pi$ such that, $\forall\, k \in K$, $\forall\, i$, one has $\pi(k)v_i = \sum_j u_{i,j}(k)v_j$, then one can find vectors $w'_i \in L^2(G)$ as in Definition 3.8 satisfying moreover $\lambda_G(k)w'_i = \sum_j u_{i,j}(k^{-1})w'_j$. For this purpose, just replace the family w_i given in Definition 3.8 by $w'_i = \int_K \sum_j u_{i,j}(k)\lambda_G(k)w_j$. □

Let us compute the function ξ_G for $G = \mathrm{SL}(2,\mathbb{R})^e$.
Let us show that for $a_t = (\mathrm{diag}(t_1^{1/2}, t_1^{-1/2}), \ldots, \mathrm{diag}(t_e^{1/2}, t_e^{-1/2})) \in A^+$, one has

$$\xi_G(a_t) = \xi(t_1)\cdots\xi(t_e). \tag{9}$$

Proof. — One can suppose $e = 1$, i.e. $G = \mathrm{SL}(2,\mathbb{R})$. Then, for

$$k = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \text{ and } a_t = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}, \text{ one has}$$

$$\rho(H(a_tk)) = \|a_tk\, e_1\|^2 = t\cos^2 s + t^{-1}\sin^2 s\ ,$$

and formula (9) is a consequence of Definition (4). □

Proposition 3.10. — *Let V be a finite dimensional representation of $G := \mathrm{SL}(2,\mathbb{R})$, without nonzero invariant vectors. Let π be an irreducible unitary representation of the semidirect product $V \rtimes G$ such that $\mathcal{H}_\pi^V = 0$. Then the restriction of π to G is weakly contained in the sum $\infty\lambda_G$ of infinitely many copies of the regular representation λ_G.*

Sketch of proof of Proposition 3.10. — By Mackey's theorem (see [**30**] or [**13**]), such a representation π of $V \rtimes G$ is induced from an irreducible representation σ of a proper subgroup H of $V \rtimes G$ containing V. Such a subgroup is solvable hence amenable, so σ is weakly contained in the regular representation λ_H (see Definition 4.1). Therefore π is weakly contained in $\lambda_{V\rtimes G}$. □

Proof of Theorem 3.7 We will prove this theorem only for $SL(d,\mathbb{R})$, but with the bound given by (6). The proof in the general case is similar.

Proof. — Let $e := [\frac{d}{2}]$ and

$$S := S_1 \times \cdots \times S_e \subset G = \mathrm{SL}(d,\mathbb{R}) ,$$

where $S_i \simeq \mathrm{SL}(2,\mathbb{R})$ is the subgroup whose Lie algebra has basis $E_{i,j}, E_{j,i}, E_{i,i} - E_{j,j}$ with $j = n+1-i$. These subgroups commute. Let

$$B := A \cap S = \{a = \mathrm{diag}(t_1, \dots, t_d) \in A \;/\; t_i t_{n+1-i} = 1 \;\; \forall\, i \leq e\} \;\text{ and}$$

$$C := \{a = \mathrm{diag}(t_1, \dots, t_d) \in A \;/\; t_i = t_{n+1-i} \;\; \forall\, i \leq e\} .$$

For $g = k_1 a k_2 \in G = KA^+K$, write $a = bc$ with $b \in B$, $c \in C$.

Note that G contains some subgroups $G_i = V_i \rtimes S_i$ where $V_i \simeq \mathbb{R}^2$ has a nontrivial S_i-action. According to Lemma 3.6, $\mathcal{H}_\pi$ does not contains any V_i-invariant vector. Thus, by Proposition 3.10, the restriction $\pi|_{S_i}$ is weakly contained in the infinite sum of regular representation $\infty\lambda_{S_i}$, hence $\pi|_S$ is also weakly contained in $\infty\lambda_S$. Therefore, one can apply Proposition 3.9 and formulas (6) and (9) to get the following upper bound, where $K_S := K \cap S$,

$$\begin{aligned} |<\pi(g)v, w>| &= |<\pi(b)\pi(ck_2)v, \pi(k_1^{-1})w>| \\ &\leq \xi_S(b)\, \delta_{K_S}(\pi(ck_2)v)\, \delta_{K_S}(\pi(k_1^{-1})w) \\ &= \eta_G(a)\, \delta_{K_S}(\pi(k_2)v)\, \delta_{K_S}(\pi(k_1^{-1})w) \\ &\leq \eta_G(a)\, \delta_K(v)\, \delta_K(w) , \end{aligned}$$

because S and C commute. □

3.6. Property T. —

Definition 3.11. — *One says that a continuous representation of a locally compact group G in a Banach space B almost has invariant vectors if, one has*

$$\forall \varepsilon > 0,\ \forall C \text{ compact in } G,\ \exists v \in B \;/\; \|v\| = 1 \text{ and } \forall g \in C,\ \|g\,v - v\| \leq \varepsilon .$$

Such a vector v is called (ε, C)-invariant.

One says that G has Kazhdan's property T if every unitary representation of G which almost has invariant vectors actually has nonzero invariant vectors.

The main motivation for this definition are the following three propositions, which are essentially due to Kazhdan.

Proposition 3.12. — **(Kazhdan)** *Let G be a connected quasisimple real Lie group with finite center. If $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then G has property T.*

Proposition 3.13. — *Let Γ be a lattice in a locally compact group G.*
If G has property T, then Γ also has property T.

Proposition 3.14. — *Let* Γ *be a discrete group with property* T*. Then*
a) the group Γ *is finitely generated,*
b) the quotient $\Gamma/[\Gamma,\Gamma]$ *is finite.*

As a consequence of these three propositions, one gets the main result of this section.

Corollary 3.15. — *Let* Γ *be lattice in a connected quasisimple real Lie group with finite center. Suppose that* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$*. Then* Γ *is finitely generated and* $\Gamma/[\Gamma,\Gamma]$ *is finite.*

Proof of Proposition 3.12. — Let π be a unitary representation which almost has G-invariant vectors. This means that for every $\varepsilon > 0$ and every compact C of G, one can find a $(\varepsilon/2, C)$-invariant vector v in $\mathcal{H}_\pi$. One may suppose that $\varepsilon < 1$ and that C is K-biinvariant, containing K and containing an element g with $\eta_G(g) \leq 1/2$.

The average $w := \int_K \pi(k)v\,dk$ is a K-invariant vector of norm $\|w\| \geq 1/2$ such that, for all g in C, $\|\pi(g)w - w\| \leq \varepsilon/2$. Hence the vector $v' := w/\|w\|$ is K-invariant and (ε, C)-invariant.

If $\mathcal{H}_\pi$ does not have G-invariant vector, this contradicts the bound $c_{v',v'}(g) \leq \xi(g) \leq 1/2$ given by Theorem 3.7. □

Proof of Proposition 3.13. — Let π be a unitary representation of Γ which almost has Γ-invariant vectors. We want to prove that it actually has a Γ-invariant vector. For that we will construct a representation σ of G, the *induced representation*, and show that it almost has G-invariant vectors.

Let μ be the right Haar measure on G such that $\mu(G/\Gamma) = 1$. Note that, by conservation of the volume, this measure is also left invariant. One can choose a Borel subset $F \subset G$ such that the map $m : F \times \Gamma \to G; (x,\gamma) \mapsto x\gamma$ is a bijection. It is easy to choose F such that, for any compact C of G, $m^{-1}(C)$ is relatively compact in $G \times \Gamma$. Let us write, for $g \in G$ and $x \in G$,

$$m^{-1}(gx) = (x_g, g_x)\ .$$

Define a representation σ by

$$\mathcal{H}_\sigma = \{f : G \to \mathcal{H}_\pi \text{ measurable} \quad / \quad \forall g \in G\ ,\ \forall \gamma \in \Gamma\ ,\ f(g) = \pi(\gamma)f(g\gamma) \text{ and } \int_F \|f(x)\|^2_{\mathcal{H}_\pi} d\mu < \infty\},$$

$$(\sigma(g^{-1})f)(x) := f(gx) = \pi(g_x)f(x_g).$$

Let C be a compact of G and $\varepsilon > 0$. Let $F_1 \subset F$ be a relatively compact subset of G such that $\mu(F - F_1) < \varepsilon^2/8$, and Γ_1 be the finite set $\Gamma_1 := \{g_x\ /\ g \in C\ ,\ x \in F_1\}$. Let $v \in \mathcal{H}_\pi$ be a $(\varepsilon/2, \Gamma_1)$-invariant vector and define $f \in \mathcal{H}_\sigma$ by $f|_F = v$. This vector f is

(ε, C)-invariant, since one has the majoration, for all g in C,

$$\begin{aligned}\|\sigma(g^{-1})f - f\|^2_{\mathcal{H}_\sigma} &\leq \int_{F-F_1} 4\, d\mu(x) + \int_{F_1} \|\pi(g_x)f(x_g) - f(x)\|^2_{\mathcal{H}_\pi} d\mu(x) \\ &\leq \varepsilon^2/2 + (\varepsilon/2)^2 \leq \varepsilon^2 .\end{aligned}$$

Since G has property T, $\mathcal{H}_\sigma$ contains a nonzero G-invariant vector f_0. This function is almost always equal to a nonzero vector $v_0 \in \mathcal{H}_\pi$ which is then Γ-invariant. □

Proof of Proposition 3.14. — a) Consider the unitary representation of Γ in the Hilbert direct sum $\oplus_\Delta \ell^2(\Gamma/\Delta)$ where Δ ranges over all finitely generated subgroups of Γ. The vectors $\delta_\Delta \in \ell^2(\Gamma/\Delta) \subset \mathcal{H}$ are $(0, \Delta)$-invariant. Therefore, this representation almost has invariant vectors. Property T implies that $\mathcal{H}^\Gamma \neq 0$, hence, for some Δ, $\ell^2(\Gamma/\Delta)^\Gamma \neq 0$, Γ/Δ is finite and Γ is finitely generated.

b) Since Γ is finitely generated, if the quotient $\Gamma/[\Gamma,\Gamma]$ were infinite, there would exist a surjective morphism $\Gamma \to \mathbb{Z}$ and $\mathbb{Z}$ would have property T. However, the regular representation of $\mathbb{Z}$ in $\ell^2(\mathbb{Z})$ almost has invariant vectors, $v_n := n^{-1/2} \sum_{0\leq k<n} \delta_k$, but has no invariant vector. Contradiction. □

3.7. Ergodicity. —

One of the main applications of the decay of coefficients is the ergodicity of some flows on the quotients G/Γ of finite volume.

Proposition 3.16. — *Let G be a connected semisimple real Lie group with finite center and H be a closed subgroup of G whose images in the factors $G/G_i \neq 1$ are noncompact. Let Γ be a lattice in G and ν be the G-invariant probability measure on $X := G/\Gamma$.*

Then the action of H on X is ergodic and mixing.

Remarks - *ergodicity* means that any H-invariant measurable subset A of X satisfies $\nu(A) = 0$ or 1.
- *mixing* is a stronger property. It means that $\forall A, B \subset X$, $\lim_{h\to\infty} \nu(A \cap hB) = \nu(A)\nu(B)$.
- Note that the ergodicity of the geodesic flow or of the horocycle foliation for the associated locally symmetric space $K\backslash G/\Gamma$ is a special case of this corollary.

