MINIMAL VOLUME

by

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Abstract. — The aim of this text is to explain some rigidity results related to the minimal volume of compact manifolds. For this, we describe the natural maps of Besson-Courtois-Gallot and compactness theorem à la Gromov.

***Résumé* (Volume minimal).** — Le but de ce texte est d'expliquer quelques résultats de rigidité associés au volume minimal des variétés compactes. Au passage, on décrit les applications naturelles de Besson-Courtois-Gallot et les théorèmes de compacité à la Gromov.

Contents

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1. Introduction

1.1. Questions. — A very natural question in Riemannian geometry is the following : given a C^∞ manifold M, is there a best or distinguished metric on M ? One way to answer this question is to consider some geometric functional on the space or a subspace of all Riemannian metrics of M and look for an extremum of this functional. Here we are interested in the minimal volume, which was introduced by M. Gromov.

Definition. — Let M a C^∞ manifold. Consider on M all complete Riemannian metrics with sectional curvature satisfying

$$|K_g| \leq 1.$$

The *minimal volume* of M is defined by,

$$\mathrm{Minvol}(M) = \inf_{|K_g|\leq 1} \mathrm{vol_g}(M).$$

Remarks. — a) When such a metric does not exist (on certain non-compact manifolds) we set $\mathrm{Minvol}(M) = +\infty$.

b) We recall that the sectional curvature assigns to each 2-plane $P \subset T_xM$ of the tangent space a real number $K(P)$ whose geometric meaning is the following. Let $C(r)$ be the circle of radius r tangent to P, i.e. $C(r)$ is the set of all $exp_x(rv)$ where v is a unit vector in P. The length of this circle is then given by

$$\ell(C(r)) = 2\pi r\left(1 - \frac{K(P)}{6}r^2 + o(r^2)\right),$$

so $K(P)$ measures the defect of the perimeter of the circle to be the euclidean perimeter.

c) When we scale a metric g into λg, we have $\mathrm{vol_{\lambda g}}(M) = \lambda^{n/2}\mathrm{vol_g}(M)$ and $K_{\lambda g} = \frac{1}{\lambda}K_g$. We infer that if M is covered by a Torus (i.e. if M is compact and support a flat metric) then $\mathrm{Minvol}(M) = 0$. On the other hand, by an obvious density argument, we also see that $\mathrm{Minvol}(M) = \inf_{||K_g||_\infty = 1} \mathrm{vol_g}(M)$. Eventually, it is equivalent to consider

$$\inf_{\mathrm{vol_g}(M)=1} ||K_g||_\infty{}^{n/2} = \inf \mathrm{vol_g}(M)||K_g||_\infty{}^{n/2},$$

(which is scale invariant) where the infimum is taken among all complete metrics g with bounded curvature and finite volume.

Questions about this functional (see [**5**], question 266)
a) Is Minvol(M) zero or positive? Try to classify manifolds for which Minvol(M) is zero and those for which it is not achieved.

b) If Minvol(M) $\neq 0$, is it attained by a metric? When those best metrics exist, try to classify them.

c)] Compute Minvol(M) for various manifolds. Can we say something about the set of values when M runs through compact manifolds? Is zero an isolated point of this set?

1.2. The 2-dimensional case. — Compact surfaces

In this case we can answer all the questions a), b) and c) thanks to the Gauss-Bonnet formula. Suppose M is a nice surface, i.e. compact, oriented, without boundary. We can see K_g as a function on M. Let us suppose that $-1 \leq K_g \leq 1$, the Gauss-Bonnet formula then gives

$$\begin{aligned} |2-2g(M)| = |\chi(M)| &= \left|\frac{1}{2\pi}\int_M K_g(x)\ dv_g(x)\right| \\ &\leq \frac{1}{2\pi}\int_M |K_g(x)|\ dv_g(x) \\ &\leq \frac{\mathrm{vol_g}(M)}{2\pi}, \end{aligned}$$

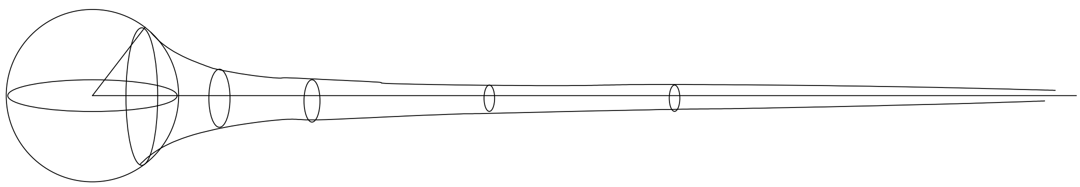
where $\chi(M)$ is the Euler characteristic and $g(M)$ is the genus of the surface. The Torus and the Klein bottle have zero minimal volume since they support some flat metric but there is obviously no metric realizing the minimal volume. In the other cases, the minimal volume is positive and achieved by metrics where K_g is constant equal to ± 1 (which exists by the uniformization Theorem). For the sphere and the projective space only the canonical metrics of curvature $+1$ realize the minimal volume. In the remainding cases, connected sum of $k \geq 3$ projectives spaces or orientable surfaces of genus ≥ 2, the optimal metrics have curvature -1. In the orientable case, the set of metrics, up to isometry, realizing the minimal volume is infinite since it coincides with the Teichmuller space, which is known to be a complex manifold of dimension $3g-3$. All this show that the image of the minimal volume functional on the compact surfaces is $2\pi\mathbb{N}$, hence is a discrete set of $\mathbb{R}$.

Non-compact surfaces

The Gauss-Bonnet formula holds for complete surface with bounded curvature and finite volume. Thus hyperbolic metric of finite volume are again mimimal for all the surfaces with $\chi(M) < 0$ and then we have $\mathrm{Minvol}(M) = -2\pi\chi(M)$. Moreover, the surface of infinite genus have infinite minimal volume. For the plane $\mathbb{R}^2$, we have the following result ([**3**]) :

$$\mathrm{Minvol}(\mathbb{R}^2) = 2\pi(1+\sqrt{2})$$

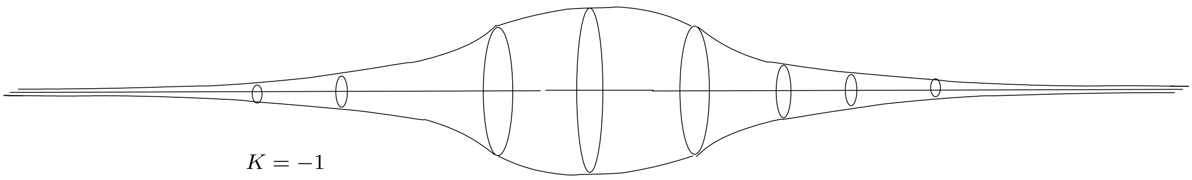
with an extremal C^1 metric obtained by pasting a spherical disk of curvature 1, and an hyperbolic cusp, both with boundary of length $\pi\sqrt{2}$. The metric is not C^2 near the gluing circle as the curvature varies from $+1$ to -1.



an extremal metric on $\mathbb{R}^2$

The cylinder $S^1 \times \mathbb{R}$ and the Möbius band have zero minimal volume.

Proof. — Indeed, on the cylinder consider a warped product metric $g = f^2(t)d\theta^2 + dt^2$, with an hyperbolic cusp of finite volume at each end. That is $f(t) = e^{-t}$ for large $t > 0$ and $f(t) = e^t$ for large $t < 0$.



The formula for the sectional curvature is

$$K_g = -\frac{f''}{f}.$$

Thus g can be choosen complete with finite volume and $-1 \leq K_g \leq 1$. Consider now any $\varepsilon > 0$ and the family of metrics

$$g_\varepsilon = \varepsilon^2 f^2(t) d\theta^2 + dt^2.$$

Clearly, the sectional curvature of g_ε is unchanged and $vol(g_\varepsilon)$ is arbitrary small. As we can choose f even we infer that the minimal volume of the Möbius band is 0. □

We will see below large generalizations of this trick.

From now on and otherwise specified, M will be oriented with dimension ≥ 3

The rest of the paper is organized as follows. In section 2 we give examples and some characterizations of manifolds whose minimal volume is zero. In section 3, we present some rigidity results involving volume, minimal volume and hyperbolic manifolds. In section 4, we explain the main ideas of the proofs of the rigidity results. The section 5 is devoted to the construction of the natural maps of Besson-Courtois-Gallot. In the section 6, we conclude the proof of the main theorem.

2. Zero or non-zero minimal volume

2.1. Manifolds with Minvol = 0. — In this section we give some examples of manifolds with zero minimal volume.

Examples. — a) M is compact and admits a flat Riemannian metric.

b) M admits a free action of the cercle S^1.
Trivial examples are the Torus or the cylinder. A non trivial example is given by the Hopf fibration $S^3 \to S^2$. Suppose that M has a Riemannian metric g of bounded curvature and finite volume. After averaging the S^1 action, we can assume it to be isometric. At each point x of M, the tangent space T_xM decomposes orthogonally into a vertical part (tangent to the S^1 orbit) and a horizontal part. Thus, we can write $g = g_v + g_h$. As above, consider for any $\varepsilon > 0$ the family of metrics

$$g_\varepsilon = \varepsilon^2 g_v + g_h.$$

Using O'Neill formulas on the Riemannian submersion $(M, g) \to (M/S^1, g_h)$ (see [**4**], chapter 9 or the Technical Chapter in [**5**]) we can show that the sectional curvatures remain bounded (it is crucial in the computation that the fibers are one dimensional). On the other hand, the volume can be made arbitrary small.

c) A generalization of the S^1 action is performed by Cheeger and Gromov with the definition of *T-structure* and *F-structure*.
An F-structure on a space M is a generalization of a torus action. Different tori (possibly of different dimensions) act locally on finite covering spaces of subsets of M. These actions satisfy a compatibility condition, which insures that M is "partitioned" in different orbits. The F-structure is said to have positive dimension if all the orbits have positive dimension. The definitions are quite technical. Here I give the formulations given by Fukaya in his survey on Hausdorff convergence ([**15**], definition 19.1, 19.2). A *T-structure* on M is a triple $(U_i, T^{k_i}, \varphi_i)$ such that

1. $\{U_i\}$ is an open covering of M,
2. T^{k_i} is a k_i-dimensional torus,
3. $\varphi_i : T^{k_i} \to Diff(U_i)$ is an effective and smooth action,
4. When $U_i \cap U_j \neq \emptyset$, $U_i \cap U_j$ is (T^{k_i}, φ_i) and (T^{k_j}, φ_j) invariant and the two actions commute.

