SYMMETRIC SPACES OF THE NON-COMPACT TYPE : LIE GROUPS

by

Paul-Emile PARADAN

Abstract. — In these notes, we give first a brief account to the theory of Lie groups. Then we consider the case of a smooth manifold with a Lie group of symmetries. When the Lie group acts transitively (e.g. the manifold is homogeneous), we study the (affine) invariant connections on it. We end up with the particuler case of homogeneous spaces which are the symmetric spaces of the non-compact type.

Résumé **(Espaces symétriques de type non-compact : groupes de Lie)**

Dans ces notes, nous introduisons dans un premier les notions fondamentales sur les groupes de Lie. Nous abordons ensuite le cas d'une variété différentiable munie d'un groupe de Lie de symétries. Lorsque le groupe de Lie agit transitivement (i.e. la variété est homogène) nous étudions les connexions (affines) invariantes par ce groupe. Finalement, nous traitons le cas particulier des espaves symétriques de type non-compact.

Contents

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1. Introduction

This note is meant to give an introduction to the subjects of Lie groups and of equivariant connections on homogeneous spaces. The final goal is the study of the Levi-Civita connection on a symmetric space of the non-compact type. An introduction to the subject of "symmetric spaces" from the point of view of differential geometry is given in the course by J. Maubon [**5**].

2. Lie groups and Lie algebras: an overview

In this section, we review the basic notions concerning the Lie groups and the Lie algebras. For a more complete exposition, the reader is invited to consult standard textbooks, for example [**1**], [**3**] and [**6**].

Definition 2.1. — *A Lie group G is a differentiable manifold*[(1)] *which is also endowed with a group structure such that the mappings*

$$\begin{aligned} G\times G \longrightarrow \ & G, \quad (x,y)\longmapsto xy \quad \text{multiplication} \\ G \longrightarrow \ & G, \quad x\longmapsto x^{-1} \quad \text{inversion} \end{aligned}$$

are smooth.

We can define in the same way the notion of a *topological group*: it is a topological space[(2)] which is also endowed with a group structure such that the 'multiplication' and 'inversion' mappings are continuous.

The most basic examples of Lie groups are $(\mathbb{R},+)$, $(\mathbb{C}-\{0\},\times)$, and the general linear group $\mathrm{GL}(V)$ of a finite dimensional (real or complex)

[(1)] All manifolds are assumed second countable in this text.

[(2)] Here "topological space" means Hausdorff and locally compact.

vector space V. The classical groups like

$$\begin{aligned} \mathrm{SL}(n,\mathbb{R}) &= \{g\in \mathrm{GL}(\mathbb{R}^n),\ \det(g)=1\},\\ \mathrm{O}(n,\mathbb{R}) &= \{g\in \mathrm{GL}(\mathbb{R}^n),\ {}^tgg=\mathrm{Id}_n\},\\ \mathrm{U}(n) &= \{g\in \mathrm{GL}(\mathbb{C}^n),\ {}^t\overline{g}g=\mathrm{Id}_n\},\\ \mathrm{O}(p,q) &= \{g\in \mathrm{GL}(\mathbb{R}^{p+q}),\ {}^tg\mathrm{I}_{p,q}g=\mathrm{I}_{p,q}\},\ \text{where } \mathrm{I}_{p,q}=\begin{pmatrix}\mathrm{Id}_p & 0\\ 0 & -\mathrm{Id}_q\end{pmatrix}\\ \mathrm{Sp}(\mathbb{R}^{2n}) &= \{g\in \mathrm{GL}(\mathbb{R}^{2n}),\ {}^tgJg=J\},\ \text{where } J=\begin{pmatrix}0 & -\mathrm{Id}_n\\ \mathrm{Id}_n & 0\end{pmatrix}\end{aligned}$$

are all Lie groups. It can be proved by hand, or one can use an old Theorem of E. Cartan.

Theorem 2.2. — *Let G be a* closed *subgroup of* $\mathrm{GL}(V)$. *Then G is an embedded submanifold of* $\mathrm{GL}(V)$*, and equipped with this differential structure it is a Lie group.*

The identity element of any group G will be denoted by e. We write the tangent spaces of the Lie groups G, H, K at the identity element e respectively as: $\mathfrak{g}=\mathbf{T}_eG,\quad \mathfrak{h}=\mathbf{T}_eH,\quad \mathfrak{k}=\mathbf{T}_eK.$

Example : The tangent spaces at the identity element of the Lie groups $\mathrm{GL}(\mathbb{R}^n), \mathrm{SL}(n,\mathbb{R}), \mathrm{O}(n,\mathbb{R})$ are respectively

$$\begin{aligned} \mathfrak{gl}(\mathbb{R}^n) &= \{\text{endomorphisms of } \mathbb{R}^n\},\\ \mathfrak{sl}(n,\mathbb{R}) &= \{X\in\mathfrak{gl}(\mathbb{R}^n),\ \mathrm{Tr}(X)=0\},\\ \mathfrak{o}(n,\mathbb{R}) &= \{X\in\mathfrak{gl}(\mathbb{R}^n),\ {}^tX+X=0\},\\ \mathfrak{o}(p,q) &= \{X\in\mathfrak{gl}(\mathbb{R}^n),\ {}^tX\mathrm{Id}_{p,q}+\mathrm{Id}_{p,q}X=0\},\ \text{where } p+q=n.\end{aligned}$$

2.1. Group action. — A morphism $\phi : G\to H$ of groups is by definition a map that preserves the product : $\phi(g_1g_2)=\phi(g_1)\phi(g_2)$.

Exercise 2.3. — *Show that $\phi(e)=e$ and $\phi(g^{-1})=\phi(g)^{-1}$.*

Definition 2.4. — A (left) action of a group G *on a set M is a mapping*

$$\alpha : G\times M\longrightarrow M \tag{2.1}$$

such that $\alpha(e,m)=m, \forall m\in M$, and $\alpha(g,\alpha(h,m))=\alpha(gh,m)$ for all $m\in M$ and $g,h\in G$.

Let $\mathrm{Bij}(M)$ be the group of all bijective maps from M onto M. The conditions on α are equivalent to saying that the map $G \to \mathrm{Bij}(M), g \mapsto \alpha_g$ defined by $\alpha_g(m) = \alpha(g,m)$ is a group morphism.

If G is a Lie (resp. topological) group and M is a manifold (resp. topological space), the action of G on M is said to be smooth (resp. continuous) if the map (2.1) is smooth (resp. continuous). When the notations are understood we will write $g \cdot m$, or simply gm, for $\alpha(g,m)$.

A *representation* of a group G on a real (resp. complex) vector space V is a group morphism $\phi : G \to \mathrm{GL}(V)$: the group G acts on V through linear endomorphisms.

NOTATION : If $\phi : M \to N$ is a smooth map between differentiable manifolds, we denote by $\mathbf{T}_m\phi : \mathbf{T}_m M \to \mathbf{T}_{\phi(m)}N$ the differential of ϕ at $m \in M$.

2.2. Adjoint representation. — Let G be a Lie group and let $\mathfrak{g}$ be the tangent space of G at e. We consider the conjugation action of G on itself, defined by

$$c_g(h) = ghg^{-1}, \quad g, h \in G.$$

The mappings $c_g : G \to G$ are smooth and $c_g(e) = e$ for all $g \in G$, so one can consider the differential of c_g at e

$$\mathrm{Ad}(g) = \mathbf{T}_e c_g \ : \mathfrak{g} \to \mathfrak{g}.$$

Since $c_{gh} = c_g \circ c_h$ we have $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$. That is, the mapping

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \qquad (2.2)$$

is a smooth group morphism which is called the *adjoint representation* of G.