Proof. — Let $A \subset X$ be a H-invariant measurable subset. Let π be the unitary representation of G in $L^2_0(X) := \{f \in L^2(X,\nu) \ / \ \int_X f d\nu = 0\}$. One has $L^2_0(X)^G = 0$. The vector $v_A = \mathbf{1}_A - \nu(A)$ is a H-invariant vector in $L^2_0(X)$. By Corollary 3.3, one has $v_A = 0$. Hence $\nu(A) = 0$ or 1. This proves ergodicity. Mixing uses the same proof with Theorem 3.2 and the equality

$$\nu(hA \cap B) - \nu(A)\nu(B) = <\pi(h)v_A, v_B>.$$

□

Corollary 3.17. — *Let* Γ *be a lattice in a quasisimple real Lie group* G *and* $a \in A^+$, $a \neq 1$. *Then, for almost all* $x \in G/\Gamma$, *the semi-orbit* $\{a^n x \ , \ n \geq 0\}$ *is dense in* G/Γ.

Proof. — The density of almost all quasi-orbits is a classical consequence of ergodicity: since G/Γ is metrisable separable, it is enough to show that, for almost all open set O, the union $\cup_{n\geq 0} a^{-n}O$ is of full measure. This follows from its a-invariance and from ergodicity. □

Remark The mixing speed can be estimated thanks to the uniform decay of coefficients given in Theorem 3.7.

4. Lecture on Boundaries

The aim of this lecture is to show how measurable Γ-equivariant maps between "boundaries" can be used to prove some algebraic properties for a lattice Γ in a higher rank simple Lie group G.

We will prove a theorem of Margulis which says that Γ is almost simple, i.e. any normal subgroup of Γ is either finite or of finite index.

Note that the same tool is at the heart of the proof of the Margulis superrigidity theorem, but we will not discuss this here.

4.1. Amenability. —

Let us recall a few different equivalent definitions of amenability.

In this section G is a locally compact (metrisable and separable) space and μ is a left Haar measure on G. Let

$$\mathrm{UCB}(G) := \{f \in L^\infty(G) \;/\; \lim_{y\to e} \sup_{x\in G} |f(yx) - f(x)| = 0\}$$

be the set of bounded functions $f : G \to \mathbb{C}$ which are left uniformly continuous.

A *mean* on $L^\infty(G)$ or on $\mathrm{UCB}(G)$ is a linear form m such that

$$m(1) = 1 \text{ and } (f \geq 0 \Rightarrow m(f) \geq 0).$$

Such a linear form m is real, i.e. $m(\mathrm{Re}(f)) = \mathrm{Re}(m(f))$ and continuous: $|m(f)| \leq \|f\|_\infty$.

Let 1_G be the trivial representation of G in $\mathbb{C}$, λ_G the regular representation of G, and $\infty\lambda_G$ the Hilbert direct sum of infinitely many copies of λ_G.

Definition 4.1. — **(Godement, Hulanicki)** *A locally compact group G is amenable if it satisfies one of the following equivalent properties.*

(1) *For every continuous action of G on a compact space X, there exists an invariant probability measure on X.*

(1′) *Same statement with X metrisable.*

(2) *For every continuous affine action of G on a compact convex subset A of a Hausdorff locally convex topological vector space E, there exists a fixed point in A.*

(2′) *Same statement with E metrisable.*

(3) $\mathrm{UCB}(G)$ *has a left-invariant mean.*

(4) $L^\infty(G)$ *has a left-invariant mean.*

(5) $L^1(G)$ *almost has left-invariant vectors.*

(6) $L^2(G)$ *almost has left-invariant vectors, i.e. $1_G \prec \lambda_G$ (see Definition 3.8).*

(7) *For every irreducible unitary representation π of G, one has $\pi_G \prec \infty\lambda_G$.*

(8) $1_G \prec \infty\lambda_G$.

At first glance, these eight properties look quite different. After rereading them carefully, one realizes that all of them have to do with fixed points or almost fixed point of some G-actions.

Using these definitions, one easily obtains the following examples:

Examples - The groups $\mathbb{Z}$ and $\mathbb{R}$ are amenable. Note that these two groups do not have invariant probability measures. Examples of invariant means on $L^\infty(\mathbb{Z})$ are given by taking limits with respect to some ultrafilter of $\mathbb{Z}$. Invariant means are not unique. Property (2) for $\mathbb{Z}$ is an old result of Kakutani, whose proof is the following sentence: one can get a fixed point as a cluster point of the sequence of barycenters of the first n points of an orbit of $\mathbb{Z}$ in A. Property (6) for $\mathbb{Z}$ has already been proven in the proof of Proposition 3.14.
- A compact group is amenable: use property (6).
- An extension of two amenable groups is amenable: use property (2).
- A solvable Lie group is amenable.
- A noncompact semisimple real Lie group is not amenable: property (1) is not satisfied since the action of G on G/P does not have any invariant probability measure.

The main tools in the proof of these equivalences are those provided by classical functional analysis.

Proof. — Let us prove $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2')$.
$(1') \Rightarrow (2')$ The barycenter of a G-invariant probability measure on A is a fixed point.
$(2') \Rightarrow (1)$ Let $F := \mathrm{C}(X) = \{$continous functions on $X\}$, $E := F^\star = \mathcal{M}(X) = \{$bounded measures on $X\}$ and $A := \mathcal{P}(X) = \{$probabilities on $X\} \subset E$. The action of G on A is continuous and affine. Here E is not assumed to be metrisable but, since G is separable, one can write $F = \cup F_\alpha$ where the F_α are the separable closed G-invariant vector subspaces of F. For all α, let $p_\alpha : E \to F_\alpha^\star$ be the restriction map from F to F_α. Since $F_\alpha^\star$ is metrisable, G has a fixed point in $p_\alpha(A)$. The intersection of the family of nonempty compact sets $A_\alpha := \{a \in A \;/\; p_\alpha(a)$ is G-invariant$\}$ is also nonempty. It is the set of fixed points of G in A.
$(1) \Rightarrow (2)$ Same as $(1') \Rightarrow (2')$ but without the metrisability hypothesis.
$(2) \Rightarrow (1')$ Same as $(2') \Rightarrow (1)$ but easier.

Let us prove the equivalences of (3) and (4) with the previous ones.
$(2) \Rightarrow (3)$ The action of G on $\mathrm{UCB}(G)$ by left-translations, given by $\pi(g)f(x) := f(g^{-1}x)$, $\forall f \in \mathrm{UCB}(G)$, $\forall g, x \in G$, is continuous. Hence the action on the set A of means on $\mathrm{UCB}(G)$ is also continuous. This set is closed, convex, and bounded, hence weakly compact. Any fixed point of this action is an invariant mean.
$(3) \Rightarrow (4)$ First recall a few basic definitions and properties of convolution.
For $\nu \in \mathcal{P}(G)$, $F \in L^\infty(G)$, $f \in \mathrm{UCB}(G)$, $\alpha \in \mathrm{C}_c(G)$ and $x \in G$, one has
$$\nu \star F\,(x) = \int_G F(y^{-1}x)d\nu(y)\ ,$$
$$\alpha \star f\,(x) = \int_G \alpha(y)f(y^{-1}x)d\mu(y).$$
Recall that one has $\nu\star F \in L^\infty(G)$ and $\alpha\star F \in \mathrm{UCB}(G)$. Moreover, for any approximation of identity $\beta_n \in \mathrm{C}_c(G)$, the sequence $\beta_n\star f$ converges uniformly to f. One has also similar statements with $F \star \alpha$ and $f \star \beta_n$.

We will show the following assertion.

Lemma 4.2. — *If* UCB(G) *has a left invariant mean m, then there exists an invariant mean $\widetilde{m}$ on $L^\infty(G)$ such that $\forall \alpha \in \mathrm{C}_c(G)$ satisfying $\int_G \alpha d\mu = 1$, $\forall F \in L^\infty(G)$, one has $\widetilde{m}(\alpha \star F) = \widetilde{m}(F)$.*

Proof. — First notice that, since $\alpha \star f$ is a uniform limit of averages of translates of f, one has $m(\alpha \star f) = m(f)$ for any $f \in \mathrm{UCB}(G)$. Then fix an element $\alpha_0 \in \mathrm{C}_c(G)$ such that $\int_G \alpha_0 d\mu = 1$ and define a mean $\widetilde{m}$ on $L^\infty(G)$ by $\widetilde{m}(F) = m(\alpha_0 \star F)$. One computes

$$\begin{aligned}\widetilde{m}(\alpha \star F) &= \lim_n m(\alpha_0 \star \alpha \star F \star \beta_n) = \lim_n m(F \star \beta_n) = \lim_n m(\alpha_0 \star F \star \beta_n) \\ &= m(\alpha_0 \star F) = \widetilde{m}(F).\end{aligned}$$

Now we deduce from this the invariance of $\widetilde{m}$. Define as above $\pi(g)F : x \to F(g^{-1}x)$, note that $\alpha \star \pi(g)F = \alpha_g \star F$ where $\alpha_g(x) := \Delta(g)^{-1}\alpha(xg^{-1})$, here Δ denotes the modulus function. Compute

$$\widetilde{m}(\pi(g)F) = \widetilde{m}(\alpha \star \pi(g)F) = \widetilde{m}(\alpha_g \star F) = \widetilde{m}(F).$$

□

(4) ⇒ (1) Let m be an invariant mean on $L^\infty(G)$. Fix a point $x_0 \in X$ and associate to any function $h \in \mathrm{C}(X)$ a function $\widetilde{h} \in L^\infty(G)$ defined by $\widetilde{h}(g) = f(g\,x_0)$. The formula $\mu(h) = m(\widetilde{h})$ then defines a G-invariant probability measure on X.

Let us prove the equivalence of (5) with the previous properties.

(3) ⇒ (5) Recall that $L^\infty(G) \simeq L^1(G)^\star$, and denote by $\varphi \to m_\varphi$ the natural injection of $L^1(G)$ in the dual of $L^\infty(G)$ given by $m_\varphi(F) = \int_G \varphi F d\mu$. Let

$$P(G) := \{\varphi \in L^1(G) \ / \ \varphi \geq 0 \ \text{ and } \ \int_G \varphi d\mu = 1\}.$$

Let us first show that $P(G)$ *is weakly dense in the set of means on* $L^\infty(G)$. To that purpose, notice that, if we endow $L^\infty(G)^\star$ with the weak topology, its dual is $L^\infty(G)$. If a mean m was not in the weak closure of the convex set $P(G)$, the Hahn-Banach theorem would give an element $F \in L^\infty(G)$ such that $m(F) > \ell$ where $\ell := \sup_{\varphi\in P(G)} \int_G \varphi F d\mu$ is the essential sup of F. Contradiction with $m(\ell - F) \geq 0$.

Choose a mean $\widetilde{m}$ on $L^\infty(G)$ as in Lemma 4.2. Let $\varphi_j \in P(G)$ be a filter such that m_{φ_j} converges to $\widetilde{m}$. Note that, since $L^\infty(G)^*$ is not metrisable, one has to use filters instead of sequences. From the equalities $m_{\alpha\star\varphi_j}(F) = m_{\varphi_j}(\alpha' \star F)$ where $\alpha'(g) = \Delta(g)^{-1}\alpha(g^{-1})$, one deduces that

$\alpha \star \varphi_j - \varphi_j$ *weakly converges to* 0 *for all* $\alpha \in \mathrm{C}_c(G)$ *satisfying* $\int_G \alpha d\mu = 1$.