Now for the *F-structure*, we have to consider finite coverings $\tilde{U}_i$ of U_i and natural actions of toruses T^{k_i} on the cover $\tilde{U}_i$ instead of U_i. By natural we means that the orbits are well defined in U_i even if the action does not descend (think for instance of the action of $\mathbb{T}^2$ on itself which induces two actions of $\mathbb{S}^1$ on $\mathbb{T}^2$, one of which descends to the Klein bottle whereas only the orbits of the second do).

The existence of a F-structure of positive dimension (that is, whose orbits have positive dimension) on M is related to the collapsing of M. One says that M is ε-collapsed if it supports a Riemannian metric g_ε with $-1 \leq K_{g_\varepsilon} \leq 1$ and injectivity radius $\leq \varepsilon$ at each point. Recall that the injectivity radius at x is the supremum of the radius $r > 0$ such that $exp_x : B(0, r) \subset T_xM \longrightarrow B(x, r) \subset M$ is a diffeomorphism. For metrics with sectional curvature between -1 and $+1$, a bound below for the volume of the unit ball $B(x, 1)$ is equivalent to a bound below for the injectivity radius at x. Thus if the minimal volume is zero, the manifold is ε-collapsed for any $\varepsilon > 0$. On the contrary, a manifold can be ε-collapsed and not have a small volume. For example, a torus $S^1 \times S^1$ where one of the circle has length ε and the other $1/\varepsilon$ is ε-collapsed but has volume $(2\pi)^2$. A fundamental result of Cheeger and Gromov ([**10**] theorem 4.1, [**11**] theorem 0.1) is

$$\begin{pmatrix} \text{M has a F-structure} \\ \text{of positive dimension} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{M is } \varepsilon\text{-collapsed} \\ \text{for any } \varepsilon > 0 \end{pmatrix}.$$

To prove $\Rightarrow$, start with metrics g_ε defined on the U_i, shrinked in certains directions tangent to the orbit. The problem is how to patch them on $U_i \cap U_j$. If the shrinking directions are different, one has to expand the metric in directions normal to both orbits to keep the curvature bounded. The volume may going to infinity. They prove a strenghtened version of the converse $\Leftarrow$, that is the existence of a universal $\varepsilon_n > 0$ such that ε_n-collapse implies the existence a F-structure of positive dimension. For the vanishing of the minimal volume, they have the following ([**10**] theorem 3.1)

$$\begin{pmatrix} \text{M has a polarized F-structure} \\ \text{of positive dimension} \end{pmatrix} \Longrightarrow \text{Minvol}(M) = 0$$

Roughly speaking, a F-structure is *polarized* if there is a collection of connected (non trivial) subgroups $H_i \subset T^{k_i}$ whose action is locally free and such that the H_i-orbit of $p \in U_i \cap U_j$ either contains or is contained in the H_j-orbit of p. Cheeger and Gromov prove also that in dimension

3, if M has a F-structure of positive dimension, it has a polarized one. Thus, if the minimal volume of a three-manifold is below some ε_n, it must be zero. X. Rong ([**25**]) has proved this true for $n = 4$ but it remains open in higher dimensions. The more general result, by Cheeger and Rong ([**12**]), is the following : there exists $\delta(n, d) > 0$ such that any n-dimensionnal Riemannian manifold with sectionnal curvatures between -1 and 1, diameter bounded by d and volume below $\delta(n, d)$ has a polarized F-structure.

d) Any product $M \times N$ where M is one of the above examples and N is arbitrary.

2.2. A criterion for $\mathrm{Minvol}(M) > 0$. — Via the generalization of the Gauss-Bonnet formula, the Euler characteristic provides, in even dimension, an obstruction to the vanishing of the minimal volume :

$$\mathrm{Minvol}(M) \geq c(n)\chi(M).$$

M. Gromov defines in [**16**] another invariant, the *simplicial volume*, as follows.

Definition. — The fondamental class $[M] \in H_n(M, \mathbb{R})$ can be reprensented by

$$c = \sum_i a_i \sigma_i$$

where a_i are reals and σ_i are singular simplices (if M is open the chain c is assumed to be locally finite). The *simplicial volume* is then defined by

$$||M|| = \inf \sum_i |a_i| \quad \in [0, +\infty]$$

where the infimum is taken among all chains $c = \sum_i a_i \sigma_i$ representing $[M]$.

We state some useful properties :

Proposition 2.1 ([**16**]). — 1) *If $f : M \to M'$ is a proper map of degree d, then*

$$||M|| \geq |d| \; ||M'||.$$

In particular, if M is compact and has a self mapping of degree $d \geq 2$ then $||M|| = 0$.

2) *If M is compact and N is arbitrary then*

$$a(n)||M||||N|| \leq ||M \times N|| \leq b(n)||M||||N||.$$

where $a(n) > 0$ and $b(n) > 0$ are positive constants depending only on $n = dim(M \times N)$.

3) *The connected sum is additive*

$$||M \sharp N|| = ||M|| + ||N||.$$

We now state some fundamental results of M.Gromov ([**16**]). Recall that the Ricci curvature is a symmetric bilinear form on TM, which can be defined as follows. Given $v \in T_xM$,

$$\mathrm{Ric_g}(v, v) = \sum_{i=1}^{n-1} K(P_{ve_i}),$$

where (e_i) is an orthonormal basis of $v^{\perp} \subset T_xM$ and $P_{ve_i} = vect(v, e_i)$. Then,

Theorem 2.2. — *If (M, g) is complete and satisfies $\mathrm{Ric_g} \geq -(n-1)g$ then*

$$\mathrm{vol_g}(M) \geq \frac{1}{(n-1)^n n!}||M||.$$

Corollary 2.3. — *Under the same assumption*

$$\mathrm{Minvol}(M) \geq \frac{1}{(n-1)^n n!}||M||.$$

Indeed, $K_g \geq -1$ implies $\mathrm{Ric_g} \geq -(n-1)g$. As a consequence, if M has a non zero degree map onto a M' with $||M'|| > 0$, then $\mathrm{Minvol}(M) > 0$. If $||M'|| > 0$ and M is an arbitrary manifold, $\mathrm{Minvol}(M \sharp M') > 0$. Related to the isolation problem, we have the following

Theorem 2.4 **(Gromov's isolation Theorem)**

There exists $\varepsilon_n > 0$ such that any complete Riemannian manifold (M^n, g) with $\mathrm{Ric_g} \geq -(n-1)g$ and $\mathrm{vol_g}(B(p, 1)) \leq \varepsilon_n$ for each $p \in M$ satisfies

$$||M|| = 0.$$

So, if the minimal volume is sufficiently small, the simplicial volume is zero. In dimension 3 (Cheeger-Gromov) and 4 (Rong [**25**]), small minimal volume implies zero minimal volume. The question is open in higher dimensions.

We can now describe a large family of manifolds for which $||M|| > 0$.

Theorem 2.5 **(Thurston's inequality)**. — *If* $K_g \leq -1$ *then*

$$||M|| \geq C(n)\mathrm{vol_g}(M).$$

We infer that a manifold which supports a metric whose sectional curvature are bounded from above by a negative constant has a non zero minimal volume. We can use the three properties of the simplicial volume recalled at the beginning of this section to produce a lot of manifolds with non-zero minimal volume (for instance a product of hyperbolic manifolds whose curvature is non positive). It is very hard to find some examples which are not of this type. Recently, J-F Lafont and B. Schmidt [**19**] have shown that all closed locally symmetric spaces of non-compact type have non zero simplicial volume.

If the metric is hyperbolic, the Thurston's inequality is strenghtened in

Theorem 2.6 **(Gromov)**. — *If* (M, g_0) *is hyperbolic, then*

$$||M|| = \frac{vol_{g_0}(M)}{V_n}$$

where V_n *is the volume of any ideal regular simplex of the* n*-hyperbolic space* $\mathbb{H}^n$.

3. Minimal volume and rigidity

We now turn to the conditions insuring the existence of a metric realizing the minimal volume. The strongest result of this kind is the following

Theorem 3.1 **(Besson, Courtois and Gallot, 1995 [8])**

Let (X, g_0) *an hyperbolic compact manifold and* g *a Riemannian metric such that*

$$\mathrm{Ric_g} \geq -(n-1)g.$$

Then

$$\mathrm{vol_g}(X) \geq vol_{g_0}(X)$$

with equality if and only if g is isometric to g_0.

In particular, the minimal volume of such manifolds is achieved by the hyperbolic metric only. In fact, this result is a corollary of a more general theorem involving the volume and the volume entropy of the metric.

Definition. — The *volume entropy* $h(g)$ of a compact Riemannian manifold (Y, g) is defined as follows. Let $\tilde{Y}$ the universal covering of Y, $y \in \tilde{Y}$ and $\tilde{g}$ the lift of g. We define

$$\begin{aligned} h(g) &= \lim_{r \to \infty} \frac{1}{r} \ln(vol_{\tilde{g}}(B_{\tilde{g}}(y, r)) \\ &= \inf \left\{ c > 0, \int_{\tilde{Y}} e^{-c.\rho(y,z)} \, \mathrm{dvol}_{\tilde{g}}(z) < \infty \right\} \end{aligned}$$

(see [**20**]).

For example, the volume of an hyperbolic ball of radius r is $\mathrm{vol}_{\mathbb{H}^n}(r) \sim c(n)e^{(n-1)r}$ as $r \to \infty$ thus $h(g_0) = n - 1$. For other locally symmetric spaces with negative curvature, that are the quotient of the complex hyperbolic space, the quaternionic hyperbolic space or the Cayley hyperbolic plane with curvatures normalized as to be pinched by -4 and -1, one has $h(g_0) = (n + d - 2)$, where n is the real dimension of the space and d is the dimension of the algebra (2 in the complex case, 4 in the quaternionic case and 8 in the Cayley case). We then have

***Theorem 3.2* (BCG)**. — *let (X, g_0) be a locally symmetric compact manifold of negative curvature and (Y, g) another compact Riemannian manifold. Let us assume that there is a map $f : Y \to X$ of degree $d \neq 0$. Then*

$$h(g)^n \mathrm{vol_g}(Y) \geq |d| h(g_0)^n vol_{g_0}(X)$$

where $h(g)$ and $h(g_0)$ are the volume entropies of the metrics g and g_0 respectively. Moreover, there is equality if and only if f is homotopic to a homothetic covering.

To obtain Theorem 3.1 from Theorem 3.2, we apply the Bishop inequality which says that if $\mathrm{Ric_g} \geq -(n-1)g$, then

$$\mathrm{vol}_{\mathbb{H}^n}(r) \geq \mathrm{vol}_{\tilde{g}}(B_{\tilde{g}}(y,r))$$

thus $h(g_0) \geq h(g)$, and take $Y = X$ and $f = \mathrm{id}_{|\mathrm{X}}$. Now the equality case in Theorem 3.1 implies also $h(g) = h(g_0)$.

