CRITICAL EXPONENTS AND RIGIDITY IN NEGATIVE CURVATURE

by

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Abstract. — The goal of this lecture is to describe a theorem of M.Bonk and B.Kleiner on the rigidity of discrete groups acting on CAT(-1)-spaces whose limit set's Hausdorff and topological dimensions coincide. We will give the proof of M.Bonk and B.Kleiner and also an alternative proof in a particular case.

***Résumé* (Exposants critiques et rigidité en courbure négative)**
Dans ces notes nous présentons un théorème de M. Bonk et B. Kleiner concernant la rigidité des groupes discrets d'isométries sur des espaces CAT(-1) dont les dimensions de Hausdorff et topologiques sont égales. Nous décrivons la preuve de M. Bonk et B. Kleiner ainsi qu'une preuve différente dans un cas particulier.

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1. Introduction

A famous theorem of G.D.Mostow states that a compact hyperbolic manifold of dimension $n \geq 3$ is determined up to isometry by its fundamental group. In other words, if Γ is a cocompact lattice in $PO(n, 1)$, with $n \geq 3$, there is a unique faithfull and discrete representation $\rho : \Gamma \to PO(n, 1)$ up to conjugacy.

On the other hand, for some lattices Γ of $PO(n, 1)$ there exist many faithfull discrete nonconjugate representations $\rho : \Gamma \to PO(m, 1)$ $2 \leq n < m$ as described in the following example.

Bendings: Let us assume that a lattice Γ in $PO(n, 1)$ is a free product $A *_C B$ of its subgroups A and B over the amalgamated subgroup C such that C cocompactly preserves a totally geodesic copy of the hyperbolic space $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$. For such a group Γ the quotient manifold $M = \mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold with a totally geodesic embedded and separating hypersurface $N = \mathbb{H}^{n-1}/C$. One can consider a Fuchsian representation $\rho_0 : \Gamma \to PO(n + 1, 1)$. A representation ρ of a lattice Γ of $PO(n, 1)$ in $PO(m, 1)$ with $2 \leq n < m$ is called fuchsian if $\rho(\Gamma)$ preserves a totally geodesic copy of the hyperbolic space $\mathbb{H}^n$ in $\mathbb{H}^m$. Let Γ be a lattice of $PO(n, 1)$, a fuchsian representation ρ_0 of Γ in $PO(m, 1)$ with $m > n$ can be obtained by this way: $\rho_0 : A \in \Gamma \to \left(\begin{array}{c|c} A & 0 \\ \hline 0 & Id \end{array}\right) \in PO(m, 1)$.

For such a fuchsian representation ρ_0 of $\Gamma = A *_C B$ in $PO(n + 1, 1)$ the group $\rho_0(C)$ preserves a totally geodesic copy of the hyperbolic space $\mathbb{H}^{n-1}$ in $\mathbb{H}^{n+1}$. The group $\rho_0(C)$ is then centralized in $PO(n + 1, 1)$ by the subgroup of rotations around $\mathbb{H}^{n-1}$ in $\mathbb{H}^{n+1}$ which is isomorphic to S^1. For $r_t = e^{it} \in S^1$, let us define $\rho_t : \Gamma \to PO(n+1, 1)$ by $\rho_t(a) = a$ for all $a \in A$ and $\rho_t(b) = r_t^{-1} b r_t$ for all $b \in B$. As r_t commutes with $\rho_0(C)$ there is no ambiguity in the definition of $\rho_t(c)$ for $c \in C = A \cap B$. It can be shown that for $t \neq 0$ small enough, the group $\rho_0(\Gamma)$ does not preserve any totally geodesic copy of $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$ and thus cannot be conjugate to ρ_0, cf. [**11**].

One way of distinguishing between a Fuchsian and a non Fuchsian representation ρ of a cocompact lattice Γ of $PO(n, 1)$ into $PO(m, 1)$, $m > n$ is to compare their limit set. Basically the size of the limit set of

$G =: \rho(\Gamma)$ for a non fuchsian representation ρ is stricly larger than the size of the limit set of $G_0 =: \rho_0(\Gamma)$ for any Fuchsian representation ρ_0.

Before going further, let us turn to a more general setting and introduce some notations.

Let X be a CAT(-1)-space, cf. [**4**]. Examples of CAT(-1)-space are Cartan Hadamard manifold of negative curvature $K \leq -1$, ie. simply connected manifolds of negative sectional curvature $K \leq -1$.

For a discrete group of isometry G of a CAT(-1)-space X, we define the limit set $\Lambda(G)$ of G as the closure of the orbit of some (and hence any) point $o \in X$ in the ideal boundary ∂X of X, namely $\Lambda(G) = \overline{Go}^{X \cup \partial X} \cap \partial X$. The convex hull of $\Lambda(G)$ is the smallest G-invariant convex subset of $X \cup \partial X$ containing $\Lambda(G)$, and we denote it by $H(G)$. A discrete group of isometry G of X is convex cocompact if $H(G)/G$ is a compact subset of X/G. The convex cocompactness is equivalent to the quasi-convex cocompactness that we define now.

A subset $Y \subset X$ is said quasi-convex if there is a constant $C > 0$ such that every geodesic segment with endpoints in Y lies in the C-neighborhood of Y.

Definition 1.1. — *Let X be a CAT(-1)-space and G a discrete group of isometries of X. The group G is said quasi-convex cocompact if there exist a G-invariant quasi-convex subset $Y \subset X$ with compact quotient Y/G.*

For example if $\rho_0 : \Gamma \to PO(n+1, 1)$ is a Fuchsian representation of a cocompact lattice Γ of $PO(n, 1)$, then $G_0 = \rho_0(\Gamma)$ is a convex cocompact group of the hyperbolic space $\mathbb{H}^{n+1}$ .The limit set $\Lambda(G_0)$ of G is the boundary $\partial\mathbb{H}^n$, the convex hull $H(G_0)$ is the totally geodesic copy of $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$ preserved by G_0 and the convex cocompactness of G_0 comes from the cocompactness of Γ. If $G_t = \rho_t(\Gamma)$ are bendings then the G_t's are convex cocompact for t small enough, and the limit set $\Lambda(G_t)$ of each such G_t is then a topological $n-1$-dimensional sphere [**17**], [**8**].

For a CAT(-1) space X, let us define a distance on the ideal boundary as follows. Let o be a fixed point in X. Let ξ, ξ' be two points in ∂X and denote by $l(\xi, \xi')$ the distance between o and the geodesic joining ξ

and ξ'. The following

$$d(\xi, \xi') =: e^{-l(\xi,\xi')} \tag{1.1}$$

is a distance on ∂X. This distance depends on the choice of the base point o but two different choices of a base point give rise to equivalent distances, [**8**].

We denote by $\delta(G)$ the Hausdorff dimension with respect to the distance d of the limit set $\Lambda(G)$. Let us recall that the d-Hausdorff measure H^d on a metric space (M, d) is defined as follows. For $A \subset M$ and $\eta > 0$, we set

$$H^d_\eta(A) = inf\{\Sigma_j (diam(E_j))^d\}$$

where the infimum is taken on all sequences $\{E_j\}$ of subset of M which cover A and whose diameter satisfies $diam E_j \leq \eta$ for all j's, and $H^d(A) = \lim_{\eta\to 0} H^d_\eta(A)$. We say that M has Hausdorff dimension δ if $H^d(M) = 0$ for $d > \delta$ and $H^d(M) = \infty$ for $d < \delta$.

The following definitions will be usefull.

Definition 1.2. — *(i) A complete metric space (M, d) of Hausdorff dimension δ is said Ahlfors regular if there is a constant $C > 0$ such that*

$$C^{-1} r^\delta \leq H^\delta(B(x, r)) \leq C r^\delta$$

for every ball $B(r)$ of radius r in (M, d).

(ii)A metric space is uniformly perfect if there exist a constant $C > 0$ such that for every $x \in M$ and $0 < r < diam M$, there is a point $y \in M$ which satisfies

$$C^{-1} r \leq d(x, y) \leq r$$

An Ahlfors regular space is automatically uniformly perfect. This can be easily deduced from the fact that $B(x, r) - B(x, C^{-1}r)$ has positive measure for C large enough.

For example the limit set $(\Lambda(G), d)$ of a quasi-convex cocompact group G acting on a CAT(-1)-space is Ahlfors regular and uniformly perfect.

Whenever the group G is quasi-convex cocompact, the Hausdorff dimension $\delta(G)$ of $\Lambda(G)$ can be defined as the critical exponent of the Poincaré series $\Sigma_{g\in G} e^{-s dist(o, \Gamma o)}$, where $dist$ stands for the distance in X, [**17**]

$$\delta(G) = \inf\{s > 0 \quad | \quad \Sigma_{g \in G} e^{-sdist(o,\Gamma o)} < \infty\}.$$

Let us remark that the critical exponent of the Poincaré series $\Sigma_{g \in G} e^{-sdist(o,\Gamma o)}$ does not depend on the choice of the point o because of the triangle inequality.

For example, if $\rho_0 : \Gamma \to PO(n+1,1)$ is a Fuchsian representation of a cocompact lattice Γ of $PO(n,1)$ and $G_0 = \rho_0(\Gamma)$ then $\delta(G_0) = n-1$. For a non fuchsian faithfull discrete convex cocompact representation $\rho : \Gamma \to PO(n+1,1)$ with $G = \rho(\Gamma)$, the limit set of G is stricly "bigger" than the limit set of G_0, namely, $\delta(G) > \delta(G_0) = n-1$. In particular for the above bendings $\delta(G_t)$ is strictly increasing. This has been first observed by H.Poincaré, then proved by R.Bowen for $n = 2$ and D.Sullivan for larger n and extended by several authors in variable curvature or without special assumption on G, [**6**], [**17**], [**1**], [**14**].

For a quasi-convex cocompact representation of a cocompact lattice of $PO(n,1)$ in a CAT(-1) space M.Bourdon proved the following

Theorem 1.3. — [**5**] *Let Γ be a cocompact lattice in $PO(n,1)$ and $\rho : \Gamma \to Isom(X)$ a discrete faithfull representation of Γ in the isometry group of a CAT(-1) space X. We assume that $G =: \rho(\Gamma)$ is quasi-convex cocompact. Then, $\delta(G) \geq n-1$ and $\delta(G) = n-1$ if and only if G preserves a totally geodesic copy H of $\mathbb{H}^n$ in X with compact quotient H/G.*

In the particular case of the above bendings ρ_t of a cocompact lattice Γ of $PO(n,1)$ in $PO(m,1)$, $m > n$, the limit set $\Lambda(G_t)$ of $G_t = \rho_t(\Gamma)$ is a $(n-1)$-dimensional topological sphere of Hausdorff dimension $\delta(G_t) \geq n-1$ for t small enough and equality $\delta(G_t) = n-1$ happens if and only if $t = 0$.