Let us show that *one can choose φ_j such that this convergence is strong.* Let $\prod_\alpha L^1(G)$ be the product of infinitely many copies of $L^1(G)$ indexed by all the test functions α whose integral equals 1. Its dual is the direct sum $\oplus_\alpha L^\infty(G)$. Let $T : L^1(G) \longrightarrow \prod_\alpha L^1(G)$ be the linear map defined by $T(\varphi)_\alpha := \alpha \star \varphi - \varphi$.

We have shown that 0 belongs to the weak closure of the convex set $T(P(G))$. The Hahn-Banach theorem implies that the weak closure of a convex set is equal to its strong closure. Hence there is a filter, still denoted by $\varphi_j \in P(G)$, such that,

(10) $\|\alpha \star \varphi_j - \varphi_j\|_{L^1}$ converges to 0 for all $\alpha \in \mathrm{C}_c(G)$ satisfying $\int_G \alpha d\mu = 1$.

Let $\varepsilon > 0$ and C be a compact of G. Recall that we want to find an element $\varphi \in P(G)$ such that, for all $g \in K$, one has $\|\pi(g)\varphi - \varphi\|_{L^1} \leq \varepsilon$. Let us fix $\beta \in P(G)$, we will show that a suitable $\varphi := \beta \star \varphi_j$ works.

By continuity of the left translation in $L^1(G)$, there exists an open neighborhood $\mathcal{U}$ of e in G such that for all $y \in \mathcal{U}$, one has $\|\pi(y)\beta - \beta\|_{L^1(G)} \leq \varepsilon/3$.

Let us choose a covering of C by finitely many translates $x_1\mathcal{U}, \ldots, x_n\mathcal{U}$. Thanks to (10), one can find j such that, $\|(\pi(x_i)\beta) \star \varphi_j - \varphi_j\|_{L^1} \leq \varepsilon/3$, $\forall\, i = 1, \ldots, n$. Writing $g = x_i y$ with $i \leq n$ and $y \in E$, one gets
$\|\pi(g)\varphi - \varphi\|_{L^1}$
$\leq \|(\pi(x_i y)\beta) \star \varphi_j - (\pi(x_i)\beta) \star \varphi_j\|_{L^1} + \|(\pi(x_i)\beta) \star \varphi_j - \varphi_j\|_{L^1} + \|\varphi_j - \beta \star \varphi_j\|_{L^1}$
$\leq \|(\pi(y)\beta) \star \varphi_j - \beta \star \varphi_j\|_{L^1} + 2\varepsilon/3 \leq \|\pi(y)\beta - \beta\|_{L^1}\, \|\varphi_j\|_{L^1} + 2\varepsilon/3 \leq \varepsilon.$

(5) $\Rightarrow$ (4) By asumption, there exists a sequence $\varphi_j \in L^1(G)$ such that $\|\varphi_j\|_{L^1} = 1$ and, $\forall g \in G$, $\lim\limits_{j\to\infty} \|\pi(g)\varphi_j - \varphi_j\|_{L^1} = 0$. Replacing φ_j by $|\varphi_j|$, one may suppose that $\varphi_j \in P(G)$. Since the set of means on $L^\infty(G)$ is closed and bounded, it is weakly compact. Any cluster value m of the sequence φ_j is an invariant mean since, $\forall F \in L^\infty(G)$, one has

$$\begin{aligned} |m(\pi(g)F - F)| &= \lim \left| m_{\varphi_j}(\pi(g)F - F)\right| = \lim \left| \int_G (\pi(g^{-1})\varphi_j - \varphi_j) F d\mu \right| \\ &\leq \overline{\lim}\, \|F\|_{L^\infty} \|\pi(g^{-1})\varphi_j - \varphi_j\|_{L^1} = 0. \end{aligned}$$

Let us prove the equivalence of (6), (7) and (8) with the previous properties.

(5) $\Rightarrow$ (6) If φ is a vector of $L^1(G)$ of norm 1 such that $\|\pi(g)\varphi - \varphi\|_{L^1} \leq \varepsilon$, then $\psi := |\varphi|^{\frac{1}{2}}$ is a vector of $L^2(G)$ of norm 1 such that $\|\pi(g)\psi - \psi\|_{L^2} \leq \varepsilon$.

(6) $\Rightarrow$ (5) If ψ is a vector of $L^2(G)$ of norm 1 such that $\|\pi(g)\psi - \psi\|_{L^2} \leq \varepsilon$, then $\varphi := |\psi|^2$ is a vector of $L^1(G)$ of norm 1 such that, by Cauchy-Schwarz:
$\|\pi(g)\varphi - \varphi\|_{L^1} \leq \|\pi(g)|\psi| + |\psi|\|_{L^2}\, \|\pi(g)|\psi| - |\psi|\|_{L^2} \leq 2\varepsilon.$

(6) $\Rightarrow$ (7) The operator $U : L^2(G) \otimes \mathcal{H}_\pi \to L^2(G, \mathcal{H}_\pi)$ given by $U(\psi \otimes v)(x) = \psi(x)\pi(x^{-1})v$ defines a unitary equivalence between $\lambda_G \otimes \pi$ and the representation of G in $L^2(G, \mathcal{H}_\pi)$ given by $(g\,F)(x) = F(g^{-1}x)$ which is equivalent to $\dim(\mathcal{H}_\pi)\,\lambda_G$. Therefore, if 1_G is weakly contained in λ_G, then $\pi = 1 \otimes \pi$ is weakly contained in $\lambda_G \otimes \pi$ hence in $\infty\,\lambda_G$.

(7) $\Rightarrow$ (8) Clear.

(8) $\Rightarrow$ (4) Let us first show that *the function* $1 \in L^\infty(G)$ *belongs to the weak closure of the set* $\mathcal{C}$ *of coefficients* $c_{\psi,\psi}$ *of functions* $\psi \in L^2(G)$, *such that* $\|\psi\|_{L^2} = 1$. Our hypothesis means that, for all $\varepsilon > 0$ and all compact C of G, there exists a sequence $\psi_i \in L^2(G)$ of elements of norm 1 and a sequence $a_i \in \mathbb{C}$ such that $\sum_i |a_i|^2 = 1$ and, for all $g \in C$, $|1 - \sum_i |a_i|^2 c_{\psi_i,\psi_i}(g)| \leq \varepsilon$. Hence the function 1 belongs to the weak closure of the closed

convex hull $co(\mathcal{C})$ of $\mathcal{C}$. But the function 1 is an extremal point in the unit ball of $L^\infty(G)$. Hence it is also an extremal point of $co(\mathcal{C})$. Such a point belongs to the weak closure of $\mathcal{C}$ (this almost means $1_G \prec \lambda_G$ but not quite).

In other words, we have found a sequence $\psi_j \in L^2(G)$ with $\|\psi_j\|_{L^2} = 1$ such that the sequence of elements of $L^\infty(G)$ given by $g \mapsto \|\pi(g)\psi_j - \psi_j\|_{L^2}$ weakly converges to 0. Let $\varphi_j := |\psi_j|^2 \in L^1(G)$. The same argument as (6) $\Rightarrow$ (5) shows that the sequence of elements of $L^\infty(G)$ given by $g \mapsto \|\pi(g)\varphi_j - \varphi_j\|_{L^1}$ weakly converges to 0. This hypothesis is enough to follow the arguments of the implication (5) $\Rightarrow$ (4). Therefore, every mean m in the closure of the sequence m_{φ_j} is G-invariant. □

4.2. The normal subgroup theorem. — The following theorem is the aim of this lecture.

Theorem 4.3. — **(Kazhdan, Margulis)** *Let G be a real linear quasisimple Lie group. If $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then any lattice Γ in G is quasisimple, i.e. every normal subgroup N of Γ is either finite or of finite index.*

Remark This theorem is still true for G semisimple if the lattice Γ is irreducible. But the following proof has to be modified when G has a factor of real rank 1.

Sketch of proof of Theorem 4.3 The main idea hidden behind the proof of this theorem is to consider the σ-algebra $\mathfrak{M}(Y)^N$ of N-invariant Borel subsets of $Y := G/P$, modulo those of measure zero. Since N is normal in Γ, this σ-algebra is Γ-invariant. One first proves that any Γ-invariant σ-algebra of G/P is the inverse image of the σ-algebra of all Borel subsets on G/P' for a bigger group $P' \supset P$ (Theorem 4.5 and Lemma 4.6).

When $\mathfrak{M}(Y)^N$ is non trivial, i.e. when $P' \neq G$, the group N acts trivially on G/P' and N is in the center of G.

When $\mathfrak{M}(Y)^N$ is trivial, one shows that Γ/N is amenable. For that purpose, one constructs, for every continuous action of Γ/N on a compact metrisable space X, a boundary map, i.e. a measurable Γ-equivariant map

$$\Phi : G/P \longrightarrow \mathcal{P}(X).$$

Since $\mathfrak{M}(Y)^N$ is trivial, such a boundary map must be constant. Its image is a Γ/N-invariant probability measure on X, which proves the amenability of Γ/N. Using property T, one deduces that Γ/N is finite.

The detailed proof will last up to the end of this lecture.

4.3. The boundary map. —

Starting from an action of Γ on a compact space X, one constructs a *boundary map.*

Proposition 4.4. — **(Furstenberg)** *Let Γ be a lattice in a semisimple Lie group G acting continuously on a compact metrisable space X and P be a minimal parabolic subgroup of G. Then there exists a measurable Γ-equivariant map $\Phi : G/P \to \mathcal{P}(X)$.*

- Recall that we set $\mathrm{C}(X) := \{\text{continuous functions on } X\}$, $\mathcal{M}(X) := \{\text{bounded measures on } X\}$ and $\mathcal{P}(X) := \{\text{probabilities on} X\}$.
- We endowed implicitly G/P with a G-quasiinvariant measure, for instance a K-invariant measure.
- *Measurable* means that for every Borel subset E of the compact metrisable space $\mathcal{P}(X)$, the pullback $\Phi^{-1}(E)$ is measurable in G/P i.e. equal to a Borel subset of G/P up to some negligeable set.
- Γ-*equivariant* means that for all γ in Γ and almost all x in G/P, one has $\Phi(\gamma x) = \gamma\Phi(x)$.
- The map Φ is called *boundary map*, since G/P may be thought of as a boundary of the symmetric space G/K.

Proof. — Let $F := L^1_\Gamma(G, \mathrm{C}(X))$ be the space of Γ-equivariant measurable maps $f : G \to \mathrm{C}(X)$ such that $\|f\| := \int_{\Gamma\backslash G} \|f(g)\|_\infty dg < \infty$. Let $E := L^\infty_\Gamma(G, \mathcal{M}(X))$ be the space of bounded, Γ-equivariant, measurable maps $m : G \to \mathcal{M}(X)$. The duality

$$<m, f>:= \int_{\Gamma\backslash G} <m(g), f(g)> dg$$

gives an identification of E with the continuous dual of F, because if Y is a fundamental domain of Γ in G, one has $F \simeq L^1(Y, \mathrm{C}(X))$ and $E \simeq L^\infty(Y, \mathrm{C}(X)^\star) \simeq F^\star$. The subset $A = L^\infty_\Gamma(G, \mathcal{P}(X)) \subset E$ is convex, closed, and bounded, hence weakly compact. Right translation on G induces continuous actions of G on F, E, and A.

Since P is a compact extension of a solvable group, it is amenable and hence has a fixed point Φ in A. This point Φ is the required measurable map, since a P-invariant element of E is almost surely equal to a measurable function which is constant on the orbits of P. □

4.4. Quotients of G/P. — For every measured space (Z, μ) where μ is a σ-finite measure, one denotes by $\mathfrak{M}(Z) = \mathfrak{M}(Z, \mu)$ the σ-algebra of measurable subsets of Z modulo those of measure zero. In other words,

$$\mathfrak{M}(Z) \simeq \{f \in L^\infty(Z) \ / \ f^2 = f\}.$$

Theorem 4.3 will be a consequence of the following theorem which we will prove in the following sections.