Remarks. — a) Theorem 3.2 gives a proof of the Mostow rigidity Theorem. Indeed, suppose that Y and X are compact locally symmetric spaces of negative curvature and $f : Y \to X$ is an homotopy equivalence. Then inequality holds in both directions and from the equality case, f is homotopic to an isometry.

a') For other locally symmetric spaces of rank 1, the existence of a metric realizing the minimal volume remains open in particular for the complex hyperbolic space.

b) An interesting consequence of Theorem 3.2 is, when g_0 is hyperbolic, the inequality

$$\mathrm{Minvol}(Y) \geq |d| \mathrm{vol_{g_0}}(X)$$

In my thesis, I proved that

Theorem 3.3 **(Bes)**. — *If (X, g_0) is compact hyperbolic, $f : Y \to X$ a map of degree $d \neq 0$ and*

$$\mathrm{Minvol}(Y) = |d| \mathrm{vol_{g_0}}(X)$$

then f is homotopic to a differentiable covering.

It has surprising consequences that exhibit the drastic difference between simplicial and minimal volume. The first one is that the minimal volume is non additive by connected sum

Corollary 3.4 **([6])**. — *Let (X, g_0) be a compact hyperbolic manifold. Then*

$$\mathrm{Minvol}(X \sharp X) > 2\mathrm{vol_{g_0}}(X).$$

Indeed, $X\sharp X$ cannot have an hyperbolic metric, otherwise its universal covering would be $\mathbb{R}^n$, contradicting the fact that $\pi_{n-1}(X\sharp X) \neq 1$.

The second difference is deeper : the minimal volume depends on the differentiable structure of the manifold. To see this, we take a differentiable compact manifold which is homeomorphic to an hyperbolic manifold but is not diffeomorphic. Such exotic differentiable structures had been constructed by Farell and Jones ([**13**]). Their idea is to do the connected sum of an hyperbolic compact manifold X, or a finite covering of it, with an exotic sphere Σ (a manifold homeomorphic to the standard sphere but not diffeomorphic). Then $X\sharp\Sigma$ is homeomorphic but not diffeomorphic to X. The existence of exotic spheres follows from the works of Kervaire-Milnor and Smale, with a dimension condition. The lower dimension is $n = 7$ where there is 28 exotic spheres. Moreover, for any $\varepsilon > 0$, Farell and Jones can construct such a manifold $X\sharp\Sigma$ with pinched negative curvature metric

$$-1 - \varepsilon \leq K \leq -1 + \varepsilon.$$

Corollary 3.5 **(**[**6**]**)**. — *For these manifolds,*

$$\mathrm{Minvol}(X\sharp\Sigma) > vol_{g_0}(X).$$

Jeff Boland, Chris Connell and Juan Souto ([**9**]) have extended BCG's Theorem 3.2 for complete hyperbolic manifold of finite volume, with the hypothesis that the map f be proper. But the Theorem 3.3 does not extend to the complete case : counter-examples are constructed in [**7**].

4. Structure of the proofs of the volume rigidity theorems

In this section we explain the ideas of the proofs of theorems 3.2 and 3.3. For the sake of simplicity, we suppose that f has degree one. The main tools are the natural maps of BCG and the Gromov theory of convergence of Riemannian manifolds.

Natural maps

Suppose we are given (X, g_0) an hyperbolic manifold (or a locally symmetric space of negative curvature), (Y, g) a Riemannian manifold, a map $f : Y \to X$ of degree one and a constant $c > h(g)$, where $h(g)$ is the volume entropy. Then there exist a map

$$F_c : Y \to X$$

with the following properties. The map is C^1, homotopic to f and for all $y \in Y$,

$$|\mathrm{Jac}\ F_c(y)| \leq \left(\frac{c}{h(g_0)}\right)^n. \tag{4.1}$$

Moreover, there is equality at one point if and only if d_yF_c is an homothety of ratio $\frac{c}{h(g_0)}$. This map is called the natural map.

This readily implies the inequality of Theorem 3.2 since

$$\begin{aligned} \mathrm{vol_g}(X) &= \int_Y F_c^*(\mathrm{dv_{g_0}})(y) \\ &= \int_Y \mathrm{Jac}(F_c)\ \mathrm{dv_g}(y) \\ &\leq \int_Y |\mathrm{Jac}(F_c)|\ \mathrm{dv_g}(y) \\ &\leq \left(\frac{c}{h(g_0)}\right)^n \mathrm{vol_g}(Y). \end{aligned} \tag{4.2}$$

Letting c tends to $h(g)$, we get

$$\mathrm{vol_{g_0}}(X) \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{vol_g}(Y).$$

Finally, when $\mathrm{Ric_g} \geq -(n-1)g$ and g_0 is hyperbolic, we have $h(g) \leq h(g_0)$ hence $\mathrm{vol_{g_0}}(X) \leq \mathrm{vol_g}(Y)$.

Rigidity cases

We first sketch the proof of the equality case in theorem 3.2. Suppose that g is normalized to have $\mathrm{vol_g}(Y) = \mathrm{vol_{g_0}}(X)$ and $h(g) = h(g_0)$. Letting c tends to $h(g_0)$ in the inequalities (4.2) gives us that $\mathrm{Jac}(F_c)$ tends to 1 in L^1 norm. We can show moreover that F_c has a uniform Lipschitz upper bound when c is close enough to $h(g_0)$. A sequence F_{c_k} then converges to a 1-Lipschitz map $F : Y \to X$. Showing F preserve the volume, we infer that it is an injective map hence open which admits an isometric derivative almost everywhere. We finish the proof by showing that F is an isometry on its image which by connexity of X implies that F is actually an isometry from Y to X.

The essential difference with the equality case of Theorem 3.3 is that we cannot suppose a priori that there exists a metric on Y which realizes the minimal volume. Instead, we have a sequence of Riemannian metrics (g_k), such that $|K(g_k)| \leq 1$ and $vol_{g_k}(Y) \to \mathrm{Minvol}(Y) = vol_{g_0}(X)$. To overcome this difficulty we will apply a compactness result, due to Gromov and Peters ([**17**]) which says for any $n \in \mathbb{N}$, $D > 0$, $v > 0$, the set

$$\mathbb{M}(n, D, v) = \left\{ \begin{array}{ccc} \text{M} & | & |K| \leq 1 \\ \text{n-riemannian} & | & diam(M) \leq D \\ \text{compact manifold} & | & vol(M) \geq v \end{array} \right\}$$

is relatively compact, for the Gromov-Hausdorff or the bilipschitz topology, in the space of n-Riemannian compact manifolds with metric of regularity $C^{1,\alpha}$ ($\alpha \in]0,1[$). This theorem will imply that a subsequence $(g_{k'})$ will converge in Lipschitz topology to a $C^{1,\alpha}$ limit metric g_∞ on Y. But if we set $c_{k'} = h(g_{k'})(1 + \frac{1}{k'})$, then theorem 3.2 applied to $F_{c_{k'}} : Y \to X$ implies that

$$\frac{\mathrm{vol}_{\mathrm{g_0}}(X)}{\mathrm{vol}_{\mathrm{g_{k'}}}(Y)} \leq \left[\frac{h(g_{k'})(1 + \frac{1}{k'})}{h(g_0)} \right]^n \leq \left(1 + \frac{1}{k'} \right)^n,$$

hence we have $h(g_\infty) = \lim h(g_{k'}) = h(g_0)$ ($= \lim c_{k'}$) and $\mathrm{vol}_{\mathrm{g_\infty}}(Y) = \lim \mathrm{vol}_{\mathrm{g_{k'}}}(Y) = \mathrm{vol}_{\mathrm{g_0}}(X)$. And then we are in the equality case of Theorem 3.2. Then, to apply the Gromov-Peters theorem we have to show that we can choose a sequence (g_k) with bounded diameter.

To do this we argue by contradiction. First remark that $||Y|| \geq ||X|| > 0$ and so, the Gromov's isolation Theorem 2.4 implies that we can choose $p_k \in Y$ such that

$$\mathrm{vol}_{\mathrm{g_k}}(B_{g_k}(p_k, 1)) \geq \varepsilon_n. \tag{4.3}$$

By the pointed version of the compactness theorem ([**17**] p387 + [**24**]) , there is a complete, non compact Riemannian n-manifold (Z^n, g_∞) and a point $p \in Z$ such that the following holds. The metric g_∞ is of class $C^{1,\alpha}$ and for any $R > 0$, there exist diffeomorphisms $\varphi_{R,k} : B(p, R) \subset Z \to \varphi_{R,k}(B(p, R)) \subset Y$ such that $||\varphi^*_{R,k} g_k - g_\infty||_{C^{1,\alpha}} \to 0$ on $B(p, R)$ as $k \to \infty$ and $\varphi_{R,k}(p) = p_k$. We will show that a subsequence of the family maps

$(F_k \circ \varphi_{R,k})_{R,k}$ will converge to an injective Lipschitz map $F : Z \to X$ with closed image. By connectedness of X, F is an homeomorphism between Z and X contradicting the compacity of X.

5. BCG's natural maps

5.1. Preliminaries. — To construct the BCG natural maps F_c, we need to recall some classical facts on visual measures and Busemann functions. Here we suppose that $\tilde{X}$ is an Hadamard space, i.e. a complete simply connected Riemannian manifold of nonpositive curvature.

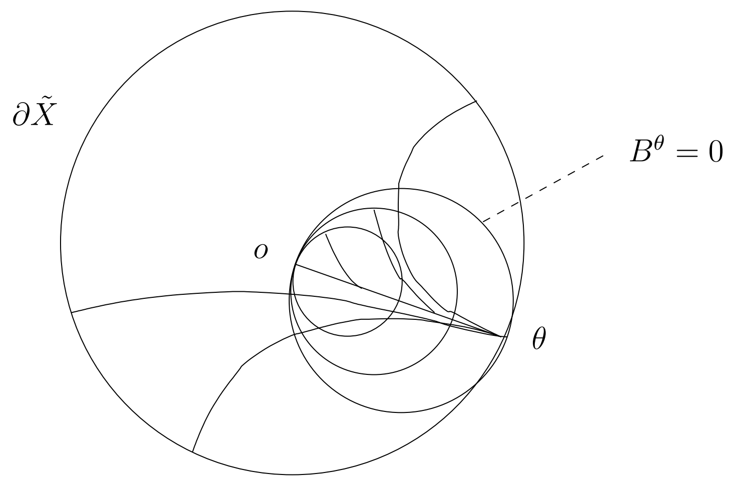
The Busemann fonctions ([**2**] and [**1**] for a more detailled exposition)

Fix a base point $o \in \tilde{X}$, $\theta \in U_0\tilde{X}$ the unit tangent sphere at o and consider a geodesic ray $\gamma_\theta(s) = exp_0(s\theta)$. For each $x \in \tilde{X}$, the function $s \to d(x, \gamma_\theta(s)) - d(o, \gamma_\theta(s))$ is monotone and bounded (by the triangle inequality) we thus define

$$B(x, \theta) = \lim_{s \to \infty} d(x, \gamma_\theta(s)) - d(o, \gamma_\theta(s)).$$

By definition, $B(o, \theta) = 0$ for any θ. The following properties are relatively easy to show ([**2**] paragraph 3). Note that, as $\tilde{X}$ is an Hadamard space, $\theta \mapsto \lim_{s\to\infty} \gamma_\theta(s)$ realizes an homeomorphism between $U_o\tilde{X}$ and $\partial\tilde{X}$. So B will be considered subsequently as defined on $\tilde{X} \times \partial\tilde{X}$.