Let us stress the fact that in the theorem 1.3, the quasi-convex cocompact group G is assumed to be isomorphic to a cocompact lattice of $PO(n,1)$.

It's worth mentioning that the same conclusion of the theorem 1.2 still holds for any convex cocompact group G in $PO(m,1)$ which is not assumed to be isomorphic to a cocompact lattice of $PO(n,1)$ but whose limit set is supposed to be homeomorphic to a standard n-sphere, $2 \leq n \leq m-1$. This was actually obseved earlier by Izeki, [**9**].

Theorem 1.4. — [**9**] *Let G be a discrete convex cocompact group of isometry of $PO(m,1)$. Let us assume that the limit set $\Lambda(G)$ of G is homeomorphic to a n-dimensionel sphere. Then, $\delta(G) \geq n$ and $\delta(G) = n$ if and only if G preserves a totally geodesic copy of $\mathbb{H}^{n+1}$ in $\mathbb{H}^m$.*

M.Bonk and B.Kleiner have extended this result to the case of a discrete group G of isometries of a CAT(-1)-space X.

Theorem 1.5. — [**3**] *Let G be a convex cocompact group of isometries of a CAT(-1) space X. Let $\delta(G)$ and $dim_{top}(\Lambda(G))$ be the Hausdorff and topological dimension of the limit set $\Lambda(G)$. We assume that $\delta(G) = dim_{top}(G) = n$ for some integer $n \geq 2$. Then, G preserves a totally geodesic copy H of the hyperbolic space $\mathbb{H}^{n+1}$ embedded in X, such that H/G is compact.In particular, G is a cocompact lattice in $PO(n,1)$.*

Remark: In the theorem 1.3, G is assumed to be isomorphic to a cocompact lattice in $PO(n,1)$. On the contrary in the theorem 1.5, G is not assumed to be a cocompact lattice in $PO(n,1)$, but this fact is a part of the conclusion. In fact the proof of the theorem 1.5 relies on the theorem 1.3: M.Bonk and B.Kleiner actually show that under the assumptions of the theorem 1.5 then G is isomorphic to a cocompact lattice in $PO(n,1)$.

The following rigidity theorem which was observed by G.Knieper is a particular case of the theorem 1.5.

Theorem 1.6. — [**10**] *Let $M = X/G$ be a $(n+1)$-dimensional compact riemannian manifold with sectional curvature $K \leq -1$, where X is the universal covering space of M and G its fundamental group. If $\delta(G) = n$ then M is hyperbolic, i.e. $K = -1$.*

As G is cocompact the limit set $\Lambda(G)$ coincides with the ideal boundary of X which is a n-dimensional topological sphere thus the theorem 1.6 follows from the theorem 1.5.

In the next section we will give an altenative proof of the theorem 1.6. In the section 3, we state a theorem of P. Tukia. In section 4, we recall definitions of a weak tangent, quasi-Möbius homeomorphisms between metric spaces and prove that the action of a quasi-convex cocompact group G of isometries of a $CAT(-1)$ space on its limit set $(\Lambda(G), d)$ is quasi-Möbius conjugate to the action of G on the one point compactification of any weak tangent of $(\Lambda(G), d)$. In section 5, we give the definitions of topological dimension, regular maps and give conditions under which a compact metric space of topological dimension n has a weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$. In section 6 we prove that a compact n-Ahlfors regular metric space whose topological dimension equals n has a weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$. The section 7 gives the proof of the theorem 1.5.

2. Alternative proof of the theorem 1.5 in a simpler case

We first give a proof of the theorem 1.6 distinct of the original one and which does not use the theorem 1.5.

We shall use the following criterium for a Cartan Hadamard manifold X to be isometric to the hyperbolic space $\mathbb{H}^{n+1}$. For $x \in X$ and $\theta \in \partial X$ let us denote $B(x, \theta)$ the Busemann function defined by

$$B(x, \theta) = \lim_{t \to \infty} dist(x, c(t)) - dist(o, c(t))$$

where o is a fixed base point in X and $c(t)$ a geodesic ray joining o to θ.

The following lemma characterizes the hyperbolic space $\mathbb{H}^{n+1}$ among Cartan Hadamard manifolds.

Lemma 2.1. — *Let X be a $(n+1)$-dimensional Cartan Hadamard manifold with Busemann function B. Then X is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ if and only if for each $\theta \in \partial X$ and $x \in X$, we have $HessB(x, \theta) + dB(x, \theta) \otimes dB(x, \theta) = g(x)$ where $HessB$ is the Hessian of B with respect to the variable x and g is the riemannian metric on X.*

This lemma amounts to saying that the horospheres of X (which are the level sets of the functions $B(., \theta)$) have their second fundamental form proportional to the metric if and only if X is of constant sectional curvature $K = -1$.

Proof. — The only if part is obvious. Let us prove the other way. Let (r,α) be the polar coordinates at the point $o \in X$ where r is the distance from o and $\alpha \in S^{n-1}$ the spherical coordinate. If the Buseman function of the Cartan Hadamard manifold X satisfies $HessB(x,\theta) + dB(x,\theta) \otimes dB(x,\theta) = g(x)$, it is easy to check that for any point $x \in X$ with polar coordinates (r,α) and any point $\theta \in \partial X$ we have $\exp B(x,\theta) = \cosh r - \cos\alpha \sinh r$, where α is the angle at o between the geodesic joining o and x and the geodesic joining o and θ. In the hyperbolic space the Buseman function $B_0(y,\xi)$ associated to an origin y_0 also satisfies the relation $\exp B_0(y,\xi) = \cosh r - \cos\alpha \sinh r$, where (r,ξ) are the polar coordinate of y at the origin y_0. The choice of an isometry between the tangent space of X at o and the tangent space of the hyperbolic space $\mathbb{H}^{n+1}$ at y_0 provides a diffeomorphism $f : X \cup \partial X \to \mathbb{H}^{n+1} \cup \partial\mathbb{H}^{n+1}$ between $X \cup \partial X$ and $\mathbb{H}^{n+1} \cup \partial\mathbb{H}^{n+1}$ which reads in polar coordinates $f(r,\alpha) = (r,\alpha)$. Since the Busemann functions B and B_0 of X and $\mathbb{H}^{n+1}$ have the same expression in polar coordinate, we get that $B(x,\theta) = B_0(f(x), f(\theta))$. Therefore for any $x, y \in X$ we have $d_X(x,y) = \sup\{B(x,\theta) - B(y,\theta) \mid \theta \in \partial X\} = \sup\{B_0(f(x), f(\theta)) - B_0(f(y), f(\theta)) \mid \theta \in \partial X\} = \sup\{B_0(f(x),\xi) - B_0(f(y),\xi) \mid \xi \in \partial\mathbb{H}^{n+1}\} = d_{\mathbb{H}^{n+1}}(f(x), f(y))$. Hence f is an isometry and X is a hyperbolic space.

□

The proof of the theorem 1.6 therefore boils down in showing that if $\delta(G) = n$ then $HessB(x,\theta) + dB(x,\theta) \otimes dB(x,\theta) = g(x)$.

For that purpose we shall construct a smooth map $F : M \to M$ homotopic to the Identity map and whose Jacobian satisfies $|JacF(x)| \leq (\det\, k(x))^{-1} \left(\frac{\delta(G)+1}{n+1}\right)^{n+1}$ where $k(x)$ is the quadratic form defined on the tangent space $T_{F(x)}M$ by

(2.1)
$$k(x)(.,.) = \int_{\partial X} HessB(\tilde{F}(\tilde{x}),\theta)(.,.) + dB(\tilde{F}(\tilde{x}),\theta)(.) \otimes db(\tilde{F}(\tilde{x}),\theta)(.) d\mu_{\tilde{x}}$$

where $\tilde{F}$ and $\tilde{x}$ stands for the lifts of F and x to the universal cover $X = \tilde{M}$ of M, and $\mu_{\tilde{x}}$ is the measure of Patterson that we will describe now.

Recall that a family of Patterson measures $(\mu_x)_{x\in X}$ associated to a discrete group of isometry G of X is a set of positive finite measures μ_x supported on ∂X, $x \in X$, such that the following holds for all $x \in X$, $\gamma \in G$,

$$\mu_{\gamma x} = \gamma_* \mu_x \tag{2.2}$$

$$\mu_x = e^{-\delta B(x,\theta)} \mu_o, \tag{2.3}$$

where $o \in X$ is a fixed origin, B the Busemann function associated to o and δ the critical exponent of G. In (2.3), δ is the critical exponent of G defined by

$$\delta = \inf\{s \in \mathbb{R} \,/\, \Sigma_{\gamma\in G} e^{d(x,\gamma y)} < \infty\}\,.$$

Let us recall that for G beeing a convex cocompact discrete group of isometries of X, the critical exponent δ of G coincides with the Hausdorff dimension $\delta(G)$ of the limit set of G, cf. [**17**]. For the existence and uniqueness of such a family of measures associated to a discrete convex cocompact group G we refer to [**12**], [**17**], [**13**]. In the sequel of this section we will write $\delta = \delta(G)$.

Before describing the construction of the map F, let us end the proof of the theorem 1.6. We assume that M is orientable (if not we replace it by a 2-fold covering).

Since F is homotopic to the Identity, it is a degree one map therefore if Ω is the volume form of M one has, (recall that $\delta = \delta(G)$),

$$volM = |\int_M F^*\Omega \quad | \leq \int_M |JacF(x)|dx \leq \Big(\int_M (\det k(x))^{-1}dx\Big)\Big(\frac{\delta+1}{n+1}\Big)^{n+1}. \tag{2.4}$$

On the other hand the sectional curvature of M satisfies $K \leq -1$, therefore by the Rauch comparison theorem we have for every $y \in X$, and $\theta \in \partial X$,

$$HessB(y,\theta) + dB_{(y,\theta)} \otimes dB_{(y,\theta)} \geq g(y), \tag{2.5}$$

therefore all eigenvalues of the quadratic form $HessB(y,\theta) + dB_{(y,\theta)} \otimes dB_{(y,\theta)}$ are greater than or equal to 1 and so are the eigenvalues of $k(x)$, hence $\det k(x) \geq 1$. We therefore get under the assumption that $\delta = n$,

$$volM \leq \int_M (detk(x))^{-1} dx \leq volM.$$

and so we get $\det k(x) = 1$. Since all eigenvalues of the quadratic form $k(x)$ are larger than or equal to 1, they are thus equal to 1. Hence we get for all $y \in X$ and $\theta \in \partial X$

$$HessB(y,\theta) + dB_{(y,\theta)} \otimes dB_{(y,\theta)} = g(y),$$

using the fact that the measures $\mu_{\tilde{y}}$ are positive on open subsets of ∂X. The lemma 2.2 then concludes the proof of the theorem 1.6.