Theorem 4.5. — **(Margulis)** *Let G be a quasisimple Lie group of real rank at least 2, Γ a lattice in G, P a minimal parabolic subgroup and $\mathfrak{M} \subset \mathfrak{M}(G/P)$ a Γ-invariant sub-σ-algebra. Then $\mathfrak{M}$ is G-invariant.*

Remarks - This theorem is still true for G semisimple if the lattice Γ is irreducible.
- Note that, by Proposition 3.16, the action of Γ on G/P is ergodic, i.e. any Γ-invariant Borel subset of G/P is of zero or full measure. Theorem 4.5 is a far-reaching extension of this assertion.
- When $\mathrm{rank}_{\mathbb{R}}(G) = 1$, any cocompact lattice Γ in G contains an infinite normal subgroup N such that Γ/N is not amenable. In this case the σ-algebra $\mathfrak{M}(G/P)^N$ is Γ-invariant but not G-invariant. The younger reader will check the existence of N when Γ is the π_1 of a compact surface, noting that such a group has a nonabelian free quotient. The more advanced reader will notice that this property is true for any Gromov hyperbolic group.

The following lemma, which is true for any separable locally compact group G, emphasizes the conclusion of Theorem 4.5.

For a closed subgroup H of G, the σ-algebra $\mathfrak{M}(G/H)$ can be identified with the σ-algebra $\mathfrak{M}(G, H)$ of Borel right H-invariant subsets of G.

Lemma 4.6. — *For every left G-invariant σ-subalgebra of $\mathfrak{M}(G)$, there exists a closed subgroup H of G such that $\mathfrak{M} = \mathfrak{M}(G, H)$.*

Remark For all closed subgroups $H_1, H_2 \subset G$, one has the equivalence

$$H_1 \subset H_2 \Longleftrightarrow \mathfrak{M}(G, H_1) \supset \mathfrak{M}(G, H_2).$$

Therefore, the σ-algebra generated by $\mathfrak{M}(G, H_1)$ and $\mathfrak{M}(G, H_2)$ is $\mathfrak{M}(G, H_1 \cap H_2)$.

Proof of Lemma 4.6. — Let $\Omega := L^\infty(G, \mathfrak{M}) \subset L^\infty(G)$ be the subspace of $\mathfrak{M}$-measurable bounded functions. Recall the *topology of convergence in measure* on $L^\infty(G)$, i.e. the topology of uniform convergence outside a suitable subset of arbitrarily small measure. The sets

$$O_{C,\varepsilon}(f) = \{g \in L^\infty(G) \;/\; \mu(\{x \in C \;/\; |g(x) - f(x)| > \varepsilon) < \varepsilon\},$$

for $C \subset G$ of finite measure and $\varepsilon > 0$, constitute a basis of neighborhoods of an element $f \in L^\infty(G)$. Let $\Omega_0 := \Omega \cap \mathrm{C}(G)$. One checks successively that
- Ω is closed in $L^\infty(G)$ for the convergence in measure;
- $\forall \varphi \in \mathrm{C}_c(G)$, $\forall f \in \Omega$, $\varphi \star f \in \Omega_0$;
- Ω_0 is dense in Ω for the convergence in measure.

For x in G, the closed subgroup $H_x := \{h \in G \;/\; f(xh) = f(x) \;\forall f \in \Omega_0\}$ does not depend on x because $\mathfrak{M}$ is G-invariant. Let us write $H := H_x$. By definition, Ω_0 is a subalgebra of $\mathrm{C}(G/H)$ which is closed for the topology of uniform convergence on all compacts and which separates points. The Stone-Weierstrass theorem shows that $\Omega_0 \simeq \mathrm{C}(G/H)$. Hence, one has $\Omega = \mathfrak{M}(G, H)$. □

Proof that Theorem 4.5 implies Theorem 4.3. — First case: Γ/N is amenable. Since $\mathit{rank}_{\mathbb{R}} G \geq 2$, G has property T. By Proposition 3.13, Γ and its quotient Γ/N also have property T. But an amenable group with property T is compact, because, since the regular representation must have an invariant vector, its Haar measure is finite. Thus Γ/N is finite.

Second case: Γ/N is not amenable. There exists a continuous action of Γ/N on a compact metrisable space X with no invariant probability measure. By Proposition 4.4, there exists a measurable Γ-equivariant map $\Phi : G/P \longrightarrow \mathcal{P}(X)$ which is not essentially constant.

Let $\mathfrak{M} := \{\Phi^{-1}(A) \ / \ A$ Borel subset of $\mathcal{P}(X)\}$ modulo the subsets of measure zero. This σ-algebra $\mathfrak{M}$ on G/P is Γ-invariant and all $M \in \mathfrak{M}$ are N-invariant. By Theorem 4.5 and Lemma 4.6, there exists a subgroup $P' \neq G$ such that $\mathfrak{M} = \mathfrak{M}(G, P')$. But then all the Borel subsets of G/P' are N-invariant, hence the action of N on G/P' is trivial and N is included in the center $Z(G) := \bigcap_{g\in G} gP'g^{-1}$ of G, since G is quasisimple. □

4.5. Contracting automorphisms. —

The proof of Theorem 4.5 relies on the following proposition, which will be proved in the next section.

Proposition 4.7. — *Let H be a separable locally compact group, $\varphi : H \to H$ a contracting automorphism and $E \subset H$ a measurable Borel subset.*

Then, for almost all h in H, one has the following convergence in measure:

$$\lim_{n\to\infty} \varphi^{-n}(hE) = \begin{cases} H & \text{if } \ hE \ni e\ , \\ \emptyset & \text{if } \ hE \not\ni e\ . \end{cases}$$

Remarks - *Contracting* means that any compact of H can be sent into any neighborhood of e by φ^n if n is sufficiently large.
- *Convergence in measure* means convergence in measure of the characteristic functions.
- To get some feeling of what happens, think of the extreme cases when hE contains or avoids a neighborhood of e.

Let us keep the notations of Section 3.3 and set $L_\theta^- = U^- \cap L_\theta$, so that one has $U^- = U_\theta^- L_\theta^-$. For every subset $E \subset U^-$, let us set

$$\psi_\theta(E) := U_\theta^- \,(E \cap L_\theta^-).$$

Corollary 4.8. — *Let $a \in A^+$, $\theta := \{\alpha \in \Pi \ / \ \alpha(a) = 1\}$, and $M \subset U^-$ be a Borel subset. Then, for almost all u in U^-, one has the following convergence in measure on U^-:*

$$\lim_{n\to\infty} a^{-n}uMa^n = \psi_\theta(uM).$$

Proof. — Since we can replace M by ℓM for ℓ in L_θ^-, it is enough to prove this assertion for almost all u in U_θ^-. Let $M_\ell := M \cap U_\theta^- \ell$ be the fibers of the projection of M on L_θ^-.

By definition, conjugation by a is a contracting automorphism of U_θ^-. By Proposition 4.7, for all ℓ in L_θ^- and almost all u in U_θ^-, one has

$$\lim_{n\to\infty} (a^{-n}uMa^n)_\ell = (\psi_\theta(uM))_\ell$$

for the convergence in measure on U_θ^-. Hence, for almost all u in U_θ^-, this assertion is true for almost all ℓ in L_θ^-. For such a u, Fubini's theorem and Lebesgue's dominated convergence theorem allow us to conclude that $\lim\limits_{n\to\infty} a^{-n}uMa^n = \psi_\theta(uM)$ for the convergence in measure on U_θ^-. □

Corollary 4.9. — *Let $\mathfrak{M}$ be a Γ-invariant sub-σ-algebra of $\mathfrak{M}(G/P)$, $M \in \mathfrak{M}$, and $\theta \subset \Pi$, with $\theta \neq \Pi$. Then, for almost all u in U^-, one has $g\psi_\theta(uM) \in \mathfrak{M}$ for all g in G.*

Corollary 4.9 is an important step towards Theorem 4.5. It allows us to construct, in any Γ-invariant σ-algebra $\mathfrak{M} \subset \mathfrak{M}(G/P)$, a G-invariant sub-σ-algebra $\mathfrak{M}_0 \subset \mathfrak{M}$. To have a chance for this σ-algebra $\mathfrak{M}_0$ to be non trivial, we will need to find $\theta \subset \Pi$ with $\emptyset \neq \theta \neq \Pi$, because for every Borel subset $E \subset G/P$, one has $\psi_\emptyset(E) = \emptyset$ or G/P.

This explains the higher-rank hypothesis in Theorem 4.5.

To prove this corollary, will need the following lemma

Lemma 4.10. — *Let Γ be a lattice in a quasisimple real Lie group G and let $a \in A^+$, $a \neq 1$. Then, for almost all $u \in U^-$, the semi-orbit $\{\Gamma u^{-1}a^n\ ,\ n \geq 0\}$ is dense in $\Gamma\backslash G$.*

Proof of Lemma 4.10. — This lemma is a consequence of Corollary 3.17, after exchanging right and left. However, one needs one more argument because the "almost all" statement is relative to the Lebesgue measure of U^-. For this, just notice that U^-P is of full measure in G, and that for any $p \in P$, since the limit $\ell := \lim\limits_{n\to\infty} a^{-n}pa^n$ exists, one has the equivalence: ($\{\Gamma ua^n\ ,\ n \geq 0\}$ is dense) $\Longleftrightarrow$ ($\{\Gamma upa^n\ ,\ n \geq 0\}$ is dense). □

Proof of Corollary 4.9. — Let $a \in A^+$ such that one has $\theta = \{\alpha \in \Pi \ /\ \alpha(a) = 1\}$. Since $\theta \neq \Pi$, one has $a \neq 1$. According to point (4) of Section 3.3, one has an identification $\mathfrak{M}(U^-) \simeq \mathfrak{M}(G/P)$. We will prove that the assertion is true for all u satisfying the conclusion of Corollary 4.8 and Lemma 4.10.

Let g be in G.One can write $g = \lim\limits_{i\to\infty} g_i$, where $g_i = \gamma_{n_i}u^{-1}a^{n_i}$, with $\gamma_{n_i} \in \Gamma$ and $n_i \to \infty$. By Corollary 4.8, the Borel subsets $a^{-n_i}uM = a^{-n_i}uMa^{n_i}$ converge in measure to $\psi_\theta(uM)$. Hence $\gamma_{n_i}M = g_ia^{-n_i}uM$ converges in measure to $g\psi_\theta(uM)$. Since $\gamma_{n_i}M$ belongs to $\mathfrak{M}$, for all i, $g\psi_\theta(uM)$ also belongs to $\mathfrak{M}$. □

Proof that Corollary 4.9 implies Theorem 4.5. — Let θ_1 be a minimal subset of Π such that $\mathfrak{M} \supset \mathfrak{M}(G/P_{\theta_1})$. Suppose that this is not an equality. Since P_{θ_1} is generated by the $P_{\{\alpha\}}$ for $\alpha \in \theta_1$, there exists $\alpha \in \theta_1$ and a subset $M \in \mathfrak{M}$ which is not right $L^-_{\{\alpha\}}$-invariant. This means that the set

$$W = \{u \in U^- \ / \ uM \cap L^-_{\{\alpha\}} \text{ and } (uM)^c \cap L^-_{\{\alpha\}} \text{ are not negligeable in } L^-_{\{\alpha\}}\}$$

is of nonzero measure in U^-. Since $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, one has $\{\alpha\} \neq \Pi$ and one can find such a u which also satisfies the conclusion of Corollary 4.9. Hence the σ-algebra $\mathfrak{M}_2$ generated by the subsets $g\psi_{\{\alpha\}}(uM)$ is a sub-σ-algebra of $\mathfrak{M}$.