1) For each $\theta \in \partial\tilde{X}$, the Buseman function $B^\theta : x \mapsto B(x, \theta)$ has regularity C^2 ([**18**]) and $\nabla B^\theta = -\theta$.
2) The Buseman function B^θ is convex on $\tilde{X}$. This follows from the convexity of the distance function in space of nonpositive curvature. Thus the sets $B^\theta_C = \{B^\theta \le C\}$, which are called *horoballs*, are convex. The level sets $S^\theta_C = \{B^\theta = C\}$, which are called horospheres, are hypersurfaces orthogonal to all geodesic rays which end in θ. We can see S^θ_C as the limit, when $s \to +\infty$ of the spheres $S(c(s), s+C)$.



Note that if $x \notin B_C^\theta$, then

$$B^\theta(x) = d(x, B_C^\theta) + C = d(x, S_C^\theta) + C.$$

In negative curvature, the distance function $t \to d(x(t), p)$ is strictly convex if $x(t)$ is a geodesic non colinear to the gradient of $q \mapsto d(q,p)$. The same holds for the the function $t \to d(x(t), W)$ if W is a convex set (disjoint from $x(t)$). It follows that in negative curvature $x \mapsto B(x,\theta)$ is strictly convex along geodesics which are not orthogonal to the horospheres. As a consequence, the horoballs are strictly convex. This implies that the restriction of DdB^θ to $(\nabla B^\theta)^\perp$, which is the second fondamental form of the horospheres, is strictly positive. For locally symmetric spaces of rank 1, the Hessian of B^θ is computed in [**8**]. We will use the following formula for the hyperbolic space :

$$DdB^\theta = g_0 - dB^\theta \otimes dB^\theta.$$

3) If $x \to \theta$ radially, then $B(x,\theta) \to -\infty$. In pinched negative curvature, if $x \to \theta_0 \neq \theta$, then $B(x,\theta) \to +\infty$. Indeed, in this case, x escapes from any horoball based at θ.

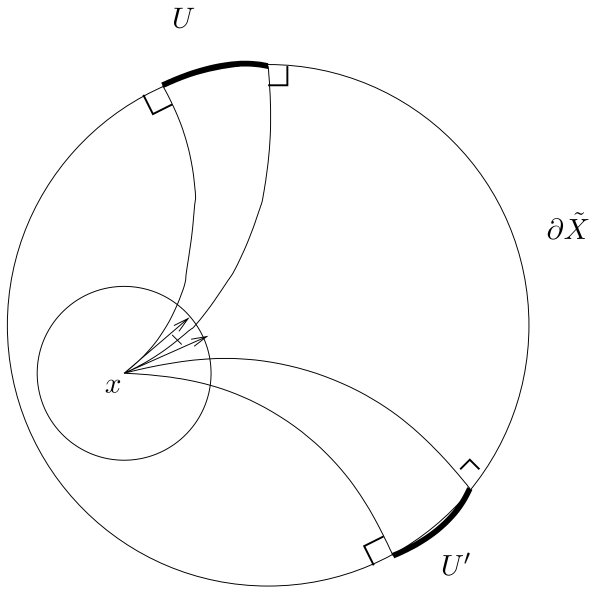
Visual measures

The visual probability measure at x is defined as follows. For each $x \in \tilde{X}$, the unit tangent sphere $U_x\tilde{X}$ is identified with the boundary $\partial\tilde{X}$

by the homeomorphism

$$v \in U_x\tilde{X} \xrightarrow{E_x} \gamma_v(\infty) \in \partial\tilde{X}.$$

The visual probability measure P_x is the push-forward by E_x of the canonical probability measure of $U_x\tilde{X}$, i.e. $P_x(U)$ is the measure of the set of vectors $v \in U_x\tilde{X}$ such that $\gamma_v(+\infty) \in U$.



We will use the following

Lemma 5.1. — 1) *P_x has no atoms on $\partial\tilde{X}$.*

2) *For each $\gamma \in Isom(\tilde{X})$,*

$$P_{\gamma x} = \gamma_* P_x.$$

3) *Let $x \in \tilde{X}$ and $\gamma \in Isom(\tilde{X})$ such that $x = \gamma^{-1}(o)$, then*

$$\frac{dP_x(\theta)}{dP_o(\theta)} = Jac(\gamma)(\theta)$$

where $Jac(\gamma)$ is the jacobian of γ acting on $\partial\tilde{X}$ by diffeomorphism.

4) *If $\tilde{X}$ is endowed with the hyperbolic metric, for any $x, o \in \tilde{X}$ we have*

$$\frac{dP_x(\theta)}{dP_o(\theta)} = e^{-h(g_0)B(x,\theta)}.$$

We note $p_o(x,\theta)$ the density of P_x with respect to P_o.

Proof. — 1) and 2) are obvious from the definition. 3) for each Borel $U \subset \partial\tilde{X}$, $P_x(U) = P_o(\gamma U)$, thus

$$P_x(U) = \int_U p_o(x,\theta)\, dP_o(\theta) = \int_{\gamma U} dP_o(\theta) = \int_U Jac(\gamma)(\theta)\, dP_o(\theta).$$

As U is arbitrary we deduce that $p_o(x,\theta) = \mathrm{Jac}(\gamma)(\theta)$.

4) one computes $p_o(x,\theta) = \mathrm{Jac}(\gamma)(\theta)$ in the half-space model of the hyperbolic space. We can assume $o = (0,\ldots,0,1)$. If x is in the horosphere $\mathbb{R}^{n-1} \times \{1\}$ based at $+\infty$, we can choose γ to be the translation $a \mapsto a + x - o$, whose Jacobian is 1 on the boundary of the space. If $x = (0,\ldots,0,x_n)$, then we choose γ to be the dilation of ratio $x_n = e^{\pm d(o,x)} = e^{B^\infty(x)}$. Hence the Jacobian of the restriction on the boundary of the space is identically $e^{(n-1)B^\infty(x)}$. The Lemma follows. □

Notations: from now, we consider P_o as a fixed probability measure on $\partial\tilde{X}$ and we just note $d\theta$ its density.
Thus $dP_x(\theta) = e^{-h(g_0)B(x,\theta)} d\theta$ is the density of P_x.

5.2. Construction of the natural maps. — We now give the construction of the BCG's natural maps. Up to homotopy, we can suppose that $f : Y \to X$ is smooth. Let us consider the universal covering $\tilde{Y}$ and $\tilde{X}$ with the lifted metrics $\tilde{g}, \tilde{g}_0$. Given $c > h(g)$, we will construct some f_*-equivariant maps $\tilde{F}_c : \tilde{Y} \longrightarrow \tilde{X}$, i.e. such that for every $[\gamma] \in \pi_1(Y)$,

$$\tilde{F}_c(\gamma.y) = [f \circ \gamma].\tilde{F}_c(y).$$

The construction has three main steps. We begin with the lift $\tilde{f} : \tilde{Y} \to \tilde{X}$. Denote $\mathcal{M}(\tilde{Y})$ the space of finite positive measures on $\tilde{Y}$ and $\mathcal{M}(\partial\tilde{X})$ the space of finite positive measure on $\partial\tilde{X}$. We will use the model of the disk D^n for the hyperbolic space $\tilde{X}$ and thus $\partial\tilde{X}$ will be identified with its boundary S^{n-1}. Fix some $c > h(g)$; in the first step, we assign to each $y \in \tilde{Y}$ a measure $\nu_y^c \in \mathcal{M}(\tilde{Y})$. In the second step, this measure is push forward to a measure on $\tilde{X}$, then to a measure $\mu_y^c \in \mathcal{M}(\partial\tilde{X})$ by convolution with the visual measures of $\tilde{X}$. In the last step, we define the barycenter map from $\mathcal{M}(\partial\tilde{X})$ to $\tilde{X}$. Finally, $\tilde{F}_c(y) = bar(\mu_y^c)$. As the construction is equivariant, we have downstairs a map $F_c : Y \to X$. Here is a picture :

$$\begin{array}{ccc} \nu_y^c \in \mathcal{M}(\tilde{Y}) & \longrightarrow & \mu_y^c \in \mathcal{M}(\partial\tilde{X}) \\ \uparrow & & \downarrow \\ y \in \tilde{Y} & \xrightarrow{\tilde{f},\tilde{F}_c} & x = bar(\mu_y^c) \in \tilde{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f \sim F_c} & X \end{array}$$

Step 1 : For each $y \in \tilde{Y}$, we define a finite positive measure ν_y^c on $\tilde{Y}$ by

$$d\nu_y^c(z) = e^{-c.\rho(y,z)} \mathrm{dv}_{\tilde{g}}(z).$$

Step 2 : This measure is pushed forward on a finite positive measure $\tilde{f}_*\nu_y^c$ on $\tilde{X}$ defined by

$$\tilde{f}_*\nu_y^c(U) = \nu_y^c(\tilde{f}^{-1}(U)),$$

for any Borel set U in $\tilde{X}$. We then defines a finite measure μ_y^c on $\partial\tilde{X}$, by doing a convolution with all probability visual measures P_x of $\partial\tilde{X}$:

$$\begin{aligned} \mu_y^c(U) &= \int_{\tilde{X}} P_x(U)\ d(\tilde{f}_*\nu_y^c)(x) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U)\ d\nu_y^c(z). \end{aligned}$$