The next step is to consider the differential of the map Ad at e:

$$\mathrm{ad} = \mathbf{T}_e\mathrm{Ad} \ : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}). \qquad (2.3)$$

This is the *adjoint representation of* $\mathfrak{g}$. In (2.3), the vector space $\mathfrak{gl}(\mathfrak{g})$ denotes the vector space of all linear endomorphisms of $\mathfrak{g}$, and is equal to the tangent space of $\mathrm{GL}(\mathfrak{g})$ at the identity.

Lemma 2.5. — *We have the fundamental relations*

- $\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}$ *for* $g \in G, X \in \mathfrak{g}$.

- $\mathrm{ad}(\mathrm{ad}(Y)X) = \mathrm{ad}(Y) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Y)$ *for* $X, Y \in \mathfrak{g}$.
- $\mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$ *for* $X, Y \in \mathfrak{g}$.

Proof. — Since Ad is a group morphism we have $\mathrm{Ad}(ghg^{-1}) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h) \circ \mathrm{Ad}(g)^{-1}$. If we differentiate this relation at $h = e$ we get the first point, and if we differentiate it at $g = e$ we get the second one.

For the last point consider two smooth curves $a(t), b(s)$ on G with $a(0) = b(0) = e$, $\frac{d}{dt}[a(t)]_{t=0} = X$, and $\frac{d}{dt}[b(t)]_{t=0} = Y$. We will now compute the second derivative $\frac{\partial^2 f}{\partial t \partial s}(0,0)$ of the map $f(t,s) = a(t)b(s)a(t)^{-1}b(s)^{-1}$. Since $f(t,0) = f(0,s) = e$, the term $\frac{\partial^2 f}{\partial t \partial s}(0,0)$ is defined in an intrinsic manner as an element of $\mathfrak{g}$. For the first partial derivatives we get $\frac{\partial f}{\partial t}(0,s) = X - \mathrm{Ad}(b(s))X$ and $\frac{\partial f}{\partial s}(t,0) = \mathrm{Ad}(a(t))Y - Y$. So $\frac{\partial^2 f}{\partial t \partial s}(0,0) = \mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$. □

Definition 2.6. — *If G is a Lie group, one defines a bilinear map, $[-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $[X,Y]_{\mathfrak{g}} = \mathrm{ad}(X)Y$. It is the* Lie bracket *of* $\mathfrak{g}$. *The vector space $\mathfrak{g}$ equipped with $[-,-]_{\mathfrak{g}}$ is called the Lie algebra of G. We have the fundamental relations*
- anti-symmetry *:* $[X,Y]_{\mathfrak{g}} = -[Y,X]_{\mathfrak{g}}$
- Jacobi identity *:* $\mathrm{ad}([Y,X]_{\mathfrak{g}}) = \mathrm{ad}(Y) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Y)$.

On $\mathfrak{gl}(\mathfrak{g})$, a direct computation shows that $[X,Y]_{\mathfrak{gl}(\mathfrak{g})} = XY - YX$. So the Jacobi identity can be rewritten as $\mathrm{ad}([X,Y]_{\mathfrak{g}}) = [\mathrm{ad}(X), \mathrm{ad}(Y)]_{\mathfrak{gl}(\mathfrak{g})}$ or equivalently as

$$[X,[Y,Z]_{\mathfrak{g}}]_{\mathfrak{g}} + [Y,[Z,X]_{\mathfrak{g}}]_{\mathfrak{g}} + [Z,[X,Y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0 \quad \text{for all } X,Y,Z \in \mathfrak{g}. \tag{2.4}$$

Definition 2.7. — • *A* Lie algebra $\mathfrak{g}$ *is a real vector space equipped with the antisymmetric bilinear map $[-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity.*

• *A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a* morphism *of Lie algebras if*

$$\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}. \tag{2.5}$$

Remark 2.8. — *We have defined the notion of a real Lie algebra. However, the definition goes through on any field k, in particular when $k = \mathbb{C}$ we shall speak of complex Lie algebras. For example, if $\mathfrak{g}$ is a real Lie* *algebra, the complexified vector space* $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ *inherits a canonical structure of complex Lie algebra.*

The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the typical example of a morphism of Lie algebras. This example generalizes as follows.

Lemma 2.9. — *Consider a smooth morphism* $\Phi : G \to H$ *between two Lie groups. Let* $\phi : \mathfrak{g} \to \mathfrak{h}$ *be its differential at* e*. Then:*
- *The map* ϕ *is* Φ*-equivariant:* $\phi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\Phi(g)) \circ \phi$.
- ϕ *is a morphism of Lie algebras.*

The proof works as in Lemma 2.5.

EXAMPLE : If G is a closed subgroup of $\mathrm{GL}(V)$, the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ is a morphism of Lie algebras. In other words, if $X, Y \in \mathfrak{g}$ then $[X,Y]_{\mathfrak{gl}(V)} = XY - YX$ belongs to $\mathfrak{g}$ and corresponds to the Lie bracket $[X,Y]_{\mathfrak{g}}$.

2.3. Vectors fields and Lie bracket. — Here we review a typical example of Lie bracket : the one of vector fields.

Let M be a smooth manifold. We denote by $\mathrm{Diff}(M)$ the group formed by the diffeomorphisms of M, and by $\mathrm{Vect}(M)$ the vector space of smooth vector fields. Even if $\mathrm{Diff}(M)$ is not a Lie group (it's not finite dimensional), many aspects discussed earlier apply here, with $\mathrm{Vect}(M)$ in the role of the Lie algebra of $\mathrm{Diff}(M)$. If $a(t)$ is a smooth curve in $\mathrm{Diff}(M)$ passing through the identity at $t = 0$, the derivative $V = \frac{d}{dt}[a]_{t=0}$ is a vector field on M.

The "adjoint" action of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ is defined as follows. If $V = \frac{d}{dt}[a]_{t=0}$ one takes $\mathrm{Ad}(g)V = \frac{d}{dt}[g \circ a \circ g^{-1}]_{t=0}$ for every $g \in \mathrm{Diff}(M)$. The definition of Ad extends to any $V \in \mathrm{Vect}(M)$ through the following expression

$$\mathrm{Ad}(g)V|_m = \mathbf{T}_{g^{-1}m}(g)(V_{g^{-1}m}), \qquad m \in M. \tag{2.6}$$

We can now define the adjoint action by differentiating (2.6) at the identity. If $W = \frac{d}{dt}[b]_{t=0}$ and $V \in \mathrm{Vect}(M)$, we take

$$\mathrm{ad}(W)V|_m = \frac{d}{dt}\left[\mathbf{T}_{b(t)^{-1}m}(b(t))(V_{b(t)^{-1}m})\right]_{t=0}, \qquad m \in M. \tag{2.7}$$

If we take any textbook on differential geometry we see that $\mathrm{ad}(W)V = -[W, V]$, where $[-,-]$ is the usual Lie bracket on $\mathrm{Vect}(M)$. To explain why we get this minus sign, consider the group morphism

$$\begin{aligned} \Phi : \mathrm{Diff}(M) &\longrightarrow \mathrm{Aut}(\mathcal{C}^\infty(M)) \\ g &\longmapsto \underline{g} \end{aligned} \tag{2.8}$$

defined by $\underline{g} \cdot f(m) = f(g^{-1}m)$ for $f \in \mathcal{C}^\infty(M)$. Here $\mathrm{Aut}(\mathcal{C}^\infty(M))$ is the group of automorphisms of the algebra $\mathcal{C}^\infty(M)$. If $b(t)$ is a smooth curve in $\mathrm{Aut}(\mathcal{C}^\infty(M))$ passing through the identity at $t = 0$, the derivative $u = \frac{d}{dt}[b]_{t=0}$ belongs to the vector space $\mathrm{Der}(\mathcal{C}^\infty(M))$ of derivations of $\mathcal{C}^\infty(M)$: $u : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a linear map and $u(fg) = u(f)g + fu(g)$. So the Lie algebra of $\mathrm{Aut}(\mathcal{C}^\infty(M))$ has a natural identification with $\mathrm{Der}(\mathcal{C}^\infty(M))$ equipped with the Lie bracket: $[u, v]_{\mathrm{Der}} = u \circ v - v \circ u$, for $u, v \in \mathrm{Der}(\mathcal{C}^\infty(M))$.