Let us now explain the construction of the map F, cf. [**1**].

We first define a map which associates a point in X to a measure μ supported on the ideal boundary ∂X whose support is not reduced to a point. Let μ be a measure supported on ∂X. Let $\mathcal{D}_\mu : X \to \mathbb{R}$ be the function defined by

$$\mathcal{D}_\mu(y) = \int_{\partial X} e^{B(y,\theta)} d\mu(\theta) \tag{2.6}$$

A computation shows that

$$HessD_\mu(y) = \int_{\partial X} (HessB_{(y,\theta)} + dB_{(y,\theta)} \otimes dB_{(y,\theta)}) e^{B(y,\theta)} d\mu(\theta). \tag{2.7}$$

By 2.5 we then get

$$Hess\mathcal{D}_\mu(y) \geq \mathcal{D}_\mu(y)\tilde{g}, \tag{2.8}$$

thus $Hess\mathcal{D}_\mu(y)$ is positive definite and $\mathcal{D}_\mu$ is strictly convex.

Claim: If the support of μ contains at least two points, we have

$$\lim_{y_k \to \partial X} \mathcal{D}_\mu(y) = +\infty.$$

Proof. — Let $y_k \in X$ a sequence such that

$$\lim_{k\to\infty} y_k = \theta_0 \in \partial X. \tag{2.9}$$

As the support of μ contains at least two points, we have $supp(\mu) \cap (\partial X - \{\theta_0\}) \neq \emptyset$, thus there exists a compact subset $K \subset \partial X - \{\theta_0\}$ such that $\mu(K) > 0$ therefore,

$$\int_{\partial X} e^{B(y_k,\theta)} d\mu \geq \int_K e^{B(y_k,\theta)} d\mu \to +\infty. \tag{2.10}$$

because for every $\theta \in K$ we have $\lim_{y_k \to \theta_0} B(y_k, \theta) = +\infty$. □

We then have the following lemma.

Lemma 2.2. — *Let μ a finite borel measure on ∂X whose support contains at least two points. The function $\mathcal{D}_\mu$ has a unique minimum. This minimum will be denoted by $\mathcal{C}(\mu)$.*

Whenever G is cocompact the support of μ_0 equals ∂X thus according to the lemma we can define the map $\tilde{F} : X \to X$ for $x \in X$ by

$$\tilde{F}(x) = \mathcal{C}(e^{-B(x,\theta)}\mu_x)\,, \tag{2.11}$$

where $\{\mu_x\}$ is the family of Patterson Sullivan measures associated to G. The map $\tilde{F}$ satisfies the following properties.

(i) $\tilde{F}$ is a smooth G-equivariant map.

(ii) $|Jac\tilde{F}(x)| \leq (det\, k(x))^{-1}\left(\frac{\delta+1}{n+1}\right)^{n+1}$

Proof. — (i) The smoothness of $\tilde{F}$ comes from the smoothness of the Busemann function $B(x,\theta)$ with respect to x for each fixed θ. From the equivariance of the family of Patterson measures, cf. 2.9 and the cocycle relation $B(\gamma y, \gamma\theta) - B(\gamma x, \gamma\theta) = B(y,\theta) - B(x,\theta)$ we get the invariance under the diagonal action of G on $X \times X$ of the function of (x, y) defined by $\mathcal{D}_{e^{-B(x,\theta)}\mu_x}(y) = \int_{\partial X} e^{B(y,\theta)-B(x,\theta)} d\mu_x(\theta)$ which implies the equivariance of $\tilde{F}$. By equivariance $\tilde{F}$ is homotopic to the Identity.

(ii) The point $\tilde{F}(x)$ is characterized by

$$\int_{\partial X} dB_{(\tilde{F}(x),\theta)} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0. \tag{2.12}$$

In order to simplify the notations we will denote ν_x the measure $e^{B(\tilde{F}(x),\theta)-B(x,\theta)}\mu_x$. We will also write $D\tilde{F}(u)$ instead of $D\tilde{F}(x)(u)$.

By differentiating 2.12 we get the following characterization of the differential of $\tilde{F}$: for $u \in T_xX$ and $v \in T_{\tilde{F}(x)}X$, one has

$$\int_{\partial X}[HessB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u),v)+dB_{(\tilde{F}(x),\theta)}(v)dB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u))]d\nu_x(\theta)$$

$$= (\delta+1)\int_{\partial X} dB_{(\tilde{F}(x),\theta)}(v)dB_{(x,\theta)}(u)d\nu_x(\theta). \tag{2.13}$$

Let us recall that we defined the quadratic forms k for $v \in T_{\tilde{F}(x)}X$ by

$$k(v,v) = \int_{\partial X}[DdB_{(\tilde{F}(x),\theta)}(v,v)+(dB_{(\tilde{F}(x),\theta)}(v))^2]d\nu_x(\theta). \tag{2.14}$$

Let us define the quadratic form h by

$$h(v,v) = \int_{\partial X} dB_{(\tilde{F}(x),\theta)}(v)^2 d\nu_x(\theta). \tag{2.15}$$

The relation 2.13 writes, for $u \in T_xX$ and $v \in T_{\tilde{F}(x)}X$:

$$k(D\tilde{F}(u),v) = (\delta+1)\int_{\partial X} dB_{(\tilde{F}(x),\theta)}(v)dB_{(x,\theta)}(u)d\nu_x(\theta). \tag{2.16}$$

We define the quadratic form h' on T_xX for $u \in T_xX$ by

$$h'(u,u) = \int_{\partial X} dB_{(x,\theta)}(u)^2 d\nu_x(\theta), \tag{2.17}$$

and one derives from 2.16

$$|k(D\tilde{F}(x)(u),v)| \leq (\delta+1)h(v,v)^{1/2}h'(u,u)^{1/2}. \tag{2.18}$$

One now can estimate the Jacobian of $\tilde{F}$. If $D\tilde{F}$ is not of rank $n+1$, then $Jac\tilde{F}(x) = 0$. Let us assume that $D\tilde{F}$ is of rank $n+1$. Let us denote by H' [resp. H and K] the selfadjoint operator (with respect to $\tilde{g}$) associated to the quadratic form h' [resp. h, k].

Let $(v_i)_{i=1}^{n+1}$ be an orthonormal basis of $T_{\tilde F(x)}X$ which diagonalizes H and $(u_i)_{i=1}^{n+1}$ an orthonormal basis of T_xX such that the matrix of $K \circ D\tilde F(x) : T_xX \to T_{\tilde F(x)}X$ is triangular. Then,

$$\det K.|Jac\tilde F(x)| \le (\delta+1)^{n+1}(\Pi_{i=1}^{n+1} h(v_i,v_i)^{1/2})(\Pi_{i=1}^{n+1} h'(u_i,u_i)^{1/2}) \tag{2.19}$$

thus,

$$\det K.|Jac\tilde F(x)| \le (\delta+1)^{n+1}\Big(\frac{TraceH}{n+1}\Big)^{(n+1)/2}\Big(\frac{TraceH'}{n+1}\Big)^{(n+1)/2}. \tag{2.20}$$

In these inequalities one can normalize the measures

$$\nu_x = e^{B(\tilde F(x),\theta)-B(x,\theta)}\mu_x$$

such that their total mass equals one, which gives

$$TraceH = \Sigma_{i=1}^{n+1} h(v_i,v_i) \le 1, \tag{2.21}$$

the last inequality coming from the fact that for all $\theta \in \partial X$,

$$\Sigma_{i=1}^{n+1} dB_{(\tilde F(x),\theta)}(v_i)^2 \le ||dB_{(\tilde F(x),\theta)}||^2 = 1 \tag{2.22}$$

and from the previous normalization.

Similarly,

$$TraceH' = \Sigma_{i=1}^{n+1} h'(u_i,u_i) \le 1. \tag{2.23}$$

We then obtain from (2.20)

$$\det K\,.|Jac\tilde F(x)| \le \Big(\frac{\delta+1}{n+1}\Big)^{n+1}, \tag{2.24}$$

therefore we get

$$|Jac\tilde F(x)| \le (\det k(x))^{-1}\Big(\frac{\delta+1}{n+1}\Big)^{n+1}. \tag{2.25}$$

This proves (ii). □

Remark : The above proof of the theorem 1.6 would extend to the case of a noncompact $M = X/G$ with finite volume if the map F would be proper.

3. A theorem of P. Tukia

The strategy of proof of theorem 1.5 is to show that under the assumptions, the group G actually is isomorphic to a cocompact lattice in $PO(n,1)$ and then apply the theorem 1.3. The way of doing this is to apply the following theorem of P.Tukia which characterizes discrete subgroups of $PO(n,1)$.