We can verify that μ_y^c has finite measure on $\partial\tilde{X}$, with norm

$$||\mu_y^c|| = \nu_y^c(\tilde{Y}) = ||\nu_y^c||.$$

Indeed, by Fubini's theorem

$$\begin{aligned} \int_{\partial\tilde{X}} \int_{\tilde{Y}} p_o(\tilde{f}(z),\theta) e^{-c.\rho(y,z)}\ \mathrm{dv}_{\tilde{g}}(z)\ d\theta &= \int_{\tilde{Y}} e^{-c.\rho(y,z)} \int_{\partial\tilde{X}} p_o(\tilde{f}(z),\theta) d\theta\ \mathrm{dv}_{\tilde{g}}(z) \\ &= \int_{\tilde{Y}} e^{-c.\rho(y,z)}\ \mathrm{dv}_{\tilde{g}}(z). \end{aligned}$$

Moreover, we have

Lemma 5.2. — *The map $y \to \mu_y^c$ is f_*-equivariant. That is for $[\gamma] \in \pi_1(Y) \subset Isom(\tilde{Y})$, $[f \circ \gamma] \in Isom(\tilde{X})$ acts on $\mathcal{M}(\partial\tilde{X})$ by the push-forward action*

$$\mu_{\gamma y}^c = (f \circ \gamma)_* \mu_y^c.$$

Proof. — Let $U \subset \partial\tilde{X}$ be a measurable set. One computes

$$\begin{aligned} \mu_{\gamma y}^c(U) &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U) e^{-c.\rho(\gamma y, z)}\, dvol_{\tilde{g}}(z) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U) e^{-c.\rho(y, \gamma^{-1} z)}\, dvol_{\tilde{g}}(z) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(\gamma z')}(U)) e^{-c.\rho(y, z')}\, dvol_{\tilde{g}}(z'), \end{aligned} \tag{5.1}$$

with the change of variable $z' = \gamma^{-1}(z)$ and the fact that $Jac(\gamma)(z) = 1$ as γ acts by isometry. Now using ii) of Lemma 5.1 with $\beta = f \circ \gamma \in Isom(\tilde{X})$,

$$\begin{aligned} (5.1) &= \int_{\tilde{Y}} (P_{\beta \tilde{f}(z)}(U) d\nu_y^c(z) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(\beta^{-1} U) d\nu_y^c(z) \\ &= \mu_y^c(\beta^{-1}(U)) \\ &= (f \circ \gamma)_* \mu_y^c(U). \end{aligned}$$

by the definition of the push-forward action. □

Step 3 : We now take the barycenter of this measure. Let us recall the definition. Let $\mu \in \mathcal{M}(\partial\tilde{X})$ be a finite positive measure, without atoms. Consider the function on $\tilde{X}$

$$\mathcal{B}(x) = \int_{\partial\tilde{X}} B(x, \theta)\, d\mu(\theta).$$

Lemma 5.3. — *We have the two following facts :*

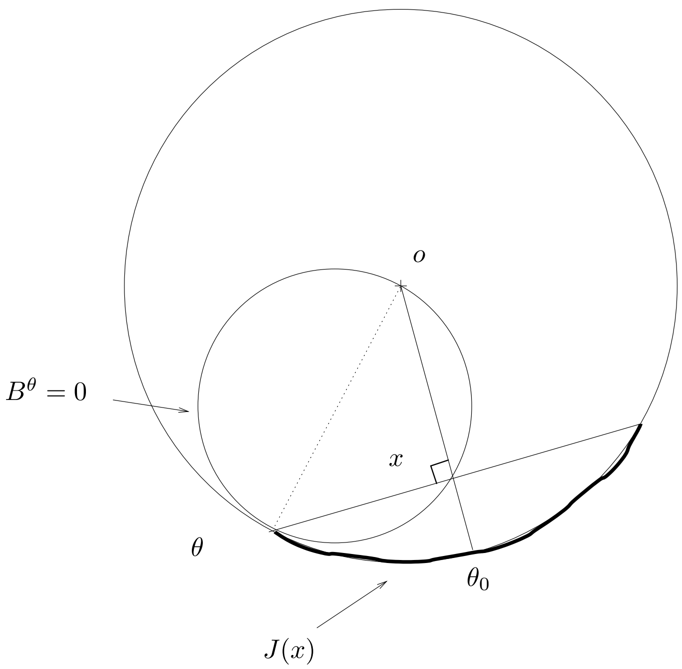
1) *$\mathcal{B}$ is strictly convex.*
2) *$\mathcal{B}(x) \to \infty$ as $x \to \theta_0 \in \partial\tilde{X}$ along a geodesic.*

Proof. — 1) It is clear because given a geodesic $x(t)$, $B(x(t), \theta)$ is convex for all $\theta \in \partial\tilde{X}$, and stricly convex for a set of θ of full μ-measure.
2) Suppose $x \to \theta_0$. Then $B(x, \theta) \to +\infty$ for a set of full μ-measure and

$B(x,\theta) \to -\infty$ on a μ-negligeable set. But we need a more quantitative argument. We consider $x \to \theta_0$ radially. Let $x_1 \in o\theta_0$ and $x \in x_1\theta_0$. By convexity of $B(.,\theta)$ and $B(o,\theta) = 0$, we have for each $\theta \in \partial\tilde{X}$

$$B(x,\theta) \geq \frac{d(o,x)}{d(o,x_1)} B(x_1,\theta).$$

Denote by $J(x) = \{\theta \in \partial\tilde{X} : B(x,\theta) \leq 0\}$. Clearly, $\mu(J(x)) \to 0$ as $x \to \theta_0$.



Let K be a compact of $\partial\tilde{X}$, such that $\mu(K) > 0$ and $\theta_0 \notin K$. Suppose x_1 sufficiently close to θ_0 such that $B(x_1,\theta) \geq C > 0$ for each $\theta \in K$. Now we compute

$$
\begin{aligned}
\mathcal{B}(x) &\geq \int_{J(x)} B(x,\theta)\ d\mu(\theta) + \int_K B(x,\theta)\ d\mu(\theta) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \int_{J(x)} B(x_1,\theta)\ d\mu(\theta) + \frac{d(o,x)}{d(o,x_1)} \int_K B(x_1,\theta)\ d\mu(\theta) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \inf_{\partial\tilde{X}}\{B(x_1,\theta)\}\mu(J(x)) + \frac{d(o,x)}{d(o,x_1)} C\mu(K) \\
&= \frac{d(o,x)}{d(o,x_1)} \left(\inf_{\partial\tilde{X}}\{B(x_1,\theta)\}\mu(J(x)) + C\mu(K) \right) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \frac{C\mu(K)}{2} \longrightarrow \infty \text{ as } x \to \theta_0.
\end{aligned}
$$

□

The function $\mathcal{B}(x)$ has a unique minimum in $\tilde{X}$, which is called the barycenter of μ and denoted by $bar(\mu)$.

Lemma 5.4. — 1) *For any $\gamma \in Isom(\tilde{X})$, $bar(\gamma_*\mu) = \gamma(bar(\mu))$.*
2) *In particular, $bar(P_x) = x$.*

Proof. — 1) The barycenter $x = bar(\mu)$ is the unique solution of the vector equation

$$\int_{\partial\tilde{X}} d_x B^\theta(u)\ d\mu(\theta) = 0, \forall u \in T_x\tilde{X}.$$

As γ acts on $\tilde{X}$ by isometry, we have $B^{\gamma(\theta)}(\gamma(x)) = B^\theta(x)$, hence $d_{\gamma(\theta)}B^{\gamma(\theta)} \circ d_x\gamma = d_x B^\theta$.

Thus, $\forall v \in T_{\gamma x}\tilde{X}$,

$$
\begin{aligned}
0 &= \int_{\partial\tilde{X}} d_x B^\theta(d_x\gamma^{-1}(v))\ d\mu(\theta) \\
&= \int_{\partial\tilde{X}} d_{\gamma(x)} B^{\gamma(\theta)}(v)\ d\mu(\theta) \\
&= \int_{\partial\tilde{X}} d_{\gamma(x)} B^{\alpha}(v) \mathrm{Jac}(\gamma^{-1})(\alpha)\ d\mu(\alpha) \\
&= \int_{\partial\tilde{X}} d_{\gamma(x)} B^{\alpha}(v)\ d(\gamma_*\mu)(\alpha),
\end{aligned}
$$

and γx is the barycenter of $\gamma_*\mu$.
2) By symmetry, this is clear for $x = o$. Then apply 1) and the fact that $\tilde{X}$ is homogeneous. □

We now define $F_c(y) = bar(\mu_y^c)$ from $\tilde{Y}$ to $\tilde{X}$.

Lemma 5.5. — *The map F_c*

1) *is C^1,*
2) *is equivariant under action of $\pi_1(Y)$ and $\pi_1(X)$,*
3) *descends in a map $F_c : Y \to X$ homotopic to f.*

Proof. — 1) See [**8**] Proposition 2.4 and 5.4.
2) Apply lemmas 5.2 and 5.4 i)
3) Consider an (equivariant) homotopy between μ_y^c and $p_o(\tilde{f}(y), \theta)d\theta$ and apply Lemma 5.4 part 2) □

5.3. Jacobian and derivative estimates. — To estimate the Jacobian of F_c, we will use two positive definite symmetric bilinear forms of trace 1. For any $y \in \tilde{Y}$ and any $v \in T_{F_c(y)}\tilde{X}$ we set

$$h_y^c(v, v) = \int_{\partial\tilde{X}} \left(d_{F_c(y)}B^\theta(v)\right)^2 \frac{d\mu_y^c(\theta)}{||\mu_y^c||} = g_0(H_y^c(v), v),$$

where H_y^c is a symmetric endomorphism of $T_{F_c(y)}\tilde{X}$ of norm and trace equal to 1 (remind that $||\nabla B^\theta|| = 1$ for all $\theta \in \partial\tilde{X}$).