Let $\mathrm{Vect}(M) \xrightarrow{\sim} \mathrm{Der}(\mathcal{C}^\infty(M))$, $V \mapsto \widetilde{V}$ be the canonical identification defined by $\widetilde{V} f(m) = \langle df_m, V_m \rangle$ for $f \in \mathcal{C}^\infty(M)$ and $V \in \mathrm{Vect}(M)$.

For the differential at the identity of Φ we get

$$d\Phi(V) = -\widetilde{V}, \quad \text{for} \quad V \in \mathrm{Vect}(M). \tag{2.9}$$

Since $d\Phi$ is an algebra morphism we have $-\widetilde{\mathrm{ad}(V)W} = [\widetilde{V}, \widetilde{W}]_{\mathrm{Der}}$. Hence we see that $[V, W] = -\mathrm{ad}(V)W$ is the traditional Lie bracket on $\mathrm{Vect}(M)$ defined by posing $\widetilde{[V, W]} = \widetilde{V} \circ \widetilde{W} - \widetilde{V} \circ \widetilde{W}$.

2.4. Group actions and Lie bracket. — Let M be a differentiable manifold equipped with a smooth action of a Lie group G. We can specialize (2.8) to a group morphism $G \to \mathrm{Aut}(\mathcal{C}^\infty(M))$. Its differential at the identity defines a map $\mathfrak{g} \to \mathrm{Der}(\mathcal{C}^\infty(M)) \xrightarrow{\sim} \mathrm{Vect}(M)$, $X \mapsto X_M$ by posing $X_M|_m = \frac{d}{dt}[a(t)^{-1} \cdot m]_{t=0}$, $m \in M$. Here $a(t)$ is a smooth curve on G such that $X = \frac{d}{dt}[a]_{t=0}$. This mapping is a morphism of Lie algebras:

$$[X, Y]_M = [X_M, Y_M]. \tag{2.10}$$

Example : Consider the actions of right and left translations R and L of a Lie group G on itself:

$$R(g)h = hg^{-1}, \quad L(g)h = gh \quad \text{for } g, h \in G. \tag{2.11}$$

Theses actions define vector fields X^L, X^R on G for any $X \in \mathfrak{g}$, and (2.10) reads

$$[X,Y]^L = [X^L, Y^L], \quad [X,Y]^R = [X^R, Y^R].$$

Theses equations can be used to define the Lie bracket on $\mathfrak{g}$. Consider the subspaces $V^L = \{X^L,\, X \in \mathfrak{g}\}$ and $V^R = \{X^R,\, X \in \mathfrak{g}\}$ of $\mathrm{Vect}(G)$. First we see that V^L (resp. V^R) coincides with the subspace of $\mathrm{Vect}(G)^R$ (resp. $\mathrm{Vect}(G)^L$) formed by the vector fields invariant by the R-action of G (resp. by the L-action of G). Secondly we see that the subspaces $\mathrm{Vect}(G)^R$ and $\mathrm{Vect}(G)^L$ are invariant under the Lie bracket of $\mathrm{Vect}(G)$. Then for any $X, Y \in \mathfrak{g}$, the vector field $[X^L, Y^L]$ belongs to $\mathrm{Vect}(G)^R$, so there exists a unique $[X,Y] \in \mathfrak{g}$ such that $[X,Y]^L = [X^L, Y^L]$.

2.5. Exponential map. — Consider the usual exponential map $\mathrm{e} : \mathfrak{gl}(V) \to \mathrm{GL}(V)$: $\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. We have the fundamental property

Proposition 2.10. — • *For any $A \in \mathfrak{gl}(V)$, the map $\phi_A : \mathbb{R} \to \mathrm{GL}(V), t \mapsto \mathrm{e}^{tA}$ is a smooth Lie group morphism with $\frac{d}{dt}[\phi_A]_{t=0} = A$.*

• *If $\phi : \mathbb{R} \to \mathrm{GL}(V)$ is a smooth Lie group morphism we have $\phi = \phi_A$ for $A = \frac{d}{dt}[\phi]_{t=0}$.*

Now, we will see that an exponential map enjoying the properties of Proposition 2.10 exists on all Lie groups.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ we consider the vector field $X^R \in \mathrm{Vect}(G)$ defined by $X^R|_g = \frac{d}{dt}[ga(t)]_{t=0},\ g \in G$. Here $a(t)$ is a smooth curve on G such that $X = \frac{d}{dt}[a]_{t=0}$. The vector fields X^R are invariant under left translation, that is

$$\mathbf{T}_g(L(h))(X^R_g) = X^R_{hg}, \quad \text{for } g, h \in G. \tag{2.12}$$

We consider now the flow of the vector field X^R. For any $X \in \mathfrak{g}$ we consider the differential equation

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,g) &= X^R(\phi(t,g)) \\ \phi(0,g) &= g. \end{aligned} \tag{2.13}$$

where $t \in \mathbb{R}$ belongs to an interval containing 0, and $g \in G$. Classical results assert that for any $g_0 \in G$ (2.13) admits a unique solution ϕ^X defined on $]-\varepsilon, \varepsilon[\times\mathcal{U}$ where $\varepsilon > 0$ is small enough and $\mathcal{U}$ is a neighborhood of g_0. Since X^R is invariant under the left translations we have

$$\phi^X(t,g) = g\phi^X(t,e). \tag{2.14}$$

The map $t \to \phi^X(t,-)$ is a 1-parameter subgroup of (local) diffeomorphisms of M: $\phi^X(t+s,m) = \phi^X(t,\phi^X(s,m))$ for t,s small enough. Eq. (2.14) gives then

$$\phi^X(t+s,e) = \phi^X(t,e)\phi^X(s,e) \quad \text{for } t,s \text{ small enough.} \tag{2.15}$$

The map $t \mapsto \phi^X(t,e)$ initially defined on an interval $]-\varepsilon,\varepsilon[$ can be extended on $\mathbb{R}$ thanks to (2.15). For any $t \in \mathbb{R}$ take $\Phi^X(t,e) = \phi^X(\frac{t}{n},e)^n$ where n is an integer large enough so that $|\frac{t}{n}| < \varepsilon$. It is not difficult to see that our definition make sense and that $\mathbb{R} \to G$, $t \mapsto \Phi^X(t,e)$ is a Lie group morphism. Finally we have proved that the vector field X^R is complete: its flow is defined on $\mathbb{R} \times G$.

Definition 2.11. — *For each $X \in \mathfrak{g}$, the element $\exp_G(X) \in G$ is defined as $\Phi^X(1,e)$. The mapping $\mathfrak{g} \to G$, $X \mapsto \exp_G(X)$ is called the exponential mapping from $\mathfrak{g}$ into G.*

Proposition 2.12. — *a) $\exp_G(tX) = \Phi^X(t,e)$ for each $t \in \mathbb{R}$.*
b) $\exp_G : \mathfrak{g} \to G$ is $\mathcal{C}^\infty$ and $\mathbf{T}_e \exp_G$ is the identity map.

Proof. — Let $s \neq 0$ in $\mathbb{R}$. The maps $t \to \Phi^X(t,e)$ and $t \to \Phi^{sX}(t\frac{X}{s},e)$ are both solutions of the differential equation (2.13): so there are equal and a) is proved by taking $t = s$. To prove b) consider the vector field V on $\mathfrak{g} \times G$ defined by $V(X,g) = (X^R(g),0)$. It is easy to see that the flow Φ^V of the vector field V satisfies $\Phi^V(t,X,g) = (g\exp_G(tX),X)$, for $(t,X,g) \in \mathbb{R} \times \mathfrak{g} \times G$. Since Φ^V is smooth (a general property concerning the flows), the exponential map is smooth. □

Proposition 2.10 take now the following form.