By construction $\mathfrak{M}_2$ is G-invariant. According to Lemma 4.6 and to point (2) of Section 3.3, there exists a subset $\theta_2 \subset \Pi$ such that $\mathfrak{M}_2 = \mathfrak{M}(G/P_{\theta_2})$. Since $\mathfrak{M}_2$ contains a non $L^-_{\{\alpha\}}$-invariant subset, α is not in θ_2. According to Lemma 4.6, the σ-algebra generated by $\mathfrak{M}(G/P_{\theta_1})$ and $\mathfrak{M}(G/P_{\theta_2})$ is $\mathfrak{M}(G/P_{\theta_3})$ where $\theta_3 = \theta_1 \cap \theta_2$. This contradicts the minimality of θ_1. □

4.6. The Lebesgue density theorem. —

The proof of Proposition 4.7 relies on a generalization of Lebesgue's density points theorem. Instead of working with small Euclidean balls associated to distances, one works with small balls associated to b-distances.

Definition 4.11. — *A b-distance on a space X is a map $d : X \times X \to [0, \infty[$ such that $\forall\, x, y, z \in X$, $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq b\,(d(x, y) + d(y, z))$.*

- In this case, one says that X is a b-metric space. Then, there exists a topology on X for which the balls $B(x, \varepsilon) := \{y \in X \ / \ d(x, y) \leq \varepsilon\}$, where $\varepsilon > 0$, form a basis of neighborhood of the points x.
- One has the inclusion $\overline{B(x, \varepsilon)} \subset B(x, b\,\varepsilon)$.
- For each subset Y of X, let $B(Y, \varepsilon) := \cup_{y \in Y} B(y, \varepsilon)$ be the ε-neighborhood of Y, and $\delta(Y) = \sup_{x,y \in Y} d(x, y)$ be the diameter of Y.

Example Let φ be a contracting automorphism of a locally compact group H. It is easy to construct a compact neighborhood C of e such that $C = C^{-1}$ and $\varphi(C) \subset C$. Choose $N \geq 1$ such that $\varphi^N(C^2) \subset C$ and define, for $x, y \in H$,

$$d_C(x, y) = 2^{-n_C(x,y)} \ , \text{ where } n_C(x, y) = \sup\{n \in \mathbb{Z} \ / \ x^{-1}y \in \varphi^n(C)\} \ .$$

This d_C is a left H-invariant 2^N-distance on H.

Let (X, d) be a locally compact b-metric space and μ be a Radon measure on X, i.e. a Borel measure which is finite on compact sets. One says that X is of *finite μ-dimension* if for all $x \in X$ and $\varepsilon > 0$, one has $\mu(B(x, \varepsilon)) > 0$ and, for all $c > 1$,

$$\sup_{x \in X} \overline{\lim_{\varepsilon \to 0}}\, \mu\left(\overline{B(x, c\varepsilon)}\right) / \mu\left(\overline{B(x, \varepsilon)}\right) < \infty \ .$$

Note that if one checks this property for some $c > 1$, it is true for all $c > 1$.

Let $E \subset X$ be a measurable subset. A point $x \in E$ is called a *density point* if

$$\lim_{\varepsilon \to 0} \mu\left(\overline{B(x,\varepsilon)} \cap E\right) / \mu\left(\overline{B(x,\varepsilon)}\right) = 1 \ .$$

Theorem 4.12. — **(Lebesgue)** *Let (X,d) be a locally compact b-metric space which is of finite μ-dimension for a Radon measure μ on X, and let E be a measurable subset of X.*

Then μ-almost every point of E is a density point.

Proof that Theorem 4.12 implies Proposition 4.7. — Let μ be a left Haar measure on H. The b-metric space (H, d_C) of the above example is of finite μ-dimension, because for all $h \in H$, one has $B(h, 2^{-n}) = h\varphi^n(C)$. Theorem 4.12 with $\varepsilon = 2^{-n}$ implies that, for almost every h in E^{-1},

$$\lim_{n \to \infty} \mu(h^{-1}\varphi^n(C) \cap E)/\mu(h^{-1}\varphi^n(C)) = 1 \ .$$

But the automorphism φ sends the Haar measure on one of its multiples. Hence

$$\lim_{n \to \infty} \mu(C \cap \varphi^{-n}(hE)))/\mu(C) = 1 \ .$$

This is true for an exhausting family of compact sets C. In other words, $\varphi^{-n}(hE)$ converges in measure to H.

The same discussion with E^c shows that for almost all h in $(E^c)^{-1}$, $\varphi^{-n}(hE)$ converges in measure to $\emptyset$. □

To prove Theorem 4.12, we will need the following two lemmas.

Lemma 4.13. — *Let (X,d) be a compact b-metric space, $Y \subset X$, and $\mathcal{F}$ be a family of closed subsets of X such that, for every $x \in X$, there exists a closed set $F \in \mathcal{F}$ containing x whose diameter is nonzero but arbitrarily small.*

Then, either Y is included in a finite disjoint union of elements of $\mathcal{F}$, or there exists a sequence $(F_n)_{n>0}$ of disjoint elements of $\mathcal{F}$ such that for all $n \geq 1$,

$$Y \ \subset \ F_1 \cup \cdots \cup F_n \cup (\cup_{k>n} B(F_k, 3b\,\delta(F_k))) \ .$$

Proof. — By induction, if $F_1, \ldots, F_k$ have been chosen and do not cover Y, the set

$$\mathcal{F}_k := \{F \in \mathcal{F} \ / \ \forall\, i \leq k \ , \ F_i \cap B(F, \delta(F)) = \emptyset\}$$

is nonempty. Let $\varepsilon_k := \sup_{F \in \mathcal{F}_k} \delta(F)$, and choose $F_{k+1} \in \mathcal{F}_k$ such that $\delta(F_{k+1}) \geq 2\varepsilon_k/3$.

It is clear that one has $\lim_k \varepsilon_k = 0$. Indeed, if this was not the case, there would be a sequence of points $p_k \in F_k$ such that $d(p_i, p_j) \geq 2\varepsilon_j/3 \ \ \forall\, i < j$ and this would contradict the compacity of X.

Let us show, by contradiction, that this sequence satisfies the required properties. Let y be a point of Y which is not in $F_1 \cup \cdots \cup F_n \cup (\cup_{k>n} B(F_k, 3b\,\delta(F_k)))$. There exists

$F \in \mathcal{F}_n$ with nonzero diameter that contains y. Let us show by induction on $k \geq n$ that F belongs to $\mathcal{F}_k$. In fact, one has

$$B(F, \delta(F) \subset B(y, 2b\,\delta(F)) \subset B(y, 2b\,\varepsilon_k) \subset B(y, 3b\,\delta(F_{k+1}))\ .$$

We chose y in such a way that this last ball does not meet F_{k+1}. Hence $B(F, \delta(F)) \cap F_{k+1} = \emptyset$ and $F \in \mathcal{F}_{k+1}$. Therefore one finds $\varepsilon_k \geq \delta(F) > 0\ \forall\, k \geq n$. Contradiction. □

Let us now suppose again that (X, d) is a locally compact b-metric space and that μ is a Radon measure on X.

A *Vitali covering* $\mathcal{F}$ of a subset Y of X is a covering of Y by closed subsets of X of non-zero measure such that $\exists\, \lambda > 1$, $\forall\, y \in Y$, $\exists F \in \mathcal{F}$ such that

$$y \in F \ \ ,\ \ \delta(F) \text{ is arbitrarily small and } \mu(B(F, 3b\,\delta(F)))/\mu(F) \leq \lambda\ .$$

Lemma 4.14. — **(Vitali)** *With these notations, for any Vitali covering $\mathcal{F}$ of a subset Y of X, there exists a sequence $(F_n)_{n>0}$ of disjoint elements of $\mathcal{F}$ such that*

$$\mu(Y - \bigcup_{n>0} F_n) = 0\ .$$

Proof. — First suppose that X is compact. Let $(F_n)_{n>0}$ be the sequence of elements of $\mathcal{F}$ given by Lemma 4.13. One has,

$$\mu(Y - \bigcup_{k \leq n} F_k) \leq \mu(\bigcup_{k>n} B(F_k, 3b\,\delta(F_k))\,) \leq \lambda \sum_{k>n} \mu(F_k) \to 0\ ,$$

for $n \to \infty$, because $\sum_k \mu(F_k) \leq \mu(X) < \infty$. Therefore, one has $\mu(Y - \bigcup_{n>0} F_n) = 0$.

When X is only locally compact, one constructs, using a proper continuous function $f : X \to [0, \infty)$, a sequence of disjoint open relatively compact sets X_i such that $\mu(X - \bigcup_{i>0} X_i) = 0$. One then applies the previous argument to each subset $Y \cap X_i$ of the compact $\overline{X}_i$, and to the covering $\mathcal{F}_i := \{F \in \mathcal{F}\ /\ F \subset X_i\}$. □

Proof of Theorem 4.12. — Let

$$B_i := \{x \in E / \underline{\lim}\ \mu(\,\overline{B(x,\varepsilon)} \cap E\,)/\mu(\,\overline{B(x,\varepsilon)}\,) < \tfrac{i-1}{i}\}.$$

It is enough to prove that $\forall\, i$, $\mu(B_i) = 0$. To that purpose, let us choose a sequence $(U_j)_{j>0}$ of open subsets of X containing E, such that $\lim\limits_{j\to\infty} \mu(U_j - E) = 0$, and let

$$\mathcal{F}^{ij} := \{\overline{B(x,\varepsilon)}\ /\ x \in E,\ \varepsilon > 0,\ \overline{B(x,\varepsilon)} \subset U_j \text{ and } \mu(\,\overline{B(x,\varepsilon)} \cap E\,)/\mu(\,\overline{B(x,\varepsilon)}\,) < \tfrac{i-1}{i}\}.$$

Since X is of finite μ-dimension, the family $\mathcal{F}^{ij}$ is a Vitali covering of B_i. Hence, there exists a sequence $(F_n^{ij})_{n>0}$ of disjoint elements of $\mathcal{F}^{ij}$ such that $\mu(B_i - \bigcup_{n>0} F_n^{ij}) = 0$. But then

$$\mu(B_i)\ \leq\ \sum_{n>0} \mu(F_n^{ij})\ \leq\ i \sum_{n>0} \mu(F_n^{ij} - E)\ \leq\ i\, \mu(U_j - E)$$

for all j. Hence $\mu(B_i) = 0$. □

5. Lecture on Local Fields

Local fields are an important tool for discrete groups. For instance, they are a decisive ingredient in the proof of the Tits alternative or of the Margulis arithmeticity theorem. We will not discuss these points here. Instead, we will show how local fields allow us to understand a larger class of groups than arithmetic groups, the so-called S-arithmetic groups.

These groups happen to be lattices in locally compact groups G which are products of real and p-adic Lie groups. Moreover, many theorems for lattices in real Lie groups can be extended to lattices in such groups G with a very similar proof. In fact, the main property of $\mathbb{R}$ used in these proofs was "locally compact field" and not "archimedean field".

Hence, this lecture will be a rereading of the proofs of the previous chapters.