Similarly, for any $y \in \tilde{Y}$ and any $u \in T_y\tilde{Y}$ we set

$$h'^c_y(u, u) = \int_{\tilde{Y}} \left(d\rho_{(y,z)}(u)\right)^2 \frac{d\nu_y^c(z)}{||\nu_y^c||} = g(H'^c_y.u, u).$$

Lemma 5.6. — *For any $y \in \tilde{Y}$, any $u \in T_y\tilde{Y}$ and any $v \in T_{F(y)}\tilde{X}$ we have*

$$\left|g_0((I - H_y^c)d_yF(u), v)\right| \leq c.g_0(H_y^c(v), v)^{1/2}.g(H'^c_y(u), u)^{1/2} \tag{5.2}$$

Proof. — From the definition of $F_c(y)$, for any $v \in T_{F_c(y)}\tilde{X}$ we have

$$\begin{aligned} 0 &= D_{F_c(y)}\mathcal{B}(v) = \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta(v)\; d\mu_y^c(\theta) \\ &= \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta(v) \left(\int_{\tilde{Y}} p_o(\tilde{f}(z), \theta)e^{-c\rho(y,z)}dvol_{\tilde{g}}(z)\right) d\theta. \qquad (5.3) \end{aligned}$$

We prolong v to a vector field V of $\tilde{X}$ by parallel translation along rays issued from $F_c(y)$. Then pick $u \in T_y\tilde{Y}$ and differentiate equation (5.3) with respect to y in the direction v we have

$$0 = \int_{\partial\tilde{X}} Dd_{F_c(y)}B^\theta(d_yF_c(u), V)d\mu_y^c(\theta) + \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta(v)\left(\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)(-cd\rho_{(y,z)}(u))d\nu_y^c(z)\right)d\theta.$$

Thus, using the Cauchy-Schwarz inequality in the second term, we have

$$\left|\int_{\partial\tilde{X}} Dd_{F_c(y)}B^\theta(d_yF_c(u), V)d\mu_y^c(\theta)\right| \leq$$

$$\int_{\partial\tilde{X}} |d_{F_c(y)}B^\theta(v)|\left(\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)d\nu_y^c(z)\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)|cd\rho_{(y,z)}(u)|^2 d\nu_y^c(z)\right)^{1/2} d\theta$$

Using again the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\int_{\partial\tilde{X}} Dd_{F_c(y)}B^\theta(d_yF_c(u), V)d\mu_y^c(\theta)\right| \leq \\
&c\left(\int_{\partial\tilde{X}} |dB_{(F(y),\theta)}(v)|^2\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)d\nu_y^c(z)d\theta\right)^{1/2}\times \\
&\left(\int_{\partial\tilde{X}}\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)|d\rho_{(y,z)}(u)|^2 d\nu_y^c(z)d\theta\right)^{1/2} \\
&= c\left(\int_{\partial\tilde{X}} |dB_{(F(y),\theta)}(v)|^2 d\mu_y^c(\theta)\right)^{1/2}\left(\int_{\tilde{Y}} |d\rho_{(y,z)}(u)|^2\ d\nu_y^c(z)\right)^{1/2} \\
&= c\ ||\nu_y^c||^{1/2}\ ||\mu_y^c||^{1/2}\ g_0(H_y^c.v,v)^{1/2}\ g({H'}_y^c.u,u)^{1/2}.
\end{aligned}$$

Now using $DdB = g_0 - dB\otimes dB$, the above integral containing DdB can be computed as

$$g_0(d_yF(u),v)||\mu_y^c|| - \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta)(d_yF_c(u))d_{F_c(y)}B^\theta)(v)\ d\mu_y^c(\theta) = g_0((I-H_y^c)d_yF_c(u),v))||\mu_y^c||,$$

and dividing by $||\mu_y^c|| = ||\nu_y^c||$, we obtain the Lemma. □

Thus d_yF_c is controled by H_y^c. Let $0 < \lambda_1^c(y) \leq ... \leq \lambda_n^c(y) < 1$ the eigenvalues of H_y^c.

Proposition 5.7. — *There exists a constant $A > 0$ such that, for any $y \in Y$,*

$$|JacF_c(y)| \leq \left(\frac{c}{h(g_0)}\right)^n \left(1 - A\sum_{i=1}^{n}(\lambda_i^c(y) - \frac{1}{n})^2\right). \tag{5.4}$$

If c is close to $h(g_0)$ and $|JacF_c(y)|$ close to 1, the eigenvalues are all close to $1/n$.

Proof. — It follows from the two following lemmas.

Lemma 5.8. — *At each point $y \in \tilde{Y}$, we have*

$$|Jac(F_c)(y)| \leq \left(\frac{c}{\sqrt{n}}\right)^n \frac{det({H_y}^c)^{1/2}}{det(I - {H_y}^c)}.$$

Proof. — Let (u_i) an orthonormal basis of $T_{F_c(y)}\tilde{X}$ which diagonalizes ${H'_y}^c$. We can suppose that d_yF_c is invertible. Let $v'_i = [(I - {H_y}^c) \circ d_yF_c]^{-1}(u_i)$. The Schmidt orthonormalization process applied to the basis (v'_i) gives an orthonormal basis (v_i) at $T_y\tilde{Y}$. The matrix of $(I - {H_y}^c) \circ d_yF_c$ in the base (v_i) and (u_i) is triangular so that

$$det(I - {H_y}^c)Jac(F_c)(y) = \prod_{i=1}^{n} g_0((I - {H_y}^c) \circ d_yF_c.v_i, u_i).$$

Thus, with (5.2),

$$\begin{aligned} det(I - {H_y}^c)|Jac(F_c)(y)| &\leq c^n \left(\prod_{i=1}^{n} g_0({H_y}^c v_i, v_i)\right)^{1/2} \left(\prod_{i=1}^{n} g({H'_y}^c u_i, u_i)\right)^{1/2} \\ &\leq c^n det({H_y}^c)^{1/2} \left[\frac{1}{n}\sum_{i=1}^{n} g({H'_y}^c u_i, u_i)\right]^{n/2}, \end{aligned}$$

and we have the desired inequality with $tr({H'_y}^c) = 1$. □

Lemma 5.9. — *Let H a symmetric positive definite $n \times n$ matrix whose trace is equal to one then, if $n \geq 3$,*

$$\frac{det(H^{1/2})}{det(I - H)} \leq \left(\frac{n}{h(g_0)^2}\right)^{n/2} \left(1 - A\sum_{i=1}^{n}(\lambda_i - \frac{1}{n})^2\right)$$

for a constant $A(n) > 0$.

Proof. — See Appendix B5 in [**8**]. □

This is the point where the rigidity of the natural maps fails in dimension 2. This completes the proof of the Proposition. □

We thus have obtained the inequality (4.1) of Theorem 3.2 . Before continuing the Proof of Theorem 3.3, we give some useful lemmas.

Lemma 5.10. — *Suppose that* $|Jac\ F_c(y)| = \left(\frac{c}{h(g_0)}\right)^n$. *Then* d_yF_c *is an homothety of ratio* $\frac{c}{h(g_0)}$.

Proof. — by Proposition 5.7 we have $H_y^c = \frac{Id}{n}$. From Lemma 5.6, we deduce that

$$||d_yF_c.u||^2 \leq n\left(\frac{c}{n-1}\right)^2 g({H'_y}^c u, u),$$

hence with $tr({H'_y}^c) = 1$ we find

$$tr(F_c^* g_0)(y) \leq n\left(\frac{c}{n-1}\right)^2. \tag{5.5}$$

Now

$$\begin{aligned}\left(\frac{c}{h(g_0)}\right)^{2n} &= det(F^*g_0) \leq \left[\frac{1}{n}tr(F^*g_0)\right]^n \\ &\leq \left(\frac{c}{n-1}\right)^{2n} = \left(\frac{c}{h(g_0)}\right)^{2n},\end{aligned}$$

which leads to $det(F^*g_0) = \left[\frac{1}{n}tr(F^*g_0)\right]^n$ and we must have ${F_c}^*g_0(y) = \left(\frac{c}{h(g_0)}\right)^2 g_y$. □

The same arguments show that

Lemma 5.11. — *For* $\varepsilon > 0$, *there exists* $\alpha(\varepsilon) > 0$ *tending to zero* 0 *with* ε *such that if*

$$\left(\frac{c}{h(g_0)}\right)^n - |\mathrm{Jac}(F_c)(y)| \leq \varepsilon$$

then for any $u \in T_y\tilde{Y}$

$$(1-\alpha(\varepsilon))\frac{c}{h(g_0)}||u|| \leq ||d_yF_c.u|| \leq (1+\alpha(\varepsilon))\frac{c}{h(g_0)}||u||.$$

We shall set $Y_{c,\varepsilon} = \left\{ y \in Y, \left(\frac{c}{h(g_0)}\right)^n - |\mathrm{Jac}(F_c)(y)| \leq \varepsilon \right\}$. Of course, it depends also on the metric g.

Lemma 5.12. — *We set* $c = (1 + \mathrm{vol}_{\mathrm{g}}(Y) - \mathrm{vol}_{\mathrm{g_0}}(X))^{\frac{1}{n}}\, h(g_0) > h(g_0) \geq h(g)$ *and*
$\varepsilon = \sqrt{\mathrm{vol}_{\mathrm{g}}(Y) - \mathrm{vol}_{\mathrm{g_0}}(X)}$. *Then we have*

$$\mathrm{vol}_{\mathrm{g}}(Y \backslash Y_{c,\varepsilon}) \leq 2\varepsilon(1 + \mathrm{vol}_{\mathrm{g}}(Y)).$$

Proof. — with our choice of c and ε we have $\varepsilon^2 = \left(\frac{c}{h(g_0)}\right)^n - 1$, hence $|\mathrm{Jac}F_c| \leq 1+\varepsilon^2-\varepsilon \leq 1-\varepsilon/2$ on $Y\backslash Y_{c,\varepsilon}$. On the other hand by Proposition 5.7, we have $|\mathrm{Jac}F_c| \leq \left(\frac{c}{h(g_0)}\right)^n = 1+\varepsilon^2$ everywhere. We infer that

$$\begin{aligned}
vol_{g_0}(X) &\leq \int_Y |\mathrm{Jac}F_c(y)|\ \mathrm{dv_g}(y) \\
&= \int_{Y_{c,\varepsilon}} |\mathrm{Jac}F_c(y)|\ \mathrm{dv_g}(y) + \int_{Y-Y_{c,\varepsilon}} |\mathrm{Jac}F_c(y)|\ \mathrm{dv_g}(y) \\
&\leq (1+\varepsilon^2) vol_g(Y_{c,\varepsilon}) + (1-\frac{\varepsilon}{2})\mathrm{vol_g}(Y - Y_{c,\varepsilon}) \\
&= \mathrm{vol_g}(Y) + \varepsilon^2 \mathrm{vol_g}(Y_{c,\varepsilon}) - \frac{\varepsilon}{2}\mathrm{vol_g}(Y - Y_{c_k,\varepsilon}),
\end{aligned}$$

hence

$$\begin{aligned}
\mathrm{vol_g}(Y - Y_{c,\varepsilon}) &\leq \frac{2}{\varepsilon}\left(\mathrm{vol_g}(Y) - \mathrm{vol_{g_0}}(X) + \varepsilon^2 \mathrm{vol_g}(Y)\right) \\
&= 2\varepsilon(1 + \mathrm{vol_g}(Y)).
\end{aligned}$$

□

This lemma says that when $\mathrm{vol_{g_k}}(Y)$ tends to $\mathrm{vol_{g_0}}(X)$ the map F_{c_k} then tends to admit a derivative more and more isometric on a set Y_{c_k,ε_k} whose relative volume in (Y, g_k) is more and more close to 1 (here, c_k and ε_k stand for the values given by lemma 5.12 associated to the metric g_k). We now show that the maps F_{c_k} admit an uniform upper bound for k large enough.