Theorem 3.1. — [**16**] *Let G be a group acting uniformly quasi-Möbius on the standard sphere (S^n, can). We assume that the induced action of G on $Tri(S^n)$ is cocompact, then the action of G on S^n is conjugate by a quasi-Möbius homeomorphism to an action by Möbius transformations of S^n.*

In the above theorem, the set $Tri(S^n)$ is defined by

$$Tri(S^n) = \{(u,v,w) \in (S^n)^3 / \quad u\,,v\,,w \quad distinct\}\,.$$

We refer to [**16**] for a complete proof of the theorem 3.1, but let us comment briefly this theorem. The sphere S^n can be considered as the boundary at infinity of the hyperbolic space $\mathbb{H}^{n+1}$. Any group G of homeomorphism of S^n naturally extends to an action on $Tri(S^n)$. There is a natural projection $p : Tri(S^n) \to \mathbb{H}^{n+1}$, where for $(u,v,w) \in Tri(S^n)$, $p\big((u,v,w)\big)$ is defined as the orthogonal projection of w onto the geodesic joining u and v. For $x \in \mathbb{H}^{n+1}$, the inverse image $p^{-1}(x)$ is homeomorphic to the set of 2-frames tangent at x, thus is compact and $Tri(S^n)$ can be thought of as an “approximation up to compact” of $\mathbb{H}^{n+1}$. The cocompactness of the action of G on $Tri(S^n)$ can then be translated into the fact that every point in S^n is a “radial point” of G. A point $u_0 \in S^n$ is a radial point of G if there exists a sequence $g_i \in G$ such that given $(u,v,w) \in Tri(S^n)$ and a geodesic line α in $\mathbb{H}^{n+1}$ with end point u_0, then the sequence $x_i =: p\big((g_iu, g_iv, g_iw)\big) \in \mathbb{H}^{n+1}$ converges to u_0. This definition corresponds to the notion of “conical points” for kleinian groups or discrete group acting on Cartan Hadamard manifolds of negative curvature. The theorem 3.1 is based on the following result, proved by P. Tukia, (c.f [**16**], theorem F): for any group G acting uniformly quasi-Möbius on S^n, there exist an invariant measurable conformal structure μ on S^n, that is a map which associates a positive definite metric $\mu(x)$ of determinant 1 to almost every $x \in S^n$. Moreover this conformal structure is approximatly constant near almost every radial point. Considering such a radial point u_0 and the associated sequence $g_i \in G$, and then transforming the action of G by these g_i's into a neighbourhood of u_0 and blowing up those neighbourhoods gives the theorem.

4. Weak tangent and self similarity of limit sets

In this section we will define the notion of weak tangent of a metric space and of quasi-Möbius homeomorphism between metric spaces. The goal of the section is to show that for any quasi-convex cocompact group G acting on a $CAT(-1)$ space and for any weak tangent (S, ρ, o) of the limit set $(\Lambda(G), d)$, then the one point compactification $(\hat{S}, \hat{\rho}_o)$ of (S, ρ, o) is quasi-Möbius homeomorphic to $(\Lambda(G), d)$, proposition 4.5. So we will be allowed to consider the conjugate action of G on $(\hat{S}, \hat{\rho}_o)$ instead of the original action of G on $(\Lambda(G), d)$, which will turn out to be particularly usefull since the coincidence between topological and Hausdorff dimension provides information on (S, ρ, o), as we will see in section 6, cf. Proposition 6.2.

Let (M_k, d_k), (M, d) be metric spaces with base points $p_k \in M_k$ and $p \in M$.

Definition 4.1. — *The sequence (M_k, d_k, p_k) is said to converge to (M, d, p) in the pointed Gromov-Hausdorff topology if $\forall R > 0$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $\exists D_k \subset B_{M_k}(p_k, R)$, $\exists D'_k \subset B_M(p, R)$, $\exists f_k : D_k \to D'_k$ such that f_k are bijections, $p_k \in D_k$, $p \in D'_k$ and for any k,*

(1) $f_k(p_k) = p$

(2) D_k is ϵ-dense in $B_{M_k}(p_k, R)$ and D'_k is ϵ-dense in $B_M(p, R)$

(3) $\forall x$, $y \in D_k$, we have $|d(f_k(x), f_k(y)) - d_k(x, y)| \leq \epsilon$.

Example: The product $\mathbb{R} \times \frac{1}{k} S^1$ of the real line with a circle of radius $\frac{1}{k}$ converges to $\mathbb{R}$ in the pointed Gromov-Hausdorff topology.

Definition 4.2. — *Let (M, d) be a metric space. A weak tangent of (M, d) is a complete metric space (S, ρ, o) with a base point $o \in S$ such that there exists a sequence $(M, \lambda_k d, p_k)$ converging in the Gromov-Hausdorff topology to (S, ρ, o) for some sequence $\lambda_k \to +\infty$.*

Example 1: Let (M, g) be a riemannian manifold; then every weak tangent at a point $x \in M$ is isometric to the tangent space $T_x M$ of M at x endowed with the euclidean distance induced by $g(x)$. For general metric spaces weak tangent are not unique.

Example 2: Let (M, d) be a metric space and (S, ρ, o) a weak tangent of (M, d) at a point $p \in M$. Let (S', ρ', o) be a weak tangent of (S, ρ, o) at o. Then (S', ρ', o) is a weak tangent of (M, d) at p.

As one see on the example 1 a weak tangent (S, ρ, o) of a metric space (M, d) may be unbounded so we shall now put a distance $\hat{\rho}$ on the one point compactification $\hat{S} = S \cup \{\infty\}$ such that the two distances ρ and $\hat{\rho}$ on S are "quasi-Möbius" equivalent. Let us now define quasi-Möbius map between metric spaces and decribe the construction of $\hat{\rho}$.

Let (M, d) be a metric space. The *cross ratio* of a four-uple of distinct points (x_1, x_2, x_3, x_4) is the real number

$$[x_1, x_2, x_3, x_4] := \frac{d(x_1, x_3) d(x_2, x_4)}{d(x_1, x_4) d(x_2, x_3)}$$

Definition 4.3. — *Let $f : (M, d) \to (M', d')$ be an injective map between two metric spaces and $\eta : [0, +\infty[\to [0, +\infty[$ a homeomorphism such that $\eta(0) = 0$. The map f is said to be η-quasi-Möbius if for any quadruple of points $x_i \in M$, $1 \leq i \leq 4$, $[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4])$.*

Remark: As a consequence of the definition one can see that a η-quasi-Möbius map $f : (M, d) \to (M', d')$ is a homeomorphism on its image $f(M)$ and that the inverse map $f^{-1} : (f(M), d') \to (M, d)$ is η'-quasi-Möbius for the homeomorphism $\eta'(t) = \eta(1/t)^{-1}$. This can be easily seen by exchanging x_1 and x_2 in the definition of a quasi-Möbius map, which gives $\eta(1/[x_1,x_2,x_3,x_4])^{-1} \leq [f(x_1),f(x_2),f(x_3),f(x_4)]$.

Example: Let G be a quasi-convex cocompact group of isometries of a CAT(-1) space X and $(\Lambda(G),d)$ the limit set of G endowed with the distance d defined in 1.1. Then G acts uniformly quasi-Möbius on $(\Lambda(G),d)$, which means that there exist an increasing homeomorphism $\eta : [0,+\infty[\to [0,+\infty[$ such that the action of each element $g \in G$ on $(\Lambda(G),d)$ is η-quasi-Möbius, cf. [**5**].

For more details on quasi-Möbius maps see the M.Bourdon's lecture.

Lemma 4.4. — *Let (S,ρ) be an unbounded metric space with a base point o and $\hat{S} = S \cup \{\infty\}$ the one point compactification of S. There exists a distance $\hat{\rho}_0$ on $\hat{S}$ inducing the topology of $\hat{S}$ such that (S,ρ) and $(S,\hat{\rho}_o)$ are η-quasi-Möbius homeomorphic for $\eta(t) = 16t$.*

Proof. — Let us consider the function $h_o : \hat{S} \to [0,+\infty[$ defined by $h_o(\infty) = 0$ and for all $x \in S$

$$h_o(x) = \frac{1}{1+\rho(o,x)}.$$

For x and $y \in S$ let us define ρ_o by $\rho_o(x,y) := h_o(x)h_o(y)\rho(x,y)$ if x and y $\in S$, $\rho_o(x,\infty) = \rho_o(\infty,x) = h_o(x)$ and $\rho_o(\infty,\infty) = 0$. If x and y $\in \hat{S}$ we define $\hat{\rho}_o(x,y) = \inf\{\Sigma_{i=0}^{k-1}\rho_o(x_i,x_{i+1})\}$ where the infimum is taken over all sequences of points $x_0,...x_k$ in $\hat{S}$ with $x_0 = x$ and $x_k = y$. The reader can easily check that the lemma is a consequence of the following inequalities

$$\frac{1}{4}\rho_o(x,y) \leq \hat{\rho}_o(x,y) \leq \rho_o(x,y) \tag{4.1}$$

. □

The next proposition is one key point in the proof of the theorem 1.5. Roughly speaking it says that the limit set $\Lambda(G)$ of a convex cocompact group G of a CAT(-1) space has a selfsimilarity property. More precisely this means that $(\Lambda(G),d)$ is quasi-Möbius homeomorphic to $(\hat{S},\hat{\rho}_o)$ for any weak tangent (S,ρ,o) of $(\Lambda(G),d)$.

Proposition 4.5. — *Let G be a convex cocompact group acting on a CAT(-1) space, $(\Lambda(G), d)$ the limit set of G endowed with the metric defined in 1.1. If (S, ρ, o) is a weak tangent of $(\Lambda(G), d)$ then $(\hat{S}, \hat{\rho}_o)$ is quasi-Möbius homeomorphic to $(\Lambda(G), d)$.*

Proof. — Let us consider a weak tangent $(S, \rho, o) = \lim_{\lambda_k \to \infty}(\Lambda(G), \lambda_k d, p_k)$ of $(\Lambda(G), d)$. In order to simplify the notations we will denote Z the limit set $\Lambda(G)$ and $\lambda_k Z$ the metric space $(\Lambda(G), \lambda_k d)$. For an arbitrary metric space (M, d) we will write $B_M(p, r)$ the ball of radius r centered at $p \in M$.

Let $\tilde{D}_k \subset B_S(o, k)$ [resp. $D_k \subset B_{\lambda_k Z}(p_k, k)$] be maximal $\frac{1}{k}$ separated subsets such that $o \in \tilde{D}_k$ and $p_k \in D_k$ and bijections $f_k : \tilde{D}_k \to D_k$ such that

$$|\lambda_k d(f_k(x), f_k(y)) - \rho(x, y)| \leq \frac{1}{k} \tag{4.2}$$

for any x, y in S.

Such sets $\tilde{D}_k$, D_k and maps f_k exist since (S, ρ, o) is the limit of $(Z, \lambda_k d)$.

It follows from 4.1 and the $\frac{1}{k}$ maximal separation of the sets $\tilde{D}_k$ and D_k that

$$\frac{1}{2}\rho(x, y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\rho(x, y) \tag{4.3}$$

thus the sequence of maps f_k is uniformly bilipschitz.

Now we can assume that $\tilde{D}_k \subset S$ contains a fixed triple of distinct points y_0, y_1, y_2 in S. Let us denote $x_i^k = f_k(y_i)$, $i = 0, 1, 2$.