Proposition 2.13. — *If $\phi : \mathbb{R} \to G$ is a ($\mathcal{C}^\infty$) one parameter subgroup, we have $\phi(t) = \exp_G(tX)$ with $X = \frac{d}{dt}[\phi]_{t=0}$.*

Proof. — If we differentiate the relation $\phi(t+s) = \phi(t)\phi(s)$ at $s = 0$, we see that ϕ satisfies the differential equation $(*)$ $\frac{d}{dt}[\phi]_t = X^R(\phi(t))$, where $X = \frac{d}{dt}[\phi]_{t=0}$. Since $t \to \Phi^X(t,e)$ is also solution of $(*)$, and $\Phi^X(0,e) = \phi(0) = e$, we have $\phi = \Phi^X(-,e)$. □

We give now some easy consequences of Proposition 2.13.

Proposition 2.14. — • *If $\rho : G \to H$ is a morphism of Lie groups and $d\rho : \mathfrak{g} \to \mathfrak{h}$ is the corresponding morphism of Lie algebras, we have* $\exp_H \circ d\rho = \rho \circ \exp_G$.

• *For* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ *we have* $\mathrm{Ad}(\exp_G(X)) = \mathrm{e}^{\mathrm{ad}(X)}$.

• $\exp_G : \mathfrak{g} \to G$ *is G-equivariant:* $\exp_G(\mathrm{Ad}(g)X) = g \exp_G(X) g^{-1}$.

• *If* $[X, Y] = 0$, *then* $\exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X) = \exp_G(X + Y)$.

Proof. — We use in each case the same kind of proof. We consider two 1-parameter subgroups $\Phi_1(t)$ and $\Phi_2(t)$. Then we verify that $\frac{d}{dt}[\Phi_1]_{t=0} = \frac{d}{dt}[\Phi_2]_{t=0}$, and from Proposition 2.13 we conclude that $\Phi_1(t) = \Phi_2(t)$, $\forall t \in \mathbb{R}$. The relation that we are looking for is $\Phi_1(1) = \Phi_2(1)$.

For the first point, we take $\Phi_1(t) = \exp_H(td\rho(X))$ and $\Phi_2(t) = \rho \circ \exp_G(tX)$: for the second point we take $\rho = \mathrm{Ad}$, and for the third one we take $\Phi_1(t) = \exp_G(t\mathrm{Ad}(g)X)$ and $\Phi_2(t) = g\exp_G(tX)g^{-1}$.

From the second and third points we have $\exp_G(X)\exp_G(Y)\exp_G(-X) = \exp_G(\mathrm{e}^{\mathrm{ad}(X)}Y)$. Hence $\exp_G(X)\exp_G(Y)\exp_G(-X) = \exp_G(Y)$ if $\mathrm{ad}(X)Y = 0$. We consider then the 1-parameter subgroups $\Phi_1(t) = \exp_G(tX)\exp_G(tY)$ and $\Phi_2(t) = \exp_G(t(X + Y))$ to prove the second equality of the last point. □

Exercise 2.15. — *We consider the Lie group* $\mathrm{SL}(2, \mathbb{R})$ *with Lie algebra* $\mathfrak{sl}(2, \mathbb{R}) = \{X \in \mathrm{End}(\mathbb{R}^2),\ \mathrm{Tr}(X) = 0\}$. *Show that the image of the exponential map* $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ *is equal to* $\{g \in \mathrm{SL}(2, \mathbb{R}),\ \mathrm{Tr}(g) \geq -2\}$.

Remark 2.16. — *The map* $\exp_G : \mathfrak{g} \to G$ *is in general not surjective. Nevertheless the set* $U = \exp_G(\mathfrak{g})$ *is a neighborhood of the identity, and* $U = U^{-1}$. *The subgroup of G generated by U, which is equal to* $\cup_{n\geq 1} U^n$, *is then a connected open subgroup of G. Hence* $\cup_{n\geq 1} U^n$ *is equal to the connected component of the identity, usually denoted by* G^o.

Exercise 2.17. — *For any Lie group G, show that* $\exp_G(X)\exp_G(Y) =$

$\exp_G(X + Y + \frac{1}{2}[X, Y] + o(|X|^2 + |Y|^2))$ *in a neighborhood of* $(0, 0) \in \mathfrak{g}^2$.

Afterward show that

$$\lim_{n\to\infty}(\exp_G(X/n)\exp_G(Y/n))^n = \exp_G(X+Y) \quad \text{and}$$

$$\lim_{n\to\infty}(\exp_G(X/n)\exp_G(Y/n)\exp_G(-X/n)\exp_G(-Y/n))^{n^2} = \exp([X,Y]).$$

2.6. Lie subgroups and Lie subalgebras. — Before giving the precise definition of a *Lie subgroup*, we look at the infinitesimal side. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ stable under the Lie bracket : $[X,Y]_{\mathfrak{g}} \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.

We have a natural extension of Theorem 2.2.

Theorem 2.18. — *Let H be a* closed *subgroup of a Lie group G. Then H is an embedded submanifold of G, and equipped with this differential structure it is a Lie group. The Lie algebra of H, which is equal to $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H$ for all $t \in \mathbb{R}\}$, is a subalgebra of $\mathfrak{g}$.*

Proof. — The two limits given in Exercise 2.17 show that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ (we use here the fact that H is closed). Let $\mathfrak{a}$ be any supplementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$: one shows that $(\exp(Y) \in H) \Longrightarrow (Y = e)$ if $Y \in \mathfrak{a}$ belongs to a small neighborhood of 0 in $\mathfrak{a}$. Now we consider the map $\phi : \mathfrak{h} \oplus \mathfrak{a} \to G$ given by $\phi(X+Y) = \exp_G(X)\exp_G(Y)$. Since $\mathbf{T}_e\phi$ is the identity map, ϕ defines a smooth diffeomorphism $\phi|_{\mathcal{V}}$ from a neighborhood $\mathcal{V}$ of $0 \in \mathfrak{g}$ to a neighborhood $\mathcal{W}$ of e in G. If $\mathcal{V}$ is small enough we see that ϕ maps $\mathcal{V} \cap \{Y = 0\}$ onto $\mathcal{W} \cap H$, hence H is a submanifold near e. Near any point $h \in H$ we use the map $\phi_h : \mathfrak{h} \oplus \mathfrak{a} \to G$ given by $\phi_h(Z) = h\phi(Z)$: we prove in the same way that H is a submanifold near h. Finally H is an embedded submanifold of G. We now look to the group operations $m_G : G \times G \to G$ (multiplication), $i_G : G \to G$ (inversion) and their restrictions $m_G|_{H\times H} : H \times H \to G$ and $i_G|_H : H \to G$ which are smooth maps. Here we are interested in the group operations m_H and i_H of H. Since $m_G|_{H\times H}$ and $i_G|_H$ are smooth we have the equivalence:

$$m_H \text{ and } i_H \text{ are smooth} \Longleftrightarrow m_H \text{ and } i_H \text{ are continuous.}$$

The fact that m_H and i_H are continuous follows easily from the fact that $m_G|_{H\times H}$ and $i_G|_H$ are continuous and that H is closed. □

Theorem 2.18 has the following important corollary

Corollary 2.19. — *If* $\phi : G \to H$ *is a* continuous *group morphism between two Lie groups, then* ϕ *is smooth.*

Proof. — Consider the graph $L \subset G \times H$ of the map ϕ : $L = \{(g, h) \in G \times H \,|\, h = \phi(g)\}$. Since ϕ is continuous L is a closed subgroup of $G \times H$. Following Theorem 2.18, L is an embedded submanifold of $G \times H$. Consider now the morphism $p_1 : L \to G$ (resp. $p_2 : L \to H$) which is respectively the composition of the inclusion $L \hookrightarrow G \times H$ with the projection $G \times H \to G$ (resp. $G \times H \to H$): p_1 and p_2 are smooth, p_1 is bijective, and $\phi = p_2 \circ (p_1)^{-1}$. Since $(p_1)^{-1}$ is smooth (see Exercise 2.24), the map ϕ is smooth. □

We have just seen the archetype of a Lie subgroup : a closed subgroup of a Lie group. But this notion is too restrictive.