Lemma 5.13. — *There exists* $r(n) > 0$, $\delta(n) > 0$ *such that if* $\mathrm{vol_g}(Y) \leq \mathrm{vol_{g_0}}(X) + \delta(n)$ *and* $y_0 \in Y_{c,\varepsilon}$ *then for any* $y \in B_g(y_0, r(n))$, *we have*

$$||d_y F_c|| \leq 2\sqrt{n}.$$

Proof. — We suppose $\delta(n)$ small enough to have $c \leq \sqrt{2}(n-1)$. The equation (5.2) allows us to control $\|d_yF_c\|$ with $\lambda_n^c(y)$, the maximal eigenvalue of H_y^c. Indeed, let $u \in U_y\tilde{Y}$ and $v = d_yF_c.u$. Then (5.2) gives us

$$(5.6)\quad (1-\lambda_n^c(y))\,|g_0(d_yF_c.u, d_yF_c.u)| \leq c.\left[\lambda_n^c(y).g_0(d_yF_c.u, d_yF_c.u)\right]^{1/2}$$

hence

$$\|d_yF_c.u\|_{g_0} \leq \frac{c\sqrt{\lambda_n^c(y)}}{1-\lambda_n^c(y)}. \tag{5.7}$$

Fix some $\eta(n) > 0$ such that if $\lambda_n^c \leq \frac{1}{n} + \eta$, the quotient above is $\leq 2\sqrt{n}$. We will show that $\lambda_n^c \leq \frac{1}{n} + \eta$ on some $B_g(y_0, r(n))$. From Proposition 5.7 and the definition of $Y_{c,\varepsilon}$, if $\delta(n)$ is sufficiently small we have $\lambda_n^c(y_0) \leq \frac{1}{n} + \frac{\eta}{2}$. We want to controll λ_n^c along small rays from y_0. Recall that H_y^c is defined by

$$h_y^c(u,v) = \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta)(u)d_{F_c(y)}B^\theta)(v)\ \frac{d\mu_y^c(\theta)}{||\mu_y^c||} = g_0(H_y^c(v), v).$$

Let u, v be two orthonormal vectors at $F_c(y_0)$, and U,V their radial parallel extensions in a neighbourhood of $F_c(y_0)$. We compute the derivative of $h_y^c(U,V)$ in a direction $w \in T_yY$. We denote $\frac{d\mu_y^c}{||\mu_y^c||} = d\sigma_y^c(\theta)$.

$$\begin{aligned} w.h_y^c(U,V) \;&=\; \int_{\partial\tilde{X}} Dd_{F_c(y)}B^\theta(d_yF_c(w), U)d_{F_c(y)}B^\theta(V)d\sigma_y^c(\theta) \\ &+\; \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(U)Dd_{F_c(y)}B^\theta(d_yF_c(w), V)d\sigma_y^c(\theta) \\ &+\; \int_{\partial\tilde{X}} d_{F_c(y)}B^\theta(U)d_{F_c(y)}B^\theta(V)w.d\sigma_y^c(\theta). \end{aligned}$$

Thus $|w.h_y^c(U,V)| \leq 2\|d_yF_c(w)\|_{g_0} + \int_{\partial\tilde{X}} |w.d\sigma_y^c(\theta)|$ since $\|DdB\| \leq 1$ and $\|dB\| \leq 1$. Recall that

$$d\sigma_y^c(\theta) \;=\; \frac{d\mu_y^c}{\mu_y^c(\partial\tilde{X})} = \frac{\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)\ d\theta}{\int_{\tilde{Y}} e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)}.$$

Thus,

$$\begin{aligned} w.d\sigma_y^c(\theta) = & \frac{\int_{\tilde{Y}} p_o(\tilde{f}(z),\theta)(-cd\rho_{(y,z)}(w))e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)\ d\theta}{\mu_y^c(\partial\tilde{X})} \\ & - \frac{d\mu_y^c}{\mu_y^c(\partial\tilde{X})^2}.\int_{\tilde{Y}}(-cd\rho_{(y,z)}(w))e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z). \end{aligned}$$

As $|cd\rho_{(y,z)}(w)| \leq c\|w\|_g$, we have

$$\int_{\partial\tilde{X}} |w.d\sigma_y^c(\theta)| \leq \int_{\partial\tilde{X}} 2c\|w\|_g d\sigma_y^c(\theta) = 2c\|w\|_g.$$

Thus, $\left|w.h_y^c(U,V)\right| \leq 2\|d_yF_c(w)\|_{g_0} + 2c\|w\|_g$. Now suppose w is normal and use (5.7):

$$\left|w.h_y^c(U,V)\right| \leq 2c\left(\frac{\sqrt{\lambda_n^c(y)}}{1-\lambda_n^c(y)}+1\right). \tag{5.8}$$

Suppose there exists $y \in Y$ such that $\lambda_n^c(y) \geq \frac{1}{n}+\eta$. Take a point y such that $\lambda_n^c(y) = \frac{1}{n}+\eta$ and $r = d(y_0,y) > 0$ is minimal. Let γ be a normal geodesic from y_0 to y. Let $U(t)$ be a parallel vector field along $F_c(\gamma)$ such that $U(r)$ is a unit eigenvector of $H^c_{\gamma(r)}$ for $\lambda_n^c(\gamma(r))$. Then, with (5.8),

$$\begin{aligned} \lambda_n^c(\gamma(r)) - \lambda_n^c(\gamma(0)) & \leq h^c_{\gamma(r)}(U(r),U(r)) - h^c_{\gamma(0)}(U(0),U(0)) \\ & \leq 2c.\int_0^r \left(\frac{\sqrt{\lambda_n^c(\gamma(t))}}{1-\lambda_n^c(\gamma(t))}+1\right)\ dt \\ & \leq 2cr\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right). \end{aligned}$$

Thus

$$\frac{\eta}{2} \leq 2cr\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right).$$

and we have a uniform bound below for $r(n)$. $\square$

We infer

Lemma 5.14. — *For any $R > 0$ and any point $p \in Y$, if*

$$\mathrm{vol_g}(Y) - \mathrm{vol_{g_0}}(X) \leq \left(\frac{\mathrm{vol_g}(B_g(p,r))\mathrm{vol}_{\mathbb{H}^n}(r(n))}{4(1+\mathrm{vol_g}(Y))e^{2nR}}\right)^2$$

then we have $||d_y F_c|| \leq 2\sqrt{n}$ for any $y \in B_g(p,R)$.

Proof. — By the Bishop-Gromov 's Theorem [**16**], for any $y \in B_g(p,R)$ we have

$$\mathrm{vol_g}(B(y,r(n)) \geq \mathrm{vol_g}(B(y,2R)\frac{\mathrm{vol}_{\mathbb{H}^n}(r(n))}{\mathrm{vol}_{\mathbb{H}^n}(2R)} \geq \mathrm{vol_g}(B(p,R)\frac{\mathrm{vol}_{\mathbb{H}^n}(r(n))}{e^{2nR}}$$

but by lemma 5.12 we have

$$\begin{aligned} \mathrm{vol_g}(Y\backslash Y_{c,\varepsilon}) &\leq 2\sqrt{\mathrm{vol_g}(Y) - \mathrm{vol_{g_0}}(X)}(1+\mathrm{vol_g}(Y)) \\ &\leq \frac{1}{2}\mathrm{vol_g}(B(p,R)\frac{\mathrm{vol}_{\mathbb{H}^n}(r(n))}{e^{2nR}}. \end{aligned}$$

So we have $\mathrm{vol_g}(Y\backslash Y_{c,\varepsilon}) < \mathrm{vol_g}(B(y,r(n))$, hence we can find $y_0 \in \mathrm{vol_g}(B(y,r(n)) \cap Y_{c,\varepsilon}$ and then the lemma 5.13 apply. □

6. End of the proof of theorem 3.3

We have left the Proof of theorem 3.3 in section 4 at the stage where we have a $C^{1,\alpha}$ complete non compact Riemannian n-manifold (Z^n, g_∞) and for any $R > 0$ a family of diffeomorphisms

$$\varphi_{R,k} : B(p,R) \subset Z \to \varphi_{R,k}(B(p,R)) \subset Y$$

such that $||\varphi^*_{R,k} g_k - g_\infty||_{C^{1,\alpha}} \to 0$ on $B(p,R)$ as $k \to \infty$. Then the family

$$(F_{c_k} \circ \varphi_{R,k})_{R,k} : B(p,R) \subset Z \to X$$

admits by diagonal extraction a subsequence that converges uniformly on compact sets to a Lipschitz map $F : (Z, g_\infty) \to (X, g_0)$.

Proposition 6.1. — *F is 1-Lipschitz and for any $B \subset Z$ measurable we have $\mathrm{vol_{g_0}}(F(B)) = \mathrm{vol}_{g_\infty}(B)$.*

Proof. — we will use the, now classical, segment inequality of Cheeger-Colding which says that on a complete manifold Y with $\mathrm{Ric}_{\mathrm{g}} \geq -(n-1)$, and for all nonnegative measurable function $h : Y \to \mathbb{R}$, we have

(6.1)
$$\int_{B(p,R)\times B(p,R)} \mathcal{F}_h(x,y)\mathrm{dxdy} \leq C(n)e^{\beta(n)R}\mathrm{vol}(B(p,R)) \int_{B(p,2R)} h(z)\mathrm{dz}$$

where $\mathcal{F}_h(x,y) = \inf_\gamma \int_\gamma h(s)ds$ and the infimum is taken over all minimizing geodesic of Y from x to y. We set $\delta_k = [\mathrm{vol}_{\mathrm{g_k}}(Y\backslash Y_{c_k,\varepsilon_k})]^{\frac{1}{2(n+1)}}$, $h = \mathbb{I}_{Y\backslash Y_{c_k,\varepsilon_k}}$ and $M_{\delta_k} = \{(y_1,y_2) \in B(p_k,R)^2 | \mathcal{F}_h \leq \frac{\delta_k}{6\sqrt{n}}\}$. By definition of M_{δ_k}, Lemma 5.11 and 5.14 we have

$$d_{g_0}(F_{c_k}(y_1), F_{c_k}(y_2)) \leq (1+\alpha(\varepsilon_k))d_{g_k}(y_1,y_2) + \frac{\delta_k}{3}$$

for any $(y_1,y_2) \in M_{\delta_k}$. Now by inequality (6.1) we get

$$\delta_k \mathrm{vol}\left[B(p_k,R)^2\backslash M_{\delta_k}\right] \leq C(n)e^{\beta(n)R}\mathrm{vol}_{\mathrm{g_k}}(Y)\mathrm{vol}_{\mathrm{g_k}}(Y\backslash Y_{c_k,\varepsilon_k})$$

hence

$$\mathrm{vol}\left[B(p_k,R)^2\backslash M_{\delta_k}\right] \leq C(n)e^{\beta(n)R}\mathrm{vol}_{\mathrm{g_0}}(X)\delta_k^{2n+1}.$$

On the other hand, for any $(y_1,y_2) \in B(p_k,R)^2$, we have by the Bishop-Gromov inequality (see the proof of Lemma 5.14)

$$\mathrm{vol}\left(B(p_k,\frac{\delta_k}{6\sqrt{n}}) \times B(p_k,\frac{\delta_k}{6\sqrt{n}})\right) \geq C(n)^2e^{-2nR}\delta_k^{2n}$$

hence for $\delta_k \leq \frac{C'(n)e^{-\beta'(n)R}}{\mathrm{vol}_{\mathrm{g_0}}(X)}$ we have $\left(B(p_k,\frac{\delta_k}{6\sqrt{n}}) \times B(p_k,\frac{\delta_k}{6\sqrt{n}})\right) \cap M_{\delta_k} \neq \emptyset$ so

$$d_{g_0}(F_{c_k}(y_1), F_{c_k}(y_2)) \leq (1+\alpha(\varepsilon_k))d_{g_k}(y_1,y_2) + \delta_k$$

$\forall (y_1,y_2) \in B(p_k,R)^2$. As $\varphi_{k,R}$ is $C^{1,\alpha}$-close to an isometry and $F_{c_k}\circ\varphi_{k,R}$ converges to F we infer that F is 1-Lipschitz.