As a by-product of this point of view, we will construct cocompact lattices in $\mathrm{SL}(d, L)$, where L is a p-adic field, and we will see that when $d \geq 3$, such lattices have property T and are quasisimple.

5.1. Examples. —

Here, as in Section 2.1, we give a few explicit examples of lattices.

Let p, p_1, p_2 be prime numbers, and $d \geq 2$, $m \geq 1$ be integers such that m is prime to p and $-m$ is a square in $\mathbb{Q}_p$ and set σ the involution of $\mathbb{Q}[\sqrt{-m}]$. Let I_d be the $d \times d$ identity matrix.

In the following examples, the embedding of Γ is the diagonal embedding.

Example 1 *The group* $\Gamma := \mathrm{SL}(d, \mathbb{Z}[\frac{1}{p}])$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{R}) \times \mathrm{SL}(d, \mathbb{Q}_p)$.

Example 2 *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\frac{1}{p_1 p_2}]) \ / \ g\,{}^t g = I_d\}$ *is a cocompact lattice in* $\mathrm{SO}(d, \mathbb{Q}_{p_1}) \times \mathrm{SO}(d, \mathbb{Q}_{p_2})$, *when* $d \geq 3$.

Example 3 *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\frac{\sqrt{-m}}{p}]) \ / \ g\,{}^t g^\sigma = I_d\}$ *is a cocompact lattice in* $\mathrm{SL}(d, \mathbb{Q}_p)$.

Example 4 Let L be a finite extension of $\mathbb{Q}_p$. One can choose a totally real algebraic integer α over $\mathbb{Z}$ of degree $[L : \mathbb{Q}_p]$ such that $L = \mathbb{Q}_p[\alpha]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\alpha, \frac{\sqrt{-m}}{p}]) \ / \ g\,{}^t g^\sigma = I_d\}$ *is a cocompact lattice in* $\mathrm{SL}(d, L)$.

Example 5 Using the two square roots of $-m$ in $\mathbb{Q}_p$, *the group* $\Gamma := \mathrm{SL}(d, \mathbb{Z}[\frac{\sqrt{-m}}{p}])$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{C}) \times \mathrm{SL}(d, \mathbb{Q}_p) \times \mathrm{SL}(d, \mathbb{Q}_p)$.

Example 6 Let $F_p((t))$ be the field of Laurent series over $\mathbb{F}_p$. *The group* $\Gamma := \mathrm{SL}(d, \mathbb{F}_p[t^{-1}])$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{F}_p((t)))$.

We will give a proof for Examples 1 to 4. The proof for the last ones is similar.

5.2. S-completions. —

Let us recall a few definitions related to the completions of $\mathbb{Q}$.

S-completions For p prime, let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$ for the absolute value $|.|_p$ such that $|p|_p = p^{-1}$. Let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \;/\; |x|_p \leq 1\}$ and let μ_p be the Haar measure on $\mathbb{Q}_p$ such that $\mu_p(\mathbb{Z}_p) = 1$.

For $p = \infty$, let $\mathbb{Q}_\infty := \mathbb{R}$ be the completion of $\mathbb{Q}$ for the usual absolute value $|.|_\infty$ such that $|x|_\infty = x$ when $x > 0$, and μ_∞ be the Haar measure on $\mathbb{Q}_\infty$ such that $\mu_\infty([0,1]) = 1$.

The $\mathbb{Q}_p$, for $p \in \mathcal{V} := \{p \in \mathbb{N}, \text{ prime}\} \cup \{\infty\}$, exhaust all the completions of $\mathbb{Q}$.

Moreover, for x in $\mathbb{Q}^\times$, one has the product formula

$$\textstyle\prod_p |x|_p = 1.$$

For $S \subset \mathcal{V}$, let

$$\mathbb{Z}_S \;:=\; \mathbb{Z}[(\tfrac{1}{p})_{p\in S-\infty}], \quad \mathbb{Q}_S := \prod_{p\in S} \mathbb{Q}_p \;\text{ and }\; \mathbb{Q}_S^\circ := \{x \in \mathbb{Q}_S^\times \;/\; \prod_{p\in S} |x_p|_p = 1\}.$$

The ring $\mathbb{Z}_S$ is a subring of the field $\mathbb{Q}$. The field $\mathbb{Q}$ will be seen as a subring of the ring $\mathbb{Q}_S$ via the diagonal embedding. When S is finite, the ring $\mathbb{Q}_S$ is locally compact.

Lemma 5.1. — *Let S be a finite subset of $\mathcal{V}$ containing ∞. Then*
a) $\mathbb{Z}_S$ is a discrete, cocompact subgroup of $\mathbb{Q}_S$;
b) $\mathbb{Z}_S^\times$ is a cocompact lattice in $\mathbb{Q}_S^\circ$;
c) for any $S' \subset S$ with $S' \neq S$, $\mathbb{Z}_S$ is dense in $\mathbb{Q}_{S'}$.

Example For p prime, the group $\mathbb{Z}[\frac{1}{p}]$ is discrete, cocompact in $\mathbb{R} \times \mathbb{Q}_p$, and dense in both $\mathbb{R}$ and $\mathbb{Q}_p$.

Proof. — a) Let $O_S := \mathbb{R} \times \prod_{p\in S-\infty} \mathbb{Z}_p$. One has $O_S + \mathbb{Z}_S = \mathbb{Q}_S$ and $O_S \cap \mathbb{Z}_S = \mathbb{Z}$.

b) Let $U_S := \mathbb{R}^\times \times \prod_{p\in S-\infty} \mathbb{Z}_p^\times$. One has $U_S\mathbb{Z}_S^\times = \mathbb{Q}_S^\circ$ and $U_S \cap \mathbb{Z}_S^\times = \{\pm 1\}$.

c) Exercise. □

Adèles and idèles The language of adèles is a way to deal with all completions of a number field which is more concise and efficient than the S-arithmetic one. For instance, it will allow us to say in a simple way that the cocompactness and density in Lemma 5.1 are uniform in S. For our purposes, the concept of adèles can be avoided and the reader may forget the following paragraph.

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the ring of adèles of $\mathbb{Q}$. It is the *restricted product* of all $\mathbb{Q}_p$. More precisely, for a finite subset $S \subset \mathcal{V}$,

$$\mathbb{A}(S) := \{x \in \prod_{p \in \mathcal{V}} \mathbb{Q}_p \;/\; \forall\, p \notin S\;,\; |x_p|_p \leq 1\} \;\text{ and }\; \mathbb{A} := \bigcup_{S\, finite} \mathbb{A}(S)\;.$$

The rings $\mathbb{A}(S)$ are endowed with the product topology and $\mathbb{A}$ is endowed with the inductive limit topology. The ring $\mathbb{A}$ of adèles is locally compact. We denote by μ its Haar measure whose restriction to the $\mathbb{A}(S)$ is the product measure of μ_p.

The subring $\mathbb{A}(\infty)$ is the ring of *integral adèles.* The field $\mathbb{Q}$ embedded diagonally in $\mathbb{A}$ is the subring of *principal adèles.*

The group of idèles is the multiplicative group $\mathbb{I} = \mathbb{A}^{\times}$ endowed with the topology induced by the embedding $\mathbb{I} \to \mathbb{A} \times \mathbb{A}\;;\; x \mapsto (x, x^{-1})$. The group $\mathbb{I}$ is locally compact. It contains the subgroup of *integral idèles* $I(\infty) := \mathbb{A}(\infty)^{\times}$, the subgroup of *principal idèles* $\mathbb{Q}^{\times}$, and

$$\mathbb{I}^{\circ} := \{x \in \mathbb{I} \;/\; |x| = 1\}, \quad \text{where} \quad |x| = \prod_{p \in \mathcal{V}} |x_p|_p\;.$$

The following lemma is a stronger version of Lemma 5.1.

Lemma 5.2. — *a)* $\mathbb{Q}$ *is a discrete cocompact subgroup of* $\mathbb{A}$.
b) $\mathbb{Q}^{\times}$ *is a cocompact lattice in* $\mathbb{I}^{\circ}$.
c) For any $p \in \mathcal{V}$, $\mathbb{Q}$ *is dense in* $\mathbb{A}_p := \mathbb{A}/\mathbb{Q}_p$.

Proof. — a) One has $\mathbb{A}(\infty) + \mathbb{Q} = \mathbb{A}$ and $\mathbb{A}(\infty) \cap \mathbb{Q} = \mathbb{Z}$.
b) One has $\mathbb{I}(\infty)\,\mathbb{Q}^{\times} = \mathbb{I}$ and $\mathbb{I}(\infty) \cap \mathbb{Q}^{\times} = \{\pm 1\}$.
c) For $p = \infty$, $\mathbb{Z}$ is dense in $\prod_{\ell \neq \infty} \mathbb{Z}_\ell$. For $p < \infty$, $\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{R} \times \prod_{\ell \neq p, \infty} \mathbb{Z}_\ell$. □

Remark There is a similar construction for any number field k using all the absolute values v of k: $\mathbb{A}_k$ is the restricted product of all the completions k_v. We will not develop this important point of view here, since thanks to Weil's restriction of scalars we will mostly work with the field $k = \mathbb{Q}$.

5.3. The space of lattices of $\mathbb{Q}_S^d$. —

In this section and the next one, we extend the results of lecture 2 to the S-arithmetic setting. The proofs are almost the same and will be only sketched.

Here is the extension of Proposition 2.1

Proposition 5.3. — *a) The group* $\mathrm{SL}(d, \mathbb{Z}_S)$ *is a lattice in* $\mathrm{SL}(d, \mathbb{Q}_S)$.
b) The group $\mathrm{SL}(d, \mathbb{Q})$ *is a lattice in* $\mathrm{SL}(d, \mathbb{A})$.

For every prime number p, the group $G_p := \mathrm{SL}(d,\mathbb{Q}_p)$ admits an Iwasawa decomposition $G_p = K_pA_pN_p$, where $K_p := \mathrm{SL}(d,\mathbb{Z}_p)$, $A_p := \{g = \mathrm{diag}(p^{n_1},\dots,p^{n_d}) \in G_p\}$, and $N_p := \{g \in G_p \,/\, g-1$ is strictly upper triangular$\}$, recalling the Iwasawa decomposition of $\mathrm{SL}(d,\mathbb{R})$ seen in Section 2.2. One also gets, for the group $G_{\mathbb{Q}_S} := \mathrm{GL}(d,\mathbb{Q}_S)$, an Iwasawa decomposition $G_{\mathbb{Q}_S} = K^{\mathbb{Q}_S}A^{\mathbb{Q}_S}N^{\mathbb{Q}_S}$. We introduce again
$A^{\mathbb{Q}_S}_s := \{a \in A^{\mathbb{Q}_S} \,/\, |a_{i,i}| \le s\,|a_{i+1,i+1}|$, for $i = 1,\dots,d-1$ $\}$, for $s \ge 1$,
$N^{\mathbb{Q}_S}_t := \{n \in N^{\mathbb{Q}_S} \,/\, |n_{i,j}| \le t$, for $1 \le i < j \le d$ $\}$, for $t \ge 0$. We define the *Siegel domain* $S^{\mathbb{Q}_S}_{s,t} := K^{\mathbb{Q}_S}A^{\mathbb{Q}_S}_sN^{\mathbb{Q}_S}_t$, and set $G_{\mathbb{Z}_S} := \mathrm{GL}(d,\mathbb{Z}_S)$.