Definition 2.20. — (H, ϕ) *is a* Lie subgroup *of a Lie group* G *if*

- H *is a Lie group,*
- $\phi : H \to G$ *is a group morphism,*
- $\phi : H \to G$ *is a one-to-one immersion.*

In the next example we consider the 1-parameter Lie subgroups of $S^1 \times S^1$: they are either closed or dense.

EXAMPLE : Consider the group morphisms $\phi_\alpha : \mathbb{R} \to S^1 \times S^1$, $\phi_\alpha(t) = (\mathrm{e}^{it}, \mathrm{e}^{i\alpha t})$, defined for $\alpha \in \mathbb{R}$. Then :

• If $\alpha \notin \mathbb{Q}$, $\mathrm{Ker}(\phi_\alpha) = 0$ and $(\mathbb{R}, \phi_\alpha)$ is a Lie subgroup of $S^1 \times S^1$ which is dense.

• If $\alpha \in \mathbb{Q}$, $\mathrm{Ker}(\phi_\alpha) \neq 0$, and ϕ_α factorizes through a smooth morphism $\widetilde{\phi_\alpha} : S^1 \to S^1 \times S^1$. Here $\phi_\alpha(\mathbb{R})$ is a closed subgroup of $S^1 \times S^1$ diffeomorphic to the Lie subgroup $(S^1, \widetilde{\phi_\alpha})$.

Let (H, ϕ) be a Lie subgroup of G, and let $\mathfrak{h}, \mathfrak{g}$ be their respective Lie algebras. Since ϕ is an immersion, the differential at the identity, $d\phi : \mathfrak{h} \to \mathfrak{g}$, is an injective morphism of Lie algebras : $\mathfrak{h}$ is isomorphic with the subalgebra $d\phi(\mathfrak{h})$ of $\mathfrak{g}$. In practice we often "forget" ϕ in our notations, and speak of a Lie subgroup $H \subset G$ with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We have to be careful : when H is not closed in G, the topology of H *is not* the induced topology.

We state now the fundamental

Theorem 2.21. — *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then there exists a unique connected Lie subgroup H of G with Lie algebra equal to $\mathfrak{h}$. Moreover H is generated by $\exp_G(\mathfrak{h})$, where $\exp_G$ is the exponential map of G.*

The proof uses Frobenius Theorem (see [**6**][Theorem 3.19]). This Theorem has an important corollary.

Corollary 2.22. — *Let G, H be two connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. If G is simply connected there exists a (unique) Lie group morphism $\Phi : G \to H$ such that $d\Phi = \phi$.*

Proof. — Consider the graph $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{h}$ of the map ϕ : $\mathfrak{l} := \{(X, Y) \in \mathfrak{g} \times \mathfrak{h} \mid \phi(X) = Y\}$. Since ϕ is a morphism of Lie algebras $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Let (L, ψ) be the connected Lie subgroup of $G \times H$ associated with $\mathfrak{l}$. Consider now the morphism $p_1 : L \to G$ (resp. $p_2 : L \to H$) which equals respectively the composition of $\phi : L \to G \times H$ with the projection $G \times H \to G$ (resp. $G \times H \to H$). The group morphism $p_2 : L \to G$ is onto with a discrete kernel since G is connected and $dp_2 : \mathfrak{l} \to \mathfrak{g}$ is an isomorphism. Hence $p_2 : L \to G$ is a covering map (see Exercise 2.24). Since G is simply connected, this covering map is a diffeomorphism. The group morphism $p_1 \circ (p_2)^{-1} : G \to H$ answers the question. □

Example : The Lie group SU(2) is composed by the 2×2 complex matrices of the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$. Hence SU(2) is simply connected since it is diffeomorphic to the 3-dimensional sphere. Since SU(2) is a maximal compact subgroup of $\mathrm{SL}(2, \mathbb{C})$, the Cartan decomposition (see Section 3.1) tells us that $\mathrm{SL}(2, \mathbb{C})$ is also simply connected.

A subset A of a topological space M is *path-connected* if any points $a, b \in A$ can be joined by a continuous path $\gamma : [0, 1] \to M$ with $\gamma(t) \in A$ for all $t \in [0, 1]$. Any connected Lie subgroup of a Lie group is path-connected. We have the following characterization of the connected Lie subgroups.

Theorem 2.23. — *Let G be a Lie group, and let H be a* path-connected *subgroup of G. Then H is a Lie subgroup of G.*

Exercise 2.24. — *Let $\rho : G \to H$ be a smooth morphism of Lie groups, and let $d\rho : \mathfrak{g} \to \mathfrak{h}$ be the corresponding morphism of Lie algebras.*

- *Show that* $\mathrm{Ker}(\rho) := \{g \in G \,|\, \rho(g) = e\}$ *is a closed (normal) subgroup with Lie algebra* $\mathrm{Ker}(d\rho) := \{X \in \mathfrak{g} \,|\, d\rho(X) = 0\}$.
- *If* $\mathrm{Ker}(d\rho) = 0$, *show that* $\mathrm{Ker}(\rho)$ *is discrete in G. If furthermore ρ is onto, then show that ρ is a covering map.*
- *If $\rho : G \to H$ is bijective, then show that ρ^{-1} is smooth.*

2.7. Ideals. — A subalgebra $\mathfrak{h}$ of a Lie algebra is called an *ideal* in $\mathfrak{g}$ if $[X, Y]_{\mathfrak{g}} \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$: in other words $\mathfrak{h}$ is a stable subspace of $\mathfrak{g}$ under the endomorphism $\mathrm{ad}(Y)$ for any $Y \in \mathfrak{g}$. A Lie subgroup H of the Lie group G is a *normal subgroup* if $gHg^{-1} \subset H$ for all $g \in G$.

Proposition 2.25. — *Let H be the connected Lie subgroup of G associated with the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The following assertions are equivalent.*

1) *H is a normal subgroup of G^o.*

2) *$\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. — **1)** $\Longrightarrow$ **2)**. Let $X \in \mathfrak{h}$ and $g \in G^o$. For every $t \in \mathbb{R}$, the element $g\exp_G(tX)g^{-1} = \exp_G(t\mathrm{Ad}(g)X)$ belongs to H: if we take the derivative at $t = 0$ we get $(*)$ $\mathrm{Ad}(g)X \in \mathfrak{h}$, $\forall g \in G^o$. If we take the differential of $(*)$ at $g = e$ we have $\mathrm{ad}(Y)X \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

2) $\Longrightarrow$ **1)**. If $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have $\exp_G(Y)\exp_G(X)\exp_G(Y)^{-1} = \exp_G(\mathrm{e}^{\mathrm{ad}Y}X) \in H$. Since H is generated by $\exp_G(\mathfrak{h})$, we have $\exp_G(Y)H\exp_G(Y)^{-1} \subset H$ for all $Y \in \mathfrak{g}$ (see Remark 2.16 and Theorem 2.21). Since $\exp_G(\mathfrak{g})$ generates G^o we have finally that $gHg^{-1} \subset H$ for all $g \in G^o$. □

Examples of Ideals : The *center* of $\mathfrak{g}$: $Z_{\mathfrak{g}} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$. The *commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$. The kernel $\ker(\phi)$ of a morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$.