As G is convex cocompact, its action on the set $Tri(Z)$ of triples of distinct points in $Z = \Lambda(G)$ is cocompact. Therefore there exist a sequence $\gamma_k \in G$ and a positive number δ such that

$$d(\gamma_k x_i^k, \gamma_k x_j^k) \geq \delta \tag{4.4}$$

and

$$\rho(y_i, y_j) \geq \delta \tag{4.5}$$

for $i, j \in \{0, 1, 2\}$.

Let us define $h_k : (\tilde{D}_k, \rho) \subset (S, \rho) \to (D'_k, d) \subset (Z, d)$ defined by $h_k := \gamma_k \circ f_k$ where $D'_k = \gamma_k(D_k)$. As G is convex cocompact it acts uniformly quasi-Möbius on $(\Lambda(G), d)$ thus there exist an increasing homeomorphism $\eta : [0, +\infty[\to [0, +\infty[$ such that the action of each element $\gamma_k \in G$ on $(\Lambda(G), d)$ is η-quasi-Möbius. Therefore, by (4.3) we get that $h_k = \gamma_k \circ f_k$ is 16η-quasi-Möbius. By the lemma 4.4, the maps h_k can be considered as 16η-quasi-Möbius maps $h_k : (\tilde{D}_k, \rho) \subset (\hat{S}, \hat{\rho}_o) \to (D'_k, d) \subset (Z, d)$.

Claim 1: The sequence $h_k : (\tilde{D}_k, \rho) \subset (\hat{S}, \hat{\rho}_o) \to (D'_k, d) \subset (Z, d)$ is equicontinuous.

Let us prove the claim. We shall prove the existence of a function $\mu : [0, \infty[\to [0, \infty[$ satisfying $\lim_{r \to 0} \mu(r) = 0$ and such that for any $x, y \in \tilde{D}_k$, $\hat{\rho}_o(x, y) = r$, then $d(h_k(x), h_k(y)) \leq \mu(r)$.

Let us consider the points y_i and x_i^k, $i = 0, 1, 2$, such that (4.4) and (4.5) hold, and denote $z_i^k = h_k(y_i) = \gamma_k(x_i^k)$. We can normalize d such that $\delta = 1$ in (4.4) and (4.5), therefore we have

$$d(z_i^k, z_j^k) \geq 1 \tag{4.6}$$

and

$$\hat{\rho}_o(y_i, y_j) \geq 1 \tag{4.7}$$

for $i, j \in \{0, 1, 2\}$.

Let $x, y \in \tilde{D}_k$, $\hat{\rho}_o(x, y) = r$. There are three cases:

1) $\hat{\rho}_o(x, y) = r \leq 1/4$ and $\hat{\rho}_o(y_1, x) \leq 1/2$

2) $\hat{\rho}_o(x, y) = r \leq 1/4$ and $\hat{\rho}_o(y_1, x) > 1/2$

3) $\hat{\rho}_o(x, y) > 1/4$.

Case 1). We have $\hat{\rho}_o(x, y_2) \geq 1/2$, $\hat{\rho}_o(x, y_3) \geq 1/2$, $\hat{\rho}_o(y, y_2) \geq 1/4$ and $\hat{\rho}_o(y, y_3) \geq 1/4$.

From these inequalities and the fact that $[h_k(x), h_k(y_2), h_k(y), h_k(y_3)] \leq \eta([x, y_2, y, y_3])$ we easily get

$$d(h_k(x), h_k(y)) \leq (diam(Z))^2 \eta(8 diam(\hat{S}) \hat{\rho}_o(x, y)). \tag{4.8}$$

Case 2). As $\hat{\rho}_o(y_2, y_3) \geq 1$ there exist $i \in \{2, 3\}$ such that $\hat{\rho}_o(y, y_i) \geq 1/2$. ¿From this and the fact that $[h_k(x), h_k(y_i), h_k(y), h_k(y_1)] \leq \eta([x, y_i, y, y_1])$ we get

$$d(h_k(x), h_k(y)) \leq (diam(Z))^2 \eta(4 diam(\hat{S}) \hat{\rho}_o(x, y)). \tag{4.9}$$

Case 3). We have $\hat{\rho}_o(x, y) \geq 1/4$, therefore

$$d(h_k(x), h_k(y)) \leq \frac{4 diam(Z)}{4} \leq 4 diam(Z) \hat{\rho}_o(x, y). \tag{4.10}$$

From (4.8), (4.9) and (4.10) we obtain the equicontinuity of the h_k's taking $\mu(r) = \inf\{(diam(Z))^2\eta(4diam(\hat{S})r), (diam(Z))^2\eta(4diam(\hat{S})r), 4diam(Z)r\}$. This ends the proof of the claim 1.

Moreover the Hausdorff distance $dist_H(\tilde{D}_k, \hat{S})$ between $(\tilde{D}_k, \hat{\rho}_o)$ and $(\hat{S}, \hat{\rho}_o)$ satisfies $\lim_{k\to\infty} dist_H(\tilde{D}_k, \hat{S}) = 0$.

Claim 2 : The Hausdorff distance $d_H(h_k(\tilde{D}_k), Z)$ between $h_k(\tilde{D}_k)$ and Z satisfies $\lim_{k\to\infty} d_H(h_k(\tilde{D}_k), Z) = 0$.

Let us prove the claim 2.

Let us recall that $h_k : (\tilde{D}_k, \hat{\rho}_o) \subset (\hat{S}, \hat{\rho}_o) \to (D'_k, d) \subset (Z, d)$ is defined by $h_k := \gamma_k \circ f_k$ where $D'_k = \gamma_k(D_k)$. We have chosen a fixed triple of distinct points y_0, y_1, y_2 in $\tilde{D}_k \subset S$ with $\rho(y_i, y_j) \geq \delta$ and such that $x_i^k = f_k(y_i)$, $i = 0, 1, 2$ satisfies $\frac{C^{-1}}{\lambda_k} \leq d(x_i^k, x_j^k) \leq \frac{C}{\lambda_k}$. The γ_k's have been chosen such that $d(z_i^k, z_j^k) \geq \delta$ where $z_i^k = \gamma_k(x_i^k)$. Let us also recall that $\tilde{D}_k$ is $\frac{1}{k}$-dense in $B_S(o, k)$ and D_k is $\frac{1}{k}$-dense in $B_{\lambda_k Z}(p, k)$. Let us write $R_k = \frac{k}{\lambda_k}$, $\epsilon_k = \frac{1}{k^2}$, $\delta_k = \frac{C^{-1}}{k}$, $\mu_k = \frac{C}{k}$. With these notations, D_k is $\epsilon_k R_k$-dense in $B_Z(p, R_k)$. Note that $x_i^k \in B_Z(p, \mu_k R_k)$, $d(x_i^k, x_j^k) \geq \delta_k R_k$ and that $\frac{\epsilon_k}{\delta_k^2}$ is bounded.

Let us now take $z \in Z$. The point z can be written $z = \gamma_k x_k$. There are two cases.

Case 1: $x_k \in B(p, R_k)$.

In that case, there exist $y_k \in D_k \cap B(p, R_k)$ such that $d(x_k, y_k) \leq \epsilon_k R_k$. Let us show now that $d(\gamma_k y_k, \gamma_k x_k) = d(\gamma_k y_k, z) \to 0$ when $k \to \infty$. This will prove the claim 2 because $\gamma_k y_k \in D'_k$. Since $d(x_i^k, x_j^k) \geq \delta_k R_k$ for $i \neq j$ then we have for, say x_1^k and x_2^k, the following estimates $d(y_k, x_1^k) \geq \frac{\delta_k R_k}{2}$ and $d(x_k, x_2^k) \geq \frac{\delta_k R_k}{2}$. By the fact that the γ_k's act uniformly quasi-Möbius

$$[\gamma_k x_k, \gamma_k x_1^k, \gamma_k y_k, \gamma_k x_2^k] \leq \eta([x_k, x_1^k, y_k, x_2^k])$$

and one can easily deduce from all this that

$$d(\gamma_k y_k, \gamma_k x_k) \leq \frac{(diam(Z))^2}{\delta_k R_k} \eta\left(\frac{4\epsilon_k \mu_k}{\delta_k^2}\right)$$

therefore $d(\gamma_k y_k, \gamma_k x_k) = d(\gamma_k y_k, z) \to 0$, which proves the claim 2 in the case 1.

Case 2: $x_k \notin B(p, R_k)$.

As (Z,d) is uniformly perfect and D_k is $\epsilon_k R_k$-dense in $B_Z(p,R_k)$, there exist a point y_k in $D_k \cap B_Z(p,R_k)$ so that $\frac{d(y_k,p)}{R_k} \geq C_0$ for a positive constant C_0 independant of k. We can assume that $\mu_k < C_0/2$ because μ_k tends to 0 when k tends to infinity. The γ_k's act uniformly quasi-Möbius thus we have

$$[\gamma_k x_k, \gamma_k x_1^k, \gamma_k y_k, \gamma_k x_2^k] \leq \eta([x_k, x_1^k, y_k, x_2^k]).$$

In our situation we get from triangle inequality
$d(x_1^k, y_k) \geq R_k(C_0 - \mu_k)$,
$d(x_2^k, x_k) \geq d(x_k, p) - \mu_k R_k$
$d(x_k, y_k) \leq 2d(x_k, p)$ and
$d(x_1^k, x_2^k) \leq 2\mu_k R_k$,
therefore, we get $\eta([x_k, x_1^k, y_k, x_2^k]) \leq \eta\left(\frac{4\mu_k d(x_k,p)}{(d(x_k,p)-\mu_k R_k)(C_0-\mu_k)}\right)$, thus

$$[\gamma_k x_k, \gamma_k x_1^k, \gamma_k y_k, \gamma_k x_2^k] \leq \eta\left(\frac{16\mu_k}{C_0}\right).$$

One then easily deduce

$$d(\gamma_k x_k, \gamma_k y_k) \leq \frac{(diam(Z))^2\, \eta\left(\frac{16\mu_k}{C_0}\right)}{\delta}$$

which proves that $d(z, \gamma_k y_k) = d(\gamma_k x_k, \gamma_k y_k)$ tends to 0 when k tends to infinity and finishes the proof of the claim 2.