Similarly, the group $G_{\mathbb{A}} = \mathrm{GL}(d,\mathbb{A})$ admits an Iwasawa decomposition $G_{\mathbb{A}} = K^{\mathbb{A}}A^{\mathbb{A}}N^{\mathbb{A}}$. We define in the same way the Siegel domain $S^{\mathbb{A}}_{s,t} := K^{\mathbb{A}}A^{\mathbb{A}}_sN^{\mathbb{A}}_t$ and set $G_{\mathbb{Q}} = \mathrm{GL}(d,\mathbb{Q})$.

Lemma 5.4. — *For $s \ge \frac{2}{\sqrt{3}}$, $t \ge \frac{1}{2}$, one has $G_{\mathbb{Q}_S} = S^{\mathbb{Q}_S}_{s,t}G_{\mathbb{Z}_S}$ and $G_{\mathbb{A}} = S^{\mathbb{A}}_{s,t}G_{\mathbb{Q}}$.*

Proof. — Same as Lemma 2.2. Just replace the norm on $\mathbb{R}^d$ by the canonical norm on $\mathbb{Q}_S^d$ or $\mathbb{A}^d$, which is the product $\|v\| = \prod_p \|v_p\|_p$ of the canonical local norms $\|w\|_p := \sup_i |w_i|_p$ for $p < \infty$ and $\|w\|_\infty := (\sum_i w_i^2)^{\frac{1}{2}}$. Note that, for all $x \in \mathbb{Q}_S$ or $\mathbb{A}$, there exists $y \in \mathbb{Q}$ such that, for all $p < \infty$, $|x_p - y|_p \le 1$ and $|x_\infty - y|_\infty \le \frac{1}{2}$. □

Proposition 5.3 is now a consequence of the following lemma, whose proof is the same as Lemma 2.3. Let $R^{\mathbb{Q}_S}_{s,t} := S^{\mathbb{Q}_S}_{s,t} \cap \mathrm{SL}(d,\mathbb{Q}_S)$ and $R^{\mathbb{A}}_{s,t} := S^{\mathbb{A}}_{s,t} \cap \mathrm{SL}(d,\mathbb{A})$.

Lemma 5.5. — *a) The volume of $R^{\mathbb{Q}_S}_{s,t}$ in $\mathrm{SL}(d,\mathbb{Q}_S)$ for the Haar measure is finite.*
a) The volume of $R^{\mathbb{A}}_{s,t}$ in $\mathrm{SL}(d,\mathbb{A})$ for the Haar measure is finite.

The set $X^{\mathbb{Q}_S}$ of lattices (i.e. discrete, cocompact subgroups) in $\mathbb{Q}_S^d$ is the quotient space $X^{\mathbb{Q}_S} := G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$. For any lattice Λ of $\mathbb{Q}_S^d$, one denotes by $d(\Lambda)$ the volume of $\mathbb{Q}_S^d/\Lambda$. It is given by the formula $d(\Lambda) = |\det(f_1,\dots,f_d)|$ for any $\mathbb{Z}_S$-basis $(f_1,\dots,f_d)$ of Λ.

Similarly $X^{\mathbb{A}} := G_{\mathbb{A}}/G_{\mathbb{Q}}$ is the set of lattices Λ in $\mathbb{A}^d$ and the covolume $d(\Lambda)$ is given by the same formula.

We still have Hermite's bound and Mahler's criterion with the same proofs.

Lemma 5.6. — *Any lattice Λ in $\mathbb{Q}_S^d$ or $\mathbb{A}^d$ contains a vector v with $0 < \|v\| \le (\frac{4}{3})^{\frac{d-1}{4}} d(\Lambda)^{\frac{1}{d}}$.*

Proposition 5.7. — *A subset $Y \subset X^{\mathbb{Q}_S}$ or $X^{\mathbb{A}}$ is relatively compact if and only if there exist constants $\alpha, \beta > 0$ such that for all $\Lambda \in Y$, one has $d(\Lambda) \le \beta$ and $\inf_{v\in\Lambda-0} \|v\| \ge \alpha$.*

5.4. Cocompact lattices. —

Let $G \subset \mathrm{GL}(d,\mathbb{C})$ be a $\mathbb{Q}$-group. Note that $G_{\mathbb{Q}_S}$ and $G_{\mathbb{A}}$ are well-defined and locally compact groups in which the respective subgroups $G_{\mathbb{Z}_S}$ and $G_{\mathbb{Q}}$, are discrete. We want to know when these subgroups are cocompact.

Godement's criterion remains the same.

Theorem 5.8. — **(Borel, Harish-Chandra)** *Let G be a $\mathbb{Q}$-group. Then one has the equivalence:*
$G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$ *is compact* $\Leftrightarrow$ $G_{\mathbb{A}}/G_{\mathbb{Q}}$ *is compact* $\Leftrightarrow$ $G_{\mathbb{R}}/G_{\mathbb{Z}}$ *is compact* $\Leftrightarrow$ G *is $\mathbb{Q}$-anisotropic.*

When G is semisimple, the proof is the same as in lecture 2, we just have to replace the intermediate lemmas and propositions by the following ones.

Lemma 5.9. — *Let $G \subset H$ be an injective morphism of $\mathbb{Q}$-groups. Suppose that G has no nontrivial $\mathbb{Q}$-character. Then the injections $G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S} \hookrightarrow H_{\mathbb{Q}_S}/H_{\mathbb{Z}_S}$ and $G_{\mathbb{A}}/G_{\mathbb{Q}} \hookrightarrow H_{\mathbb{A}}/H_{\mathbb{Q}}$ are homeomorphisms onto closed subsets.*

Lemma 5.10. — *Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space, $V = V_{\mathbb{Q}} \otimes \mathbb{C}$, $G \subset \mathrm{GL}(V)$ a $\mathbb{Q}$-subgroup without nontrivial $\mathbb{Q}$-characters. Suppose there exists a G-invariant polynomial $P \in \mathbb{Q}[V]$ such that*

$$\forall v \in V_{\mathbb{Q}}\ ,\ P(v) = 0 \Longleftrightarrow v = 0\ .$$

Then the quotients $G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$ and $G_{\mathbb{A}}/G_{\mathbb{Q}}$ are compact.

Lemma 5.11. — *Let $\varphi : G \to H$ be a $\mathbb{Q}$-isogeny between two semisimple $\mathbb{Q}$-groups. Then*
a) the groups $\varphi(G_{\mathbb{Z}_S})$ and $H_{\mathbb{Z}_S}$ are commensurable;
b) the induced map $G_{\mathbb{A}}/G_{\mathbb{Q}} \to H_{\mathbb{A}}/H_{\mathbb{Q}}$ is proper.

Corollary 5.12. — *a) In Example* 5.1.2, *Γ is cocompact in* $\mathrm{SO}(d,\mathbb{Q}_{p_1}) \times \mathrm{SO}(d,\mathbb{Q}_{p_2})$.
b) In Example 5.1.3, *Γ is cocompact in* $\mathrm{SL}(d,\mathbb{Q}_p)$.
c) In Example 5.1.4, *Γ is cocompact in* $\mathrm{SL}(d,\mathbb{L})$.

Proof of Corollary 5.12. — a) Apply Lemma 5.10 to the orthogonal $\mathbb{Q}$-group $G := SO(d,\mathbb{C}) = \{g \in \mathrm{SL}(d,\mathbb{C}) \ /\ g\,{}^tg = I_d\}$, the set $S := \{p_1, p_2, \infty\}$, the natural representation in $V_{\mathbb{Q}} = \mathbb{Q}^d$, and the polynomial $P(v) = \sum_i v_i^2$. Notice that $G_{\mathbb{R}}$ is compact.

b) Apply Lemma 5.10 to the unitary $\mathbb{Q}$-group

$$G = \left\{ \begin{pmatrix} a & -m\,b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d,\mathbb{C}) \ /\ (a+\mu\,b)\,({}^ta - \mu\,{}^tb) = I_d\ ,\ \det(a+\mu\,b) = 1 \right\},$$

where $\mu = \sqrt{-m}$, with the set $S := \{p, \infty\}$, the natural representation in $V_Q = \mathbb{Q}^d \times \mathbb{Q}^d$, and the polynomial $P(v,w) = \sum_i (v_i^2 + m\,w_i^2)$. Notice that $G_{\mathbb{R}} \simeq \mathrm{SU}(d,\mathbb{R})$ is compact.

The map $(a,b) \to a + \sqrt{-m}\,b$ gives isomorphisms $G_{\mathbb{Z}_S} \simeq \Gamma$ and $G_{\mathbb{Q}_p} \simeq \mathrm{SL}(d,\mathbb{Q}_p)$.

c) Let $S = \{p, \infty\}$, and G be the group defined in b), but considered it as a k_0-group with $k_0 = \mathbb{Q}[\alpha]$. Restrict it as a $\mathbb{Q}$-group H so that $H_{\mathbb{Z}_S} \simeq G_{\mathbb{Z}[\frac{\alpha}{p}]}$. The d real completions of k_0 give compact real unitary groups, hence $H_{\mathbb{R}}$ is compact and H is $\mathbb{Q}$-anisotropic. The field L is the only p-adic completion of k_0, hence $H_{\mathbb{Q}_p} = G_L \simeq \mathrm{SL}(d,L)$. One has just to apply Theorem 5.8. □

Remark To convince the reader that Examples c) do exist for any finite extension $L/\mathbb{Q}_p$ of degree n, we will give a construction of *a totally real algebraic integer α of degree n such that $L = \mathbb{Q}_p[\alpha]$.*

Proof. — Consider $\beta \in L$ such that $L = \mathbb{Q}_p[\beta]$. Let $Q \in \mathbb{Q}_p[X]$ be the minimal polynomial of β over $\mathbb{Q}_p$, and $R \in \mathbb{R}[X]$ a polynomial of degree n whose roots are real and distinct. Because $\mathbb{Q}$ is dense in $\mathbb{R} \times \mathbb{Q}_p$, one can find $P \in \mathbb{Q}[X]$ sufficiently near both Q and R. Take for α a suitable multiple $\alpha = N\alpha_0$ of a root α_0 of P, and note that $L = \mathbb{Q}_p[\alpha]$ by Hensel lemma. □

A general overview Let H be a $\mathbb{Q}$-group. Suppose that H is semisimple or, more generally, that H does not have any non trivial $\mathbb{Q}$-character $\chi : H \to GL(1, \mathbb{C})$. By a theorem of Borel and Harish-Chandra,

$$H_{\mathbb{Z}_S} \textit{ is a lattice in } H_{\mathbb{Q}_S} \textit{ and } H_{\mathbb{Q}} \textit{ is a lattice in } H_{\mathbb{A}}.$$

Note that, according to the weak and strong approximation theorems of Kneser and Platonov, for any semisimple simply connected $\mathbb{Q}$-group H, $p \in \mathcal{V}$ with $H_{\mathbb{Q}_p}$ noncompact, and $S \ni p$,

$$H_{\mathbb{Z}_S} \textit{ is dense in } H_{\mathbb{Q}_{S-p}} \textit{ and } H_{\mathbb{Q}} \textit{ is dense in } H_{\mathbb{A}}/H_{\mathbb{Q}_p}.$$

The examples above are the main motivation for the following definition.

Let $G = \prod_i G_i$ be a finite product of non-compact groups G_i which are the $\mathbb{Q}_{p_i}$ points of some quasisimple $\mathbb{Q}_{p_i}$-groups for some $p_i \in \mathcal{V}$.