We can associate to any Lie algebra $\mathfrak{g}$ two sequences $\mathfrak{g}_i, \mathfrak{g}^i$ of ideals of $\mathfrak{g}$. The *commutator series* of $\mathfrak{g}$ is the non increasing sequence of ideals $\mathfrak{g}^i$ with

$$\mathfrak{g}^0 = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]. \tag{2.16}$$

The *lower central series* of $\mathfrak{g}$ is the non increasing sequence of ideals $\mathfrak{g}_i$ with

$$\mathfrak{g}_0 = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]. \tag{2.17}$$

Exercise 2.26. — *Show that the* $\mathfrak{g}_i, \mathfrak{g}^i$ *are ideals of* $\mathfrak{g}$.

Definition 2.27. — *We say that* $\mathfrak{g}$ *is*

- solvable *if* $\mathfrak{g}^i = 0$ *for* i *large enough,*
- nilpotent *if* $\mathfrak{g}_i = 0$ *for* i *large enough,*
- abelian *if* $[\mathfrak{g}, \mathfrak{g}] = 0$.

Exercise 2.28. — *Let* V *be a finite dimensional vector space, and let* $\{0\} = V_0 \subset V_1 \subset \cdots V_n = V$ *be a strictly increasing sequence of subspaces. Let* $\mathfrak{g}$ *be the Lie subalgebra of* $\mathfrak{gl}(V)$ *defined by* $\mathfrak{g} = \{X \in \mathfrak{gl}(V) \mid X(V_{k+1}) \subset V_k\}$.

- *Show that the Lie algebra* $\mathfrak{g}$ *is nilpotent.*
- *Suppose now that* $\dim V_k = k$ *for any* $k = 0, \ldots, n$. *Show then that the Lie algebra* $\mathfrak{h} = \{X \in \mathfrak{gl}(V) \mid X(V_k) \subset V_k\}$ *is solvable.*

Exercise 2.29. — *For a group* G, *the subgroup generated by the commutators* $ghg^{-1}h^{-1}$, $g, h \in G$ *is the* derived subgroup, *and is denoted by* G'.

- *Show that* G' *is a normal subgroup of* G.
- *If* G *is a connected Lie group, show that* G' *is* the *connected Lie subgroup associated with the ideal* $[\mathfrak{g}, \mathfrak{g}]$.

Exercise 2.30. — • *For any Lie group* G, *show that its center* $Z_G := \{g \in G \mid hg = hg \ \forall h \in G\}$ *is a closed normal subgroup with Lie algebra* $Z_{\mathfrak{g}} := \{X \in \mathfrak{g} \mid [X, Y] = 0, \ \forall Y \in \mathfrak{g}\}$.

- *Show that a Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is solvable.*
- *Let* $\mathfrak{h}$ *be the Lie algebra defined in Exercise 2.28. Show that* $[\mathfrak{h}, \mathfrak{h}]$ *is nilpotent, and that* $\mathfrak{h}$ *is not nilpotent.*

2.8. Group actions and quotients. — Let M be a set equipped with an action of a group G. For each $m \in M$ the *G-orbit through m* is defined as the subset

$$G \cdot m = \{g \cdot m \mid g \in G\}. \tag{2.18}$$

For each $m \in M$, the *stabilizer group* at m is

$$G_m = \{g \in G \mid g \cdot m = m\}. \tag{2.19}$$

The G-action is *free* if $G_m = \{e\}$ for all $m \in M$. The G-action is *transitive* if $G \cdot m = M$ for some $m \in M$. The set-theoretic quotient M/G corresponds to the quotient of M by the equivalence relation $m \sim n \Longleftrightarrow G \cdot m = G \cdot n$. Let $\pi : M \to M/G$ be the canonical projection.

Topological side : Suppose now that M is a topological space equipped with a continuous action of a topological[3] group G. Note that in this situation the stabilizers G_m are closed in G. We define for any subsets A, B of M the set

$$G_{A,B} = \{g \in G \mid (g \cdot A) \cap B \neq \emptyset\}.$$

Exercise 2.31. — *Show that $G_{A,B}$ is closed in G when A, B are compact in M.*

We take on M/G the *quotient topology*: $\mathcal{V} \subset M/G$ is open if $\pi^{-1}(\mathcal{V})$ is open in M. It is the smallest topology that makes π continuous. Note that $\pi : M \to M/G$ is then an *open map* : if $\mathcal{U}$ is open in M, $\pi^{-1}(\pi(\mathcal{U})) = \cup_{g \in G} g \cdot \mathcal{U}$ is also open in M, which means that $\pi(\mathcal{U})$ is open in M/G.

Definition 2.32. — *The (topological) G-action on M is* proper *when the subsets $G_{A,B}$ are compact in G whenever A, B are compact subsets of M.*

This definition of a proper action is equivalent to the condition that the map $\psi : G \times M \to M \times M$, $(g, m) \mapsto (g \cdot m, m)$ is *proper*, i.e. $\psi^{-1}(\text{compact}) = \text{compact}$. Note that the action of a compact group is always proper.

[3] Here again, the topological spaces are assumed Hausdorff and locally compact.

Proposition 2.33. — *If a topological space M is equipped with a* proper *continuous action of a topological group G, the quotient topology is Hausdorff and locally compact.*

The proof is left to the reader. The main result is the following

Theorem 2.34. — *Let M be a manifold equipped with a smooth, proper and free action of a Lie group. Then the quotient M/G equipped with the quotient topology carries the structure of a smooth manifold. Moreover the projection $\pi : M \to M/G$ is smooth, and any $n \in M/G$ has an open neighborhood $\mathcal{U}$ such that*

$$\begin{array}{rcl} \pi^{-1}(\mathcal{U}) & \xrightarrow{\sim} & \mathcal{U} \times G \\ m & \longmapsto & (\pi(m), \phi_{\mathcal{U}}(m)) \end{array}$$

is a G-equivariant diffeomorphism. Here $\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to G$ is an equivariant map : $\phi_{\mathcal{U}}(g \cdot m) = g\phi_{\mathcal{U}}(m)$.

For a proof see [**1**][Section 2.3].

Remark 2.35. — *Suppose that G is a discrete group. For a proper and free action of G on M we have: any $m \in M$ has a neighborhood $\mathcal{V}$ such that $g\mathcal{V} \cap \mathcal{V} = \emptyset$ for every $g \in G$, $g \neq e$. Theorem 2.34 is true when G is a discrete group. The quotient map $\pi : M \to M/G$ is then a covering map.*

The typical example we are interested in is the action by translation of a closed subgroup H of a Lie group G: the action of $h \in H$ is $G \to G, g \mapsto gh^{-1}$. It is an easy exercise to see that this action is free and proper. The quotient space G/H is a smooth manifold and the action of translation $g \mapsto ag$ of G on itself descends to a smooth action of G on G/H. This action being transitive, the manifolds G/H are thus 'G-homogeneous'.