Let us summarize what we have obtained so far. We have an equicontinuous sequence of maps $h_k : (\tilde{D}_k, \hat{\rho})_o) \subset (\hat{S}, \hat{\rho}_o) \to (D'_k, d) \subset (Z,d)$ such that the Hausdorff distance $d_H(h_k(\tilde{D}_k), Z)$ between $h_k(\tilde{D}_k)$ and Z satisfies $\lim_{k\to\infty} d_H(h_k(\tilde{D}_k), Z) = 0$. Moreover as $\tilde{D}_k$ is $\frac{1}{k}$-dense in (S,ρ) it is easy to check that the Hausdorff distance $dist_H(\tilde{D}_k, \hat{S})$ tends to 0 when k tends to ∞ using the relation (4.1) between ρ and $\hat{\rho}_o$. By an argument similar to the Ascoli's theorem, one can then show that a subsequence h_{k_n} uniformly converges to a quasi-Möbius map $h : (\hat{S}, \hat{\rho}_o) \to (Z,d)$, i.e. $\lim_{n\to\infty} dist(h_{k_n}, h|_{\tilde{D}_{k_n}}) = 0$, where $dist(h_{k_n}, h|_{\tilde{D}_{k_n}}) = sup\{d(h_{k_n}x), h(x)) \mid x \in \tilde{D}_{k_n}\}$. The same arguments also hold for the sequence $h_{k_n}^{-1} : (D'_k, d) \subset (Z,d) \to (\tilde{D}_k, \hat{\rho}_o) \subset (\hat{S}, \hat{\rho}_o)$ which therefore uniformly subconverges to a quasi-Möbius map

$f : (Z, d) \to (\hat{S}, \hat{\rho}_o)$. Clearly one has $f \circ h = Id_{\hat{S}}$ and $h \circ f = Id_Z$ which concludes the proof of the proposition 4.5. □

5. Topological dimension and regular maps

In this section we define the topological dimension of compact metric spaces and and the notion of regular map from a metric space to $\mathbb{R}^n$. The main result of the section gives a condition under which a compact metric space of topological dimension n has a weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$, corollary 5.8.

In this section S^n , $\mathbb{R}^n$ will denote the standard n-dimensional sphere and euclidean space endowed with their natural distances d_{S^n}, $d_{\mathbb{R}^n}$ and (Z, d), (S, ρ) compact metric spaces. Let us recall that if $f$and g are two continuous maps from Z to S^n, the distance $dist(f, g)$ between f and g is defined by

$$dist(f, g) = sup\{d_{S^n}(f(x), g(x)) \quad | \quad x \in Z\}.$$

Definition 5.1. — *Let $f : Z \to S^n$ be a continuous map. A point $y \in Imf \subset S^n$ is called a stable value of f if there exist $\epsilon > 0$ such that for each continuous map $g : Z \to S^n$ satisfying $dist(f, g) \leq \epsilon$ we have $y \in Img$.*

Example: Let us consider the unit circle $S^1 \subset \mathbb{C}$ and the map $f : S^1 \to S^1$ defined as the identity on $S^1 \cap \mathcal{I}mz > 0$ and as the symmetry through the real axis on $S^1 \cap \mathcal{I}mz \leq 0$. The points $+1$, -1 are not stable values and any other point of $S^1 \cap \mathcal{I}mz > 0$ is a stable value.

Let us summarize in the following lemma two properties concerning stable values.

Lemma 5.2. — *(i) The set of stable values of a continuous map $f : Z \to S^n$ is an open subset of S^n.*

(ii) If $f : Z \to \mathbb{R}^n$ is a continuous map, then $y \in \mathbb{R}^n$ is a stable value if and only if y is a stable value of the restriction $f_{|f^{-1}(W)}$ of f to $f^{-1}(W)$ for any open neighborhood W of y.

Proof. — (i) Let $y \in Imf$ be a stable value of f and ϵ coming from the stability of f. For any $y' \in S^n$ so that $d_{S^n}(y, y') \leq \epsilon$ we pick up a small rotation Φ of S^n such that $\Phi(y') = y$. It is possible to choose Φ such that $dist(\Phi, Id_{S^n}) \leq \epsilon$, therefore we have $dist(\Phi \circ f, f) \leq \epsilon$ so that there exist $x \in Z$ with $\Phi \circ f(x) = y$ and thus $f(x) = y'$.

(ii) Let us assume that $y \in \mathbb{R}^n$ is not a stable value of $f_{|U}$ where $U = f^{-1}(W)$ and W is some open neighborhood of y. For $\delta > 0$ such that $\bar{B}(y, \delta) \subset W$ let us denote $V = f^{-1}(\mathbb{R}^n - \bar{B}(y, \delta))$ and (ρ_1, ρ_2) a partition of unity subordinate to the open cover (U, V) of $\mathbb{R}^n$. By our assumption for any $\epsilon > 0$ there exist a continuous map $g_U : U \to \mathbb{R}^n$ such that $d(f, g_U) \leq inf(\epsilon, \delta)$ and $y \notin Im(g_U)$. The map $g = \rho_1 g_U + \rho_2 f$ is a continuous map such that $d(g, f) \leq \epsilon$ and $y \notin Im(g)$ contradicting the stability of y for f.

□

Definition 5.3. — *A metric space Z has topological dimension $\geq n$ if there exist a Lipschitz map $f : Z \to S^n$ which has a stable value.*

There is another equivalent definition of the topological dimension obtained by considering open covers. Let us recall that the order of an open cover $\mathcal{U} = \{U_i \quad | i \in I\}$ is the supremum of all numbers $\#I'$, where $I' \subset I$, for which $\cap_{i \in I'} U_i \neq \emptyset$.

Definition 5.4. — *A metric space Z has topological dimension $\leq n$ if and only if every open cover of Z has an open refinement of order at most $n + 1$.*

We will also need the following

Definition 5.5. — *Let Z be a topological space, and $\mathcal{U} = \{U_i\,; i \in I\}$ a cover of Z by open subsets U_i. The nerve $Ner(\mathcal{U})$ of $\mathcal{U}$ is the simplicial complex whose simplices corresponds to the subsets $I' \subset I$ such that $\bigcap_{i \in I'} U_i \neq \emptyset$*

Regular maps and maps of bounded multiplicity are playing an important role in the proof of the theorem 1.5. Let us define it now.

Definition 5.6. — *A map $g : (S, \rho) \to \mathbb{R}^n$ is regular if it is Lipschitz and if there exist $N \in \mathbb{N}$ such that for any $R > 0$ the inverse image $g^{-1}(B(R))$ of any ball of radius R in $\mathbb{R}^n$ can be covered by at most N balls of radius R in S.*

Definition 5.7. — *A map $g : (S, \rho) \to \mathbb{R}^n$ has bounded multiplicity if there exist $N \in \mathbb{N}$ such that for any $y \in \mathbb{R}^n$ we have $\#g^{-1}(y) \leq N$.*

Remark: If $g : (S, \rho) \to \mathbb{R}^n$ is regular, it has bounded multiplicity.

The following statement describe the regular maps $f : (Z, d) \to \mathbb{R}^n$ between a compact metric space (Z, d) all of whose open subsets have topological dimension $\geq n$ and $\mathbb{R}^n$. Such maps are almost covering maps. As a corollary we get that such compact metric spaces which admit regular maps into $\mathbb{R}^n$ have weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$.

Proposition 5.8. — *Let (Z, d) be a compact metric space such that every non empty subset has topological dimension $\geq n$, $f : Z \to \mathbb{R}^n$ a regular map. Then there exist an open subset $V \subset Imf$ with $\bar{V} = Imf$ such that $U := f^{-1}(V)$ is dense in Z and $f|_U : U \to V$ is a covering locally bilipschitz map.*

Corollary 5.9. — *Let (Z, d) be a compact metric space such that every non empty subset has topological dimension $\geq n$. Let us suppose that there exist a regular map $f : Z \to \mathbb{R}^n$. Then (Z, d) has a weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$.*

Before sketching the proof of this proposition, we need some preliminaries about stability.

Let (Z, d) be a compact metric space of topological dimension n and $f : (Z, d) \to \mathbb{R}^n$ a continuous map. Without further assumption the map f may have no stable value (for example a constant map). But when f is regular then f possesses stable values.

Lemma 5.10. — *Let (Z, d) be a compact metric space of topological dimension at least n and $f : (Z, d) \to \mathbb{R}^n$ a regular map. Then f has stable values.*

Proof. — By regularity the preimage of all points of $\mathbb{R}^n$ is finite so that for any $y \in \mathbb{R}^n$ and any $\epsilon > 0$ there exist $r_y > 0$ such that $f^{-1}(B(y, r_y))$ is a finite disjoint union of open subsets of diameter less than ϵ (located in disjoint neighbourhoods of the points of $f^{-1}(y)$). Let us consider an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of $\mathbb{R}^n$ such that each $U_i \in \mathcal{U}$ whose intersection with the image $Im(f)$ of f in nonempty is a subset of some

$B(y, r_y)$ and such that the order of $\mathcal{U}$ is equal to $n+1$, (for example $\mathcal{U}$ can be chosen as the open star cover associated to a fine enough triangulation of $\mathbb{R}^n$). By construction for any $U_i \in \mathcal{U}$ the open subset $f^{-1}(U_i)$ is a finite disjoint union of open subsets of diameter less than ϵ in Z. These open subsets of small diameter yield an open covering $\mathcal{V} = \{V_j \quad | \quad j \in J\}$ of Z. By construction, for $j \in J$, the subset V_j is appearing in the finite decomposition of $f^{-1}(U_{\alpha(j)})$ for some $U_{\alpha(j)}$ in $\mathcal{U}$. This defines a map $\alpha : I \to J$. If $\alpha(j) = \alpha(j')$ for distinct $j, j' \in J$, then $V_j \cap V_{j'} = \emptyset$ since V_j and $V_{j'}$ are distinct parts of the decomposition of $f^{-1}(U_{\alpha(j)}) = f^{-1}(U_{\alpha(j')})$ therefore α induces a simplicial map between the nerves of U and V, (see definition 5.5), $\Phi : Ner(\mathcal{V}) \to Ner(\mathcal{U})$ which sends k-simplex to k-simplex. In particular the order of $\mathcal{V}$ is less than or equal to $n+1$. We consider now a partition of unity $\rho = \{\rho_i \quad | \quad i \in I\}$ of $\mathbb{R}^n$ subordinate to $\mathcal{U}$ and the partition of unity $\eta = \{\eta_j \quad | \quad j \in J\}$ of Z defined by $\eta_j := \chi_j.(\rho_{\alpha(j)} \circ f)$ subordinate to $\mathcal{V}$ where χ_j is the characteristic function of V_j. Considering ρ_i and η_j as barycentric coordinates of $\mathbb{R}^n$ and Z in $\mathcal{U}$ and $\mathcal{V}$ respectively, we get continuous maps $\rho : \mathbb{R}^n \to Ner(\mathcal{U})$ and $\eta : Z \to Ner(\mathcal{V})$ such that $\Phi \circ \eta = \rho \circ f$. Therefore $f = \rho^{-1} \circ \Phi \circ \eta$ because ρ is clearly an homeomorphism. As Φ is a simplicial map, it is easy to see that if there exist a stable value ξ of η in the interior of some n-simplex of $Ner(\mathcal{V})$, then $\rho^{-1} \circ \Phi(\xi)$ is a stable value of f. We conclude the proof of the lemma by proving that η posseses such a stable value ξ. Namely let us assume by contradiction that it is not true. Then there is a collection S of points in the interior of each n-simplex of $Ner(\mathcal{V})$ which are not stable values of η. It is therefore possible to perturb η on a neighbourhood of S to get a map η' such that $Im(\eta') \cap S = \emptyset$. We may then compose η' by the "radial projection" from S onto the $(n-1)$-skeleton of $Ner(\mathcal{V})$ to get a map η'' whose barycentric coordinates still are subordiante to $\mathcal{V}$. We then pull back the open star cover of $Ner(\mathcal{V})$ and get a refinement of $\mathcal{V}$ of order les than or equal to n which contradicts the assumption on the topological dimension of Z. $\square$

The proof of the proposition 5.7 is based on the stable points of f that we describe now.