An irreducible subgroup Γ of G is said to be arithmetic if there exists an algebraic group H defined over $\mathbb{Q}$, a finite set $S \subset \mathcal{V}$, and a group morphism $\pi : H_{\mathbb{Q}_S} \to G$ with compact kernel and cocompact image such that the groups Γ and $\pi(H_{\mathbb{Z}_S})$ are commensurable

Note that, in this case, π is automatically "algebraic".

The classification of all arithmetic groups Γ, up to commensurability, relies again on the classification of all algebraic absolutely simple groups defined over a number field k. According to a theorem of Borel and Harder ([**7**]),

for any semisimple $\mathbb{Q}_p$-group H, the group $H_{\mathbb{Q}_p}$ contains at least one lattice.

Note that, since $G = H_{\mathbb{Q}_p}$ contains a compact open subgroup U without torsion,

any lattice Γ in $H_{\mathbb{Q}_p}$ is cocompact

(to prove this basic fact, just check that U acts freely on G/Γ and hence that all U-orbits in G/Γ have same volume).

Margulis arithmeticity theorem also applies to this case:

if G has no compact factors and the total rank of G is at least 2, then all irreducible lattices Γ of G are arithmetic groups

(see [**15**]). Here the total rank of G is the sum of all the $\mathbb{Q}_{p_i}$-ranks of the G_i.

5.5. Decay of coefficients. —

The decay of coefficients and the uniform decay of coefficients are still true for p-adic semisimple Lie groups.

Theorem 5.13. — *Let $G = \prod_i G_i$ be a product of groups G_i which are the $\mathbb{Q}_{p_i}$-points of quasisimple $\mathbb{Q}_{p_i}$-groups for some $p_i \in \mathcal{V}$. Let π be a unitary representation of G such that $\mathcal{H}_\pi^{G'} = 0$ for all non-compact normal subgroup G' of G. Then, for all $v, w \in \mathcal{H}_\pi$, one has* $\lim\limits_{g\to\infty} <\pi(g)v, w> = 0$.

Corollary 5.14. — *Let G be quasisimple simply connected $\mathbb{Q}_p$-group, π a unitary representation of $G_{\mathbb{Q}_p}$ without nonzero $G_{\mathbb{Q}_p}$-invariant vectors, and H a non-relatively compact subgroup of $G_{\mathbb{Q}_p}$. Then $\mathcal{H}_\pi^H = 0$.*

A group G as in Theorem 5.13 still has maximal compact subgroups K, Cartan subspaces A, restricted roots Δ, Weyl chambers A^+, and parabolic subgroups P_θ as in Section 3.3. And those share almost all the same properties as in the real case... Well... the maximal compact subgoups are not all conjugate, the Cartan subspaces have to be replaced by the product of $\mathbb{Q}_{p_i}$-points of maximally $\mathbb{Q}_{p_i}$-split tori, A^+ is not always a subsemigroup of A... These are technical details I do not want to enter into. The recipe is: for what we want, it works the same. Let us just give an example.

For $G = \mathrm{SL}(d, \mathbb{Q}_p)$, one can take $K = \mathrm{SL}(d, \mathbb{Z}_p)$, $A = \{a = \mathrm{diag}(p^{-n_1}, \ldots, p^{-n_d}) \in G\}$, $A^+ = \{a \in A \,/\, n_1 \geq \cdots \geq n_d\}$. Then Δ, Δ^+, Π, $\mathfrak{u}^+$, $\mathfrak{p}$, $\mathfrak{u}^-$, $\mathfrak{u}_\theta^+$, $\mathfrak{p}_\theta$, $\mathfrak{u}_\theta^-$, $\mathfrak{l}_\theta$, A_θ and A_θ^+ are given by the same formulas as in Section 3.3, and one has $G = KA^+K$.

For every vector v of a unitary representation $\mathcal{H}_\pi$ of G, one sets $\delta(v) = \delta_K(v) := (\dim <Kv>)^{1/2}$.

Theorem 5.15. — *Let $G = \prod_i G_i$ be as in Theorem 5.13. Suppose that* $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2$ *$\forall\, i$. Then there exists a K-biinvariant function $\eta_G \in \mathrm{C}(G)$ satisfying $\lim\limits_{g\to\infty} \eta_G(g) = 0$, and such that for every unitary representations π of G with $\mathcal{H}_\pi^{G_i} = 0\ \forall\, i$, and for every $v, w \in \mathcal{H}_\pi$ with $\|v\| = \|w\| = 1$, one has $| <\pi(g)v, w> | \leq \eta_G(g)\delta(v)\delta(w)$. for every $g \in G$.*

Remark The function η_G has also been computed by H.Oh in the p-adic case (see [**18**]), using Harish-Chandra's function

$$\xi(|p^{-n}|) \;=\; \xi(p^n) \;=\; \frac{(p-1)n + (p+1)}{(p+1)p^{n/2}}$$

For instance, for $G = \mathrm{SL}(d, \mathbb{Q}_p)$, $d \geq 3$, and $a = \mathrm{diag}(t_1, \ldots, t_d)$ with $|t_1| \geq \cdots \geq |t_d|$,

$$\eta_G(a) = \prod_{1 \leq i \leq [n/2]} \xi(|\tfrac{t_i}{t_{n+1-i}}|) .$$

The proofs of these theorems are the same as in lecture 3, we just have to replace the intermediate lemmas and propositions by the following lemmas with the same proofs.

Lemma 5.16. — *Let π be a unitary representation of $G = \mathrm{SL}(2, \mathbb{Q}_p)$, and $v \in \mathcal{H}_\pi$. If v is either A-invariant, U^+-invariant or U^--invariant then it is G-invariant.*

For $g = kan \in G$, let us set $H(g) = a$, and reintroduce the *Harish-Chandra spherical function* ξ_G given by $\xi_G(g) = \int_K \rho(H(gk))^{-1/2} dk$ where $\rho(a) = \det_{\mathfrak{n}}(\mathrm{Ad} a)$.

Lemma 5.17. — *Let $G = \prod_i G_i$ be as in Theorem 5.13 and let π be a unitary representation of G which is weakly contained in λ_G. For every $v, w \in \mathcal{H}_\pi$ with $\|v\| = \|w\| = 1$, and every $g \in G$, one has $| < \pi(g)v, w > | \leq \xi_G(g)\delta_K(v)\delta_K(w)$.*

Lemma 5.18. — *Let V be a $\mathbb{Q}$-representation of the $\mathbb{Q}$-group $G := \mathrm{SL}(2)$, without nonzero invariant vectors. Let π be an irreducible unitary representation of the semidirect product $V_{\mathbb{Q}_p} \rtimes G_{\mathbb{Q}_p}$, without $V_{\mathbb{Q}_p}$-invariant vectors. Then the restriction of π to $G_{\mathbb{Q}_p}$ is weakly contained in the regular representation of $G_{\mathbb{Q}_p}$.*

5.6. Property T and normal subgroups. —

As in Section 3.6, the previous control on the coefficients of unitary representations of G leads to algebraic properties for the lattices of G.

Proposition 5.19. — *Let $G = \prod_i G_i$ be as in Theorem 5.13, with $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2 \ \forall i$. Then*
a) G has property T;
b) any lattice Γ in G is finitely generated and has a finite abelianization $\Gamma/[\Gamma, \Gamma]$.

Remark For a nontrivial $\mathbb{Q}$-group G, the groups $G_{\mathbb{Q}}$ and $G_{\mathbb{A}}$ never have property T, because $G_{\mathbb{Q}}$ is not finitely generated and $G_{\mathbb{A}}$ is not compactly generated.

If one adapts the arguments of lecture 4 to G, one gets the following result.

Theorem 5.20. — *Let $G = \prod_i G_i$ be as in Theorem 5.13, and let Γ be a lattice in G. If $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2 \ \forall i$, then Γ is quasisimple.*

Remark One can weaken the rank assumption in Theorem 5.20 in : total $\mathrm{rank}(G) \geq 2$.

If the reader wants to know more on one of these five lectures, he should read [26] for lecture 1 or, respectively, [5], [13], [15], and [19] for lectures 2, 3, 4, and 5.

References

[1] N.A'Campo, M.Burger - Réseaux arithmétiques et commensurateur d'après G. A. Margulis, Invent. Math. 116 (1994) p.1-25.

[2] Y.Benoist - Pavages du plan, in Journées X-UPS 2001,Editions de l'Ecole Polytechnique.

[3] Y.Benoist - Convexes divisibles I, preprint (2001).

[4] A.Borel - Compact Clifford-Klein forms of symmetric spaces, Topology 2 (1963) p.111-122.

[5] A.Borel - Introduction aux groupes arithmétiques, Hermann (1969).

[6] A.Borel - Linear algebraic groups, GTM 126 Springer (1991).

[7] A.Borel, G.Harder - Existence of discrete cocompact subgroups of reductive groups over local fields, J. Reine Angw. Math. 298 (1978) p.53-64.

[8] N.Bourbaki - Groupes et algèbres de Lie, Paris (1975).

[9] F.Greenleaf - Invariant means on topological groups and their applications (1969).

[10] M.Gromov, P.Pansu - Rigidity of lattices: an introduction, Lect. Notes in Math. 1504 (1991) p.39-137.

[11] P.de la Harpe, A.Valette - La propriété T de Kazhdan, Asterisque (1989).

[12] S.Helgason - Differential geometry, Lie groups and symmetric spaces, Acad. Press (1978).

[13] R.Howe, E.Tan - Non-abelian harmonic analysis, Universitext Springer (1992).

[14] V.Kac, E.Vinberg - Quasihomogeneous cones, Math. Notes 1 (1967) p.231-235.

[15] G.Margulis - Discrete subgroups of semisimple Lie groups, Springer Ergebnisse (1991).

[16] B.Maskit- Kleinian groups, Springer GTM (1988).

[17] H.Oh - Tempered subgroups and representations with minimal decay of matrix coefficients, Bull. Soc. Math. Fr. 126 (1998) p.355-380.

[18] H.Oh - Uniform pointwise bounds for matrix coefficients, Duke Math. J. 113 (2002) p.133-192.

[19] V.Platonov, A.Rapinchuk - Algebraic groups and number theory, Ac. Press (1994).

[20] H.Poincaré - Théorie des groupes Fuchsiens, Acta Mathematica 1 (1882) p.1-62.

[21] M.Raghunathan - Discrete subgroups of Lie groups, Springer(1972).

[22] J-P. Serre - Cohomologie des groupes discrets, Ann. of Math. Studies 70,p.77-169.

[23] J.Tits - Sur la classification des groupes algébriques semi-simples, C. R. Acad. Sci. 249 (1959) p.1438-1440.

[24] J.Tits - Structure des groupes de Weyl, Séminaire Bourbaki 288 (1965).

[25] E.Vinberg - Discrete linear groups that are generated by reflections, Izv. Akad. Nauk SSSR 35 (1971) p.1072-1112.

[26] E.Vinberg - Geometry II, Encyclopedia of Math. Sc. 29 Springer (1993).

[27] E.Vinberg, V.Gorbatsevich, O.Shvartsman - Discrete subgroups of Lie groups, in Encyclopedia of Math. Sc. 21 Springer (2000).

[28] A.Weil - Algebras with involutions and the classical groups, J. Indian Math. Soc. 24 (1961) p.589-623.

[29] D.Witte-Morris - Introduction to arithmetic groups, preprint (2003).

[30] R.Zimmer - Ergodic theory and semisimple groups, Birkhauser (1984).

Yves Benoist, ENS-CNRS 45 rue d'Ulm, Paris • *E-mail :* Yves.Benoist@ens.fr
Url : www.dma.ens.fr/~benoist