Stiefel manifolds, Grassmannians : Let V be a (real) vector space of dimension n. For any integer $k \leq n$, let $\mathrm{Hom}(\mathbb{R}^k, V)$ be the vector space of homomorphisms equipped with the following (smooth) $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$-action: for $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$ and $f \in \mathrm{Hom}(\mathbb{R}^k, V)$, we take $(g, h) \cdot f(x) = g(f(h^{-1}x))$ for any $x \in \mathbb{R}^k$. Let $S_k(V)$ be the open subset of $\mathrm{Hom}(\mathbb{R}^k, V)$ formed by the one-to-one linear map : we have a natural identification of $S_k(V)$ with the set of families

$\{v_1, \ldots, v_k\}$ of linearly independent vectors of V. Moreover $S_k(V)$ is stable under the $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$-action : the $\mathrm{GL}(V)$-action on $S_k(V)$ is *transitive*, and the $\mathrm{GL}(\mathbb{R}^k)$-action on $S_k(V)$ is *free and proper*. The manifold $S_k(V)/\mathrm{GL}(\mathbb{R}^k)$ admits a natural identification with the set $\{E \text{ subspace of } V \mid \dim E = k\}$: it is the Grassmanian manifold $\mathrm{Gr}_k(V)$. On the other hand the action of $\mathrm{GL}(V)$ on $\mathrm{Gr}_k(V)$ is transitive so that

$$\mathrm{Gr}_k(V) \cong \mathrm{GL}(V)/H$$

where H is the closed Lie subgroup of $\mathrm{GL}(V)$ that fixes a subspace $E \subset V$ of dimension k.

2.9. Adjoint group. — Let $\mathfrak{g}$ be a (real) Lie algebra. The automorphism group of $\mathfrak{g}$ is

$$\mathrm{Aut}(\mathfrak{g}) := \{\phi \in \mathrm{GL}(\mathfrak{g}) \mid \phi([X,Y]) = [\phi(X), \phi(Y)],\ \forall X, Y \in \mathfrak{g}\}. \tag{2.20}$$

It is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra equal to

$$\mathrm{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D([X,Y]) = [D(X), Y] + [X, D(Y)],\ \forall X, Y \in \mathfrak{g}\}. \tag{2.21}$$

The subspace $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is called the set of *derivations* of $\mathfrak{g}$. Thanks to the Jacobi identity we know that $\mathrm{ad}(X) \in \mathrm{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$. So the image of the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, that we denote $\mathrm{ad}(\mathfrak{g})$, is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$.

Definition 2.36. — *The adjoint group* $\mathrm{Ad}(\mathfrak{g})$ *is the connected Lie subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *associated to the Lie subalgebra of* $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$. *As an abstract group, it is the subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *generated by the elements* $\mathrm{e}^{\mathrm{ad}(X)}$, $X \in \mathfrak{g}$.

Consider now a connected Lie group G, with Lie algebra $\mathfrak{g}$, and the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. In this case, $\mathrm{e}^{\mathrm{ad}(X)} = \mathrm{Ad}(\exp_G(X))$ for any $X \in \mathfrak{g}$, so the image of G by Ad is equal to the group $\mathrm{Ad}(\mathfrak{g})$. If $g \in G$ belongs to the kernel of Ad, we have $g \exp_G(X) g^{-1} = \exp_G(\mathrm{Ad}(g)X) = \exp_G(X)$, so g commutes with any element of $\exp_G(\mathfrak{g})$. But since G is connected, $\exp_G(\mathfrak{g})$ generates G. Finally we have proved that the kernel of Ad is equal to the center Z_G of the Lie group G.

It is worth to keep in mind the following exact sequence of Lie groups

$$0 \longrightarrow Z_G \longrightarrow G \longrightarrow \mathrm{Ad}(\mathfrak{g}) \longrightarrow 0. \tag{2.22}$$

2.10. The Killing form. — We have already defined the notions of solvable and nilpotent Lie algebra (see Def. 2.27). We have the following "opposite" notion.

Definition 2.37. — *Let $\mathfrak{g}$ be a (real) Lie algebra.*

- $\mathfrak{g}$ *is* simple *if* $\mathfrak{g}$ *is not abelian and does not contain any ideal distinct from* $\{0\}$ *and* $\mathfrak{g}$.
- $\mathfrak{g}$ *is* semi-simple *if* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ *where the* $\mathfrak{g}_i$*'s are ideals of* $\mathfrak{g}$ *which are simple (as Lie algebras).*

The following results derive directly from the definition and give a first idea of the difference between "solvable" and "semi-simple".

Exercise 2.38. — *Let $\mathfrak{g}$ be a (real) Lie algebra.*

- *Suppose that $\mathfrak{g}$ is solvable. Show that $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$, and that $\mathfrak{g}$ possesses a non-zero abelian ideal.*
- *Suppose that $\mathfrak{g}$ is semi-simple. Show that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, and show that $\mathfrak{g}$ does not possess non-zero abelian ideals : in particular the center $Z_{\mathfrak{g}}$ is reduced to $\{0\}$.*

In order to give the characterization of semi-simplicity we define the *Killing form* of a Lie algebra $\mathfrak{g}$. It is the symmetric $\mathbb{R}$-bilinear map $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$B_{\mathfrak{g}}(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)), \tag{2.23}$$

where $\mathrm{Tr} : \mathfrak{gl}(\mathfrak{g}) \to \mathbb{R}$ is the canonical trace map.

Proposition 2.39. — *For $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $D \in \mathrm{Der}(\mathfrak{g})$ we have*

- $B_{\mathfrak{g}}(\phi(X),\phi(Y)) = B_{\mathfrak{g}}(X,Y)$, *and*
- $B_{\mathfrak{g}}(DX,Y) + B_{\mathfrak{g}}(X,DY) = 0$ *for all* $X, Y \in \mathfrak{g}$.
- *We have* $B_{\mathfrak{g}}([X,Z],Y) = B_{\mathfrak{g}}(X,[Z,Y])$ *for all* $X,Y,Z \in \mathfrak{g}$.

Proof. — If ϕ is an automorphism of $\mathfrak{g}$, we have $\mathrm{ad}(\phi(X)) = \phi \circ \mathrm{ad}(X) \circ \phi^{-1}$ for all $X \in \mathfrak{g}$ (see (2.20)). Then a) follows and b) comes from the derivative of a) at $\phi = e$. For c) take $D = \mathrm{ad}(Z)$ in b). □

We recall now the basic interaction between the Killing form and the ideals of $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then

• the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{h} \times \mathfrak{h}$ is the Killing form of $\mathfrak{h}$,

• the subspace $\mathfrak{h}^{\perp} = \{X \in \mathfrak{g} \,|\, B_{\mathfrak{g}}(X, \mathfrak{h}) = 0\}$ is an ideal of $\mathfrak{g}$.

• the intersection $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is an ideal of $\mathfrak{g}$ with a Killing form identically equal to 0.

It was shown by E. Cartan that the Killing form gives a criterion for semi-simplicity and solvability.

Theorem 2.40. — *(Cartan's Criterion for Semisimplicity) Let $\mathfrak{g}$ be a (real) Lie algebra. The following statements are equivalent*

a) $\mathfrak{g}$ is semi-simple,

b) the Killing form $B_{\mathfrak{g}}$ is non degenerate,

c) $\mathfrak{g}$ does not have non-zero abelian ideals.

The proof of Theorem 2.40 needs the following characterization of solvability. The reader will find a proof of the following theorem in [**3**][Section I].

Theorem 2.41. — *(Cartan's Criterion for Solvability) Let $\mathfrak{g}$ be a (real) Lie algebra. The following statements are equivalent*

• *$\mathfrak{g}$ is solvable,*

• *$B_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

We will not prove Theorem 2.41, but only use the following easy consequence.

Corollary 2.42. — *If $\mathfrak{g}$ is a (real) Lie algebra with $B_{\mathfrak{g}} = 0$, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.*

Before giving a proof of Theorem 2.40 let us show how Corollary 2.42 gives the implication $b) \Rightarrow a)$ in Theorem 2.41.