Definition 5.11. — *Let $f : Z \to \mathbb{R}^n$ be a continuous map between a topological space Z and $\mathbb{R}^n$. A point $x \in Z$ is called a stable point of f if $f(x)$ is a stable value of the restriction f_U of f to U for any open neighborhood U of x.*

A map f possesses stable points when it has bounded multiplicity:

Lemma 5.12. — *If (Z, d) is a compact metric space and $f : Z \to \mathbb{R}^n$ a regular map, then the preimage of a stable value contains a stable point.*

Proof. — We just sketch the proof. Since f is a regular map it is of bounded multiplicity. Let us denote $f^{-1}(y) = \{x_1, ..., x_k\}$. Assume the lemma is not true, then there exist open disjoint balls $B(x_i, r_i)$ such that for each $i = 1, .., k$, y is not a stable value of $f_{|B(x_i,r_i)}$. For $\delta > 0$ small enough, we have $f^{-1}(B(y, \delta)) = \cup_{i=1,..,k} U_i$ where the U_i's are open subsets of $B(x_i, r_i)$ and then y is not a stable value of $f_{|f^{-1}(B(y,\delta))}$, which contradict the lemma 5.2 (ii).

□

The stable multiplicity function is the function $\mu : \mathbb{R}^n \to \mathbb{N}$ defined by $\mu(y) =$ the number of stable points in $f^{-1}(y)$.

Lemma 5.13. — *Let (Z, d) be a compact metric space such that the topological dimension of all nonempty open subsets of Z is $\geq n$ and $f : (Z, d) \to \mathbb{R}^n$ a regular map. Let $V \subset \mathbb{R}^n$ be the subset of points $y \in \mathbb{R}^n$ where the stable multiplicity μ is locally maximal and $U := f^{-1}(V)$. Then,*

(i) V is an open dense subset of $Im(f)$ on which the multiplicity function μ is locally constant.

(ii) The restriction $f_{|U}$ of f to U is a local homeomorphism.

Proof. — The proof is a consequence of the two following claims.

Claim 1: Let us consider $y \in Im(f)$ and $\epsilon > 0$, then there exist $\delta > 0$ such that for all $y' \in B(y, \delta)$ and all stable points $x \in f^{-1}(y)$, there is a stable point in $f^{-1}(y') \cap B(x, \epsilon)$.

Claim 2: Let $y \in Im(f)$ such that μ is locally maximal at y. Then every $x \in f^{-1}(y)$ is a stable point.

Since f is of bounded multiplicity and μ takes integer values then V is clearly a dense subset of $Im(f)$. By the claim 1, V is an open subset of $\mathbb{R}^n$ and μ is locally constant on V. This proves (i). By the claim 2, the map $y \to \# f^{-1}(y)$ is locally constant on V and by claim 1 f is therefore locally injective on U which proves (ii). Let us prove the two claims.

Claim 1: Let x be a stable point in $f^{-1}(y)$. For all $\epsilon > 0$, y is a stable value of $f_{|\bar{B}(x,\epsilon)}$, and since the set of stable values is an open set there exist a $\delta(x) > 0$ such that all $y' \in B(y, \delta(x))$ are regular values of $f_{|\bar{B}(x,\epsilon)}$. The lemma 5.11 provides a stable point $x' \in f^{-1}(y') \cap \bar{B}(x, \epsilon)$ and by regularity there is finitely many (stable) point in $f^{-1}(y)$ so that we can choose $\delta = inf(\delta(x))$.

Claim 2: Let W be a relatively compact open neighborhood of y such that $\mu(y)$ is maximal on $\bar{W}$. Assume the claim is not true. There exist $x \in f^{-1}(y)$ such that $x \notin \{x_1, ..., x_k\}$ where $\{x_1, ..., x_k\}$ denotes the set of stable points of $f^{-1}(y)$ and $\epsilon > 0$ such that $B(x, \epsilon), B(x_1, \epsilon), .., B(x_k, \epsilon)$ are disjoint. By lemma 5.9 there exist a stable value $y' \in K := \bar{W} \cap \bar{B}(y, \delta)$ of $f_{|\bar{B}(x,\epsilon) \cap f^{-1}(K)}$ where $\delta = \delta(y, \epsilon)$ comes from the claim 1. Therefore by claim 1 and lemma 5.11 respectively there are stable points of $f^{-1}(y')$ in each of the balls $B(x_1, \epsilon), .., B(x_k, \epsilon)$ and $B(x, \epsilon)$, contradicting our assumption.

□

We now can end the proof of the proposition 5.7: let $f : Z \to \mathbb{R}^n$ a regular map where (Z, d) be a compact metric space such that the topological dimension of all nonempty open subsets of Z is $\geq n$ and $V \subset \mathbb{R}^n$ be the subset of points $y \in \mathbb{R}^n$ where the stable multiplicity μ is locally maximal, then according to the lemma 5.12 it remains to prove that $U := f^{-1}(V)$ is dense in Z and that f is locally bilipschitz. We first prove the density of U: let us consider a nonempty open set O in Z. By the lemma 5.9 $f(O)$ has nonempty interior and since V is dense in $Im(f)$, $f(O) \cap V \neq \emptyset$ therefore O meets $U := f^{-1}(V)$ which proves the density of U in Z.

Let us now prove that $g =: f_{|U}$ is locally bilipschitz. Since g is Lipschitz we have to prove that the inverse of g is Lipschitz. We can argue locally and restrict U so that $U \subset f^{-1}(B)$ where B is an open ball in $\mathbb{R}^n$. We have to prove the existence of a constant C such that for all $x \neq$

$y \in U$, $d(x,y) \leq C|f(x) - f(y)|$. Let us consider the euclidean ball $B'(f(x), R)$ centered at $f(x)$ where $R := 2|f(x) - f(y)|$ and denote Δ the euclidean segment joining $f(x)$ and $f(y)$. Then, $S := g^{-1}(\Delta)$ is a compact connected subset of U containing x, y and such that $S \subset f^{-1}(B'(f(x), R))$. By regularity of f, we see that S is covered by N open balls of radius R of Z. We then get easily that $d(x,y) \leq diam(S) \leq 2NR \leq 4N|f(x) - f(y)|$. This proves that $f_{|U}$ is bilipschitz and ends the proof of the proposition 5.7.

6. Topological dimension and Hausdorff dimension

For a metric space (X,d), the n-Ahlfors regularity says that the n-dimensional Hausdorff measure of metric balls behaves coarsely like the Lebesgue measure of n-dimensional euclidean balls. This property will be crucial in finding a weak tangent of (X,d) which is homeomorphic to $\mathbb{R}^n$.

Definition 6.1. — *A metric space (X,d) is Ahlfors regular if there exists a constant $C > 0$ such that*

$$C^{-1} r^n \leq H^n(B(x,r)) \leq C r^n,$$

for any ball $B(x,r)$ of (X,d), where H^n stands for the Hausdorff measure of (X,d).

Let (X,d) be a compact metric space whose topological and Hausdorff dimension coincide. Let us write n this dimension. We also will assume that the space (X,d) is n-Ahlfors regular.

In general a topological space of topological dimension n may not contain any subset homeomorphic to an open subset of $\mathbb{R}^n$. We will see however that if (X,d) is a compact n-Ahlfors regular metric space of topological dimension n then one can find a weak tangent of (X,d) which is bilipschitz homeomorphic to $\mathbb{R}^n$.

Proposition 6.2. — *Let (X,d) be a compact n-Ahlfors regular metric space whose topological dimension is equal to n, then (X,d) has a weak tangent bilipschitz homeomorphic to $\mathbb{R}^n$.*

Proof. — The proof boils down in proving the existence of a weak tangent (S,ρ) of (X,d) and a regular map $g:(S,\rho)\to\mathbb{R}^n$. The corollary 5.8 then implies the existence of a weak tangent (T,σ) of (S,ρ) which is bilipschitz homeomorphic to $\mathbb{R}^n$. Since weak tangent of a weak tangent of a space (X,d) is a weak tangent of (X,d), this ends the proof of the proposition. Let us prove now the existence of a weak tangent (S,ρ) of (X,d) and a regular map $g:(S,\rho)\to\mathbb{R}^n$.