Since $\mathrm{dv}_{\mathrm{g}_\infty}$ and $\mathrm{dv}_{\mathrm{g_0}}$ are the n-dimensional Hausdorff measure associated to d_{g_∞} and d_{g_0}, we have $\mathrm{vol}_{\mathrm{g_0}}(F(B)) \leq \mathrm{vol}_{\mathrm{g}_\infty}(B)$ for any measurable subset $B \subset Z$. On the other hand, for any compact set $B \subset Z$ we set $B_k = \varphi_{k,R}(B)$ for some large enough $R > 0$, thus by the Lebesgue dominated convergence theorem we have

$$\mathrm{vol}_{\mathrm{g_0}}(F(B)) = \lim_{k\to\infty} \mathrm{vol}_{\mathrm{g_0}}(B(F_{c_k}(B_k),\frac{1}{k}) \geq \liminf_{k\to\infty} \mathrm{vol}_{\mathrm{g_0}}(F_{c_k}(B_k)).$$

Now by the aera formula ([**23**] th. 3.7) we have

$$\begin{aligned}\int_Y |Jac(F_{c_k})| d\mathrm{vol}_{\mathrm{g_k}} &= \int_X card(F_{c_k}^{-1}(x))\ d\mathrm{vol}_{\mathrm{g_0}}(x)\\ &\geq \mathrm{vol}_{\mathrm{g_0}}(X) + \mathrm{vol}_{\mathrm{g_0}}\{x|card(F_{c_k}^{-1}(x)) > 1\}.\end{aligned}$$

Here we have used that F_{c_k} is surjective since it is of non zero degree. We set $X_k = \{x|card(F_{c_k}^{-1}(x)) > 1\}$. Recall from the proof of Lemma 5.12 that $|\mathrm{Jac}F_{c_k}| \leq 1 + \varepsilon_k^2$ everywhere, thus

$$\int_Y |Jac(F_{c_k})| d\mathrm{vol}_{\mathrm{g_k}} \leq (1+\varepsilon_k^2)\mathrm{vol}_{\mathrm{g_k}}(Y) = (1+\varepsilon_k^2)(\mathrm{vol}_{\mathrm{g_0}}(X) + \varepsilon_k^2)$$

hence $\mathrm{vol}_{\mathrm{g_0}}\{x|card(F_{c_k}^{-1}(x)) > 1\} \to 0$ as $k \to \infty$. We now set $X'_k = X\backslash(X_k \cup F_{c_k}(Y\backslash Y_{c_k,\varepsilon_k}))$. Then let $G_k : X'_k \to Y$, the inverse of F_{c_k} is well defined and $\frac{1}{1-\alpha(\varepsilon_k)}$-Lipschitz. Since $\mathrm{vol}_{\mathrm{g_0}}(F_{c_k}(Y\backslash Y_{c_k,\varepsilon_k})) \leq (1+\varepsilon_k)^2\mathrm{vol}_{\mathrm{g_k}}(Y\backslash Y_{c_k,\varepsilon_k})) \to 0$ as $k \to \infty$, we have $\mathrm{vol}_{\mathrm{g_0}}(X'_k) \to \mathrm{vol}_{\mathrm{g_0}}(X)$. Moreover since $\mathrm{vol}_{\mathrm{g_0}}(X'_k) = \mathrm{vol}_{\mathrm{g_0}}(F_{c_k}G_k(X'_k)) \leq (1+\varepsilon_k^2)\mathrm{vol}_{\mathrm{g_k}}(G_k(X'_k))$, we get $|\mathrm{vol}_{\mathrm{g_k}}(G_k(X'_k)) - \mathrm{vol}_{\mathrm{g_k}}(Y)| \to 0$ as $k \to \infty$. We then have

$$\begin{aligned}\mathrm{vol}_{\mathrm{g_0}}(F_{c_k}(B_k)) &\geq \mathrm{vol}_{\mathrm{g_0}}[F_{c_k}(B_k \cap X'_k)]\\ &\geq (1-\alpha(\varepsilon_k))^n \mathrm{vol}_{\mathrm{g_k}}[G_k(F_{c_k}(B_k \cap X'_k))]\\ &\geq (1-\alpha(\varepsilon_k))^n \mathrm{vol}_{\mathrm{g_k}}[B_k \cap G_k(X'_k)]\\ &\to \mathrm{vol}_{\mathrm{g_\infty}}(B)\end{aligned}$$

as $k \to \infty$ hence

$$\lim_{k\to\infty} \mathrm{vol}_{\mathrm{g_0}}(F_{c_k}(B_k)) \geq \mathrm{vol}_{\mathrm{g_\infty}}(B)$$

which ends the proof of the Lemma. □

Lemma 6.2. — *F is an homeomorphism from Z to X.*

Note that this is a contradiction with the fact that Z is non compact. We infer that there exists $D > 0$ and $\varepsilon > 0$ such that if $\mathrm{vol}_{\mathrm{g}}(Y) \leq \mathrm{vol}_{\mathrm{g_0}}(X) + \varepsilon$ and $|K_g| \leq 1$ then $\mathrm{Diam}(Y) \leq D$. We can conclude the proof of theorem 3.3 thanks to the Gromov's $C^{1,\alpha}$ Precompactness theorem which insures us that there is on Y a metric g_∞ which realizes the equality in theorem 3.2.

Proof. — We will show that F is injective and $F(Z) = X$. It will be sufficient since by the invariance of domain theorem of Brouwer, any injective and continuous map between topological manifolds of same dimension is open (see [**22**] 7.12). So F^{-1} is continuous and F is an homeomorphism.

The two properties can be shown by the same trick. We show first the injectivity. If $F(z_1) = x = F(z_2)$, then $F(B(z_1, r) \cup B(z_2, r)) \subset B(x, r)$. Hence if $z_1 \neq z_2$ then $B(z_1, r) \cap B(z_2, r)$ for r small enough and we get

$$\begin{aligned} \mathrm{vol}_{\mathrm{g}_0}(B(x,r)) &\geq \mathrm{vol}_{\mathrm{g}_0}(F(B(z_1,r) \cup B(z_2,r)) \\ &= \mathrm{vol}_{\mathrm{g}_\infty}(B(z_1,r) \cup B(z_2,r)) \\ &= \mathrm{vol}_{\mathrm{g}_\infty}(B(z_1,r)) + \mathrm{vol}_{\mathrm{g}_\infty}(B(z_2,r)) \end{aligned}$$

but we have

$$\mathrm{vol}_{\mathrm{g}_0}(B(x,r)) \sim c(n)r^n \sim \mathrm{vol}_{\mathrm{g}_\infty}(B(z_1,r)) \sim \mathrm{vol}_{\mathrm{g}_\infty}(B(z_1,r))$$

for r small (all $C^{1,\alpha}$ metrics behave has the Euclidean one at small scale). Which gives the injectivity by contradiction.

Let $\Omega = F(Z)$, Ω is an open set. Suppose that $\Omega \neq X$ and set x_∞ a point of the compact $X \backslash \Omega$ which minimizes the distance to $F(p) \in \Omega$. We set γ a normal minimizing geodesic on X from $F(p)$ to x_∞. Note that $\gamma([0, d(F(p), x_\infty)[) \subset \Omega$. We denote by $G : \Omega \to Z$ the (continuous) inverse of F. We now show, that G is locally 2-Lipschitz in Ω. Let $x \in \Omega$ a fixed point. If G is not 2-Lipschitz near x then we could find a sequence $(x_j) \in \Omega^{\mathbb{N}}$ with $x_j \to x$ and $d_{g_\infty}(G(x), G(x_j)) \geq 2d_{g_0}(x, x_j) > 0$ but then the balls $B(G(x), \frac{d_{g_\infty}(G(x),G(x_j)}{2})$ and $B(G(x_j), \frac{d_{g_\infty}(G(x),G(x_j)}{2})$ are disjoints and each of volume $\sim c(n)\left(\frac{d_{g_\infty}(G(x),G(x_j))}{2}\right)^n$ (for all the $G(x_j)$ are in a commun compact subset of Z) but their image by F has a controlled overlapping that contradicts the preservation of the volume by F.

Now it is easy to show (by taking a sufficiently fine regular subdivision) that for any $\varepsilon > 0$, $G \circ \gamma$ is 2-Lipschitz on $[0, d(F(p), x_\infty) - \varepsilon]$ hence $d_{g\infty}(p, G \circ \gamma(t)) \leq 2d(F(p), x_\infty)$ for any $t \in [0, d(F(p), x_\infty)[$. We then have a contradiction since $B_{g_\infty}(p, 2d(F(p), x_\infty))$ is relatively compact, so there exists $t_k \to 0$ and $z_\infty \in Z$ such that $G \circ \gamma(d(F(p), x_\infty) - t_k) \to z_\infty$. Since F is continuous we have $F(z_\infty) = \lim_{k\to\infty} F \circ G \circ \gamma(d(F(p), x_\infty) - t_k) = x_\infty$. $\square$

Note that we could refine the arguments above to show that F is an isometry from Z to X, which gives the conclusion of the Proof of theorem 3.3, without using the equality case of 3.2 ([**6**]).

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