If $\mathfrak{g}$ is a Lie algebra with $B_{\mathfrak{g}} = 0$, then Corollary 2.42 tells us that $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ distinct from $\mathfrak{g}$ with $B_{\mathfrak{g}^1} = 0$. If $\mathfrak{g}^1 \neq 0$, we iterate: $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1]$ is an ideal of $\mathfrak{g}^1$ distinct from $\mathfrak{g}^1$ with $B_{\mathfrak{g}^2} = 0$. This induction ends after a finite number of steps: let $i \geq 0$ be such that $\mathfrak{g}^i \neq 0$ and $\mathfrak{g}^{i+1} = 0$. Then $\mathfrak{g}^i$ is an abelian ideal of $\mathfrak{g}$, and $\mathfrak{g}$ is solvable.

In the situation $b)$ of Theorem 2.41, we have then that $[\mathfrak{g}, \mathfrak{g}]$ is solvable, so $\mathfrak{g}$ is also solvable.

Proof. — Proof of Theorem 2.40 using Corollary 2.42
$c) \Longrightarrow b)$. The ideal $\mathfrak{g}^{\perp} = \{X \in \mathfrak{g} \,|\, B_{\mathfrak{g}}(X, \mathfrak{g}) = 0\}$ of $\mathfrak{g}$ has a zero Killing form. If $\mathfrak{g}^{\perp} \neq 0$ we know from the preceding remark that there exists $i \geq 0$ such that $(\mathfrak{g}^{\perp})^i \neq 0$ and $(\mathfrak{g}^{\perp})^{i+1} = 0$. We see easily that $(\mathfrak{g}^{\perp})^i$ is also an ideal of $\mathfrak{g}$ (and is abelian). This gives a contradiction, hence $\mathfrak{g}^{\perp} = 0$: the Killing form $B_{\mathfrak{g}}$ is non-degenerate.

$b) \Longrightarrow a)$. We suppose now that $B_{\mathfrak{g}}$ is non-degenerate. It gives first that $\mathfrak{g}$ is not abelian. Then we use the following dichotomy:

i) either $\mathfrak{g}$ does not have ideals different from $\{0\}$ and $\mathfrak{g}$, hence $\mathfrak{g}$ is simple,

ii) either $\mathfrak{g}$ have an ideal $\mathfrak{h}$ different from $\{0\}$ and $\mathfrak{g}$.

In case *i*) we have finished. In case *ii*), let us show that $\mathfrak{h} \cap \mathfrak{h}^{\perp} \neq 0$: since $B_{\mathfrak{g}}$ is non-degenerate, it will imply that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. If $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{h}^{\perp} \neq 0$, the Killing form on $\mathfrak{a}$ is equal to zero. Following Corollary 2.42 there exists $i \geq 0$ such that $\mathfrak{a}^i \neq 0$ and $\mathfrak{a}^{i+1} = 0$. Moreover since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}^i$ is also an ideal of $\mathfrak{g}$. By considering a supplementary F of $\mathfrak{a}^i$ in $\mathfrak{g}$, every endomorphism $\mathrm{ad}(X), X \in \mathfrak{g}$, has as the following matrix expression

$$\mathrm{ad}(X) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

with $A : \mathfrak{a}^i \to \mathfrak{a}^i$, $B : F \to \mathfrak{a}^i$, and $D : F \to F$. The zero term is due to the fact that $\mathfrak{a}^i$ is an ideal of $\mathfrak{g}$. If $X_o \in \mathfrak{a}^i$, then

$$\mathrm{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

because $\mathfrak{a}^i$ is an abelian ideal. Finally for every $X \in \mathfrak{g}$,

$$\mathrm{ad}(X)\mathrm{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

and then $B_{\mathfrak{g}}(X, X_o) = 0$. It is a contradiction since $B_{\mathfrak{g}}$ is non-degenerate.

So if $\mathfrak{h}$ is an ideal different from $\{0\}$ and $\mathfrak{g}$, we have the $B_{\mathfrak{g}}$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Since $B_{\mathfrak{g}}$ is non-degenerate we see that $B_{\mathfrak{h}}$ and $B_{\mathfrak{h}^{\perp}}$ are non-degenerate, and we apply the dichotomy to the Lie algebras

$\mathfrak{h}$ and $\mathfrak{h}^{\perp}$. After a finite number of steps we obtain a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_k$ are simple ideals of $\mathfrak{g}$.

a) $\Longrightarrow$ *c)*. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projections relatively to a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ into simple ideals: the p_k are Lie algebra morphisms. If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, each $p_k(\mathfrak{a})$ is an abelian ideal of $\mathfrak{g}_k$ which is equal to $\{0\}$ since $\mathfrak{g}_k$ is simple. It proves that $\mathfrak{a} = 0$. □

Exercise 2.43. — • *For the Lie algebra* $\mathfrak{sl}(n, \mathbb{R})$ *show that* $B_{\mathfrak{sl}(n,\mathbb{R})}(X, Y) = 2n\mathrm{Tr}(XY)$. *Conclude that* $\mathfrak{sl}(n, \mathbb{R})$ *is a semi-simple Lie algebra.*

• *For the Lie algebra* $\mathfrak{su}(n)$ *show that* $B_{\mathfrak{su}(n)}(X, Y) = 2n\mathrm{Re}(\mathrm{Tr}(XY))$. *Conclude that* $\mathfrak{su}(n)$ *is a semi-simple Lie algebra.*

Exercise 2.44. — $\mathfrak{sl}(n, \mathbb{R})$ *is a simple Lie algebra.*

Let $(E_{i,j})_{1\le i,j\le n}$ *be the canonical basis of* $\mathfrak{gl}(\mathbb{R}^n)$. *Consider a non-zero ideal* $\mathfrak{a}$ *of* $\mathfrak{sl}(n, \mathbb{R})$. *Up to an exchange of* $\mathfrak{a}$ *with* $\mathfrak{a}^{\perp}$ *we can assume that* $\dim(\mathfrak{a}) \geq \frac{n^2-1}{2}$.

• *Show that* $\mathfrak{a}$ *possesses an element* X *which is not diagonal.*

• *Compute* $[[X, E_{i,j}], E_{i,j}]$ *and conclude that some* $E_{i,j}$ *with* $i \neq j$ *belongs to* $\mathfrak{a}$.

• *Show that* $E_{k,l}, E_{k,k} - E_{l,l} \in \mathfrak{a}$ *when* $k \neq l$. *Conclude.*

2.11. Complex Lie algebras. — We have worked out the notions of solvable, nilpotent, simple and semi-simple *real* Lie algebras. The definitions go through for Lie algebras defined over any field k, and all results of Section 2.10 are still true for $k = \mathbb{C}$.

Let $\mathfrak{h}$ be a *complex* Lie algebra. The Killing form is here a symmetric $\mathbb{C}$-bilinear map $B_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by (2.23), where $\mathrm{Tr} : \mathfrak{gl}_{\mathbb{C}}(\mathfrak{h}) \to \mathbb{C}$ is the trace defined on the $\mathbb{C}$-linear endomorphism of $\mathfrak{h}$.

Theorem 2.40 is valid for the complex Lie algebras: a *complex* Lie algebra is a direct sum of simple ideals if and only if its Killing form is non-degenerate.

A useful tool is the complexification of *real* Lie algebras. If $\mathfrak{g}$ is a real Lie algebra, the complexified vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ carries a canonical structure of complex Lie algebra. We see easily that the Killing forms $B_{\mathfrak{g}}$ and $B_{\mathfrak{g}_{\mathbb{C}}}$ coincide on $\mathfrak{g}$:

(2.24) $$B_{\mathfrak{g}_{\mathbb{C}}}(X,Y) = B_{\mathfrak{g}}(X,Y) \quad \text{for all } X,Y \in \mathfrak{g}.$$

With (2.24) we see that a real Lie algebra $\mathfrak{g}$ is semi-simple if and only if the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is semi-simple.

3. Semi-simple Lie groups

Definition 3.1. — *A connected Lie group G is* semi-simple *(resp.* simple*) if its Lie algebra $\mathfrak{g}$ is semi-simple (resp.* simple*).*

If we use Theorem 