Let $f:(X,d)\to S^n$ be a Lipschitz map with a regular value. Such a map actually exist because of the assumption on the topological dimension of X. We will find the regular map g among the "weak tangents" of the map f. Let us define what a weak tangent of the map f is. We consider weak tangent $(S,\rho,o)=\lim_{k\to\infty}(X,\lambda_k d,p)$ and $(\mathbb{R}^n,eucl,0)=\lim_{k\to\infty}(S^n,\lambda_k can,p_0)$ of (X,d) and (S^n,can,p_0) respectively, where $(\mathbb{R}^n,eucl,0)$ and (S^n,can,p_0) are the standard euclidean space and the standard sphere. We recall that all weak tangent of the standard sphere are isometric to the euclidean space. By definition there exist sequences of maps $\Psi_k:(S,\rho,o)\to(X,\lambda_k d,p)$, $\Phi_k:(X,\lambda_k d,p)\to(S,\rho,o)$, $\bar\Psi_k:(\mathbb{R}^n,eucl,0)\to(S^n,\lambda_k can,p_0)$, and $\bar\Phi_k:(S^n,\lambda_k can,p_0)\to(\mathbb{R}^n,eucl,0)$ which are "almost isometries" on balls of radius R for all fixed R. A map $g:(S,\rho,o)\to(\mathbb{R}^n,eucl,0)$ such that $g=\lim_{k\to\infty}\bar\Phi_k\circ f\circ\Psi_k$ is called a weak tangent of $f:(X,d)\to S^n$. Note that weak tangents of a Lipschitz map always exist for any choice of fixed marked points p and p_0. Here we want to find such a weak tangent which is regular. For that purpose we will have to choose p in an appropriate way. More precisely we want to show the existence of an $N\in\mathbb{N}$ such that for each ball $B_{\mathbb{R}^n}(R)$ of radius R in $\mathbb{R}^n$, $g^{-1}(B_{\mathbb{R}^n}(R))$ can be covered by N balls of radius CR in (S,ρ) where C is a constant independant of R. By approximation this is equivalent to saying that for each ball $B_{\lambda_k S^n}(R)$ of radius R of $(S^n,\lambda_k can)$, $f^{-1}(B_{\lambda_k S^n}(R))$ can be covered by N balls of $\lambda_k X$ of radius CR, where $\lambda_k X$ the space X endowed with the homothetic metric $\lambda_k d$. Equivalently this means that for each ball $B_{S^n}(\frac{R}{\lambda_k})$ in (S^n,can), $f^{-1}(B_{S^n}(\frac{R}{\lambda_k}))$ can be covered by N balls of radius $C\frac{R}{\lambda_k}$ in (X,d). The end of the proof of the proposition would be very easy under the following assumption:

($\star$) for any $r>0$, $H^n_X(f^{-1}(B_{S^n}(r)))\le C'r^n$ for some constant C'.

Namely, let us choose $\{x_i\}_{i\in I}$ a maximal set of points in $f^{-1}(B_{S^n}(r))$ such that $B(x_i,r)\cap B(x_j,r)=\emptyset$ for all $i\neq j$ in I. Then by Ahlfors regularity we have $H^n_X(\cup B_X(x_i,r)\geq C|I|r^n$ for some constant C, where $|I|$ is the cardinal of I. On the other hand if L denotes the Lipschitz constant of f, then $B_X(x_i,\frac{r}{L+1})\subset f^{-1}(B_{S^n}(2r))$, therefore $H^n_X(\cup B_X(x_i,\frac{r}{L+1})\leq H^n_X(f^{-1}(B_{S^n}(2r)))\leq 2^nC'r^n$ thus $|I|\leq\frac{2^nC'}{C}$.

Unfortunalely the assumption $(\star)$ does not a priori hold, thus we have to look at bad points. For $\delta>0$ and $\lambda>0$ let us define $L_\delta(x)=Sup_{0\leq t\leq\delta}\frac{H^n_X(f^{-1}(B(x,t)))}{t^n}$ and $E_{\delta,\lambda}=\{x\in S^n \quad | \quad L_\delta(x)\geq\lambda\}$.

We claim that one can choose δ small enough and λ large enough such that $H^n_{S^n}(E_{\delta,\lambda})\leq\epsilon$ for an arbitrary small number ϵ.

Let us assume the claim and finish the proof of the proposition. Let us denote $U=X-f^{-1}(E_{\delta,\lambda})$ so that $f(U)=f(X)-E_{\delta,\lambda}$. Since f has a regular value, $f(X)$ contains an open subset of S^n therefore $H^n_{S^n}(f(X))>0$. By the claim we can choose δ and λ such that $f(U)=f(X)-E_{\delta,\lambda}$ has positive Hausdorff measure $H^n_{S^n}(f(U))>0$. Since f is Lipschitz , we then get that $H^n_X(U)>0$.

We choose a point of density p of U and a weak tangent $g:(S,\rho,o)\to(\mathbb{R}^n,eucl,0)$ of $f:(X,d,p)\to(S^n,can,p_0)$ where $(S,\rho,o)=\lim_{k\to\infty}(X,\lambda_k d,p)$ and $(\mathbb{R}^n,eucl,0)=\lim_{k\to\infty}(S^n,\lambda_k can,p_0)$.

Let us prove the regularity of g. We have to prove the existence of $N\in\mathbb{N}$ such that for any $R>0$, $n\in\mathbb{N}$ and any ball $B_{S^n}(\frac{R}{\lambda_k})$ of radius $\frac{R}{\lambda_k}$ centered at an arbitrary point, the set $A_k=:f^{-1}(B_{S^n}(\frac{R}{\lambda_k}))\cap B(p,\frac{nR}{\lambda_k})$ can be covered by N balls of radius $C\frac{R}{\lambda_k}$ in (X,d).

Since p is a point of density of U we have $\lim_{\lambda_k\to\infty}\lambda_k^nH^n_X(B(p,\frac{R}{\lambda_k})-U)=0$ and therefore by Ahlfors regularity of (X,d) we easily get the existence of a point $x_k\in U$ such that $dist(x_k,f^{-1}(B_{S^n}(\frac{2R}{\lambda_k}))\cap B(p,\frac{nR}{\lambda_k}))\leq\frac{R}{\lambda_kL}$ where L is the Lipschitz constant of f. Setting $A'_k=f^{-1}(B_{S^n}(\frac{2R}{\lambda_k}))\cap B(p,\frac{nR}{\lambda_k})$ we therefore get

$$A'_k\subset f^{-1}\big(B_{S^n}\big(f(x_k),\frac{5R}{\lambda_k}\big)\big)\cap B(p,\frac{nR}{\lambda_k}).$$

Since $x_k\in U$, then $f(x_k)\notin E_{\delta,\lambda}$ hence

$$H^n_X(f^{-1}(B_{S^n}(f(x_k),\frac{5R}{\lambda_k})))\leq(\frac{5CR}{\lambda_k})^n$$

and therefore
$$H^n_X(A'_k) \leq (\frac{5CR}{\lambda_k})^n.$$
From this we can argue like in the case when the assumption $(\star)$ holds and conclude.

We now prove the claim.

Let us denote L the function defined on $Im(f) \subset S^n$ by
$$L(x) = \lim sup_{t\to 0} t^{-n} H^n_X(f^{-1}(B(x,t))).$$
The family $E_{\delta,\lambda}$ is a decreasing family of measurable subsets when δ tends to 0 and $\cap_{\delta>0} E_{\delta,\lambda} = \{x \in Im(f) \quad | \quad L(x) > \lambda\}$, therefore $\lim_{\delta\to 0} H^n_{S^n}(E_{\delta,\lambda}) = H^n_{S^n}(\{x \quad | \quad L(x) > \lambda\}$.

The claim will then follow from the estimate:
$$H^n_{S^n}(\{x \quad | \quad L(x) > \lambda\}) \leq \frac{C}{\lambda},$$
for some constant C.

Let us prove this estimate. Let us denote $E_\lambda = \{x \in Im(f) \quad | \quad L(x) > \lambda\}$. For each $x \in E_\lambda$, there exist t, $0 < t < \delta$, such that $H^n_X(f^{-1}(B(x,t))) \geq \lambda t^n$. This defines a set $\mathcal{B}$ of ball of radii less than or equal to δ which cover E_λ. By a Vitali's covering lemma one can find a sequence B_j of balls of $\mathcal{B}$ of radii r_j tending to 0 such that the B_j are pairwise disjoint, $E_\lambda \subset \cup B_j(5r_j)$, where $B_j(5r_j)$ is the ball with same center as B_j of radius $5r_j$. The construction of this sequence B_j goes as follows. Let R_1 be the supremum of the radii of the balls in $\mathcal{B}$. We first take a maximal set of disjoint balls of $\mathcal{B}$ of radii r such that $R_1/2 \leq r \leq R_1$. Then we continue the same way with the supremum R_2 of the radii of balls in $\mathcal{B}$ which are disjoint of the ones already chosen. It is then easy to check that E_λ is covered by $\cup B_j(5r_j)$ and that the radii of the B_j's tends to zero because (S^n, can) is Ahlfors regular.

By construction we have
$$\Sigma_j r_j^n \leq \lambda^{-1} H^n_{S^n}(\cup B_j) \leq \lambda^{-1} H^n(S^n).$$
Therefore, we get
$$H^n_{S^n}(E_\lambda) \leq \Sigma_j (diam(B_j(5r_j)))^n \leq (10)^n \lambda^{-1} H^n(S^n),$$
which proves the estimate.

This ends the proof of the proposition.

7. Proof of the theorem 1.5

Let G be a quasi-convex cocompact group acting on a $CAT(-1)$-space X. As was recalled in section 2, G is acting uniformly quasi-Möbius on the limit set $(\Lambda(G), d)$ of G, cf. the example following definition 4.3. We assume that the Hausdorff dimension $\delta(G)$ and the topological dimension $dim_{top}\Lambda(G)$ of the limit set of G coincide. Let us write $n = dim_{top}\Lambda(G) = \delta(G)$. As mentioned in the introduction, the limit set of G is n-Ahlfors regular, thus by proposition 6.2, $(\Lambda(G), d)$ has a weak tangent (S, ρ) bilipschitz homeomorphic to $\mathbb{R}^n$. In particular, the one point compactification $(\hat{S}, \hat{\rho}_o)$ of (S, ρ) is quasi-Möbius homeomorphic to the standard sphere S^n. On the other hand, by proposition 4.5, the one point compactification $(\hat{S}, \hat{\rho}_o)$ of (S, ρ) is quasi-Möbius homeomorphic to $(\Lambda(G), d)$, hence after conjugation we can assume that G is acting uniformly quasi-Möbius on S^n. Since G is quasi-convex cocompact, G acts cocompactly and properly discontinuously on the set of triples of distinct points $Tri(\Lambda(G))$ of $\Lambda(G)$ thus the conjugate action of G on S^n also acts cocompactly and properly discontinuously on the set of triples of distinct points $Tri(S^n)$ of S^n, hence the theorem 3.1 of Tukia asserts that G is isomorphic to a group of Möbius transformation of S^n. The corresponding action of G on $Tri(S^n)$ is proper discontinuous and cocompact, hence the action of G on S^n extends to a discrete cocompact action on the hyperbolic space $\mathbb{H}^{n+1}$. By the theorem 1.3 we then conclude the existence of a totally geodesic G invariant copy H of $\mathbb{H}^{n+1}$ embedded in X such that H/G is compact. This concludes the proof of theorem 1.5. $\square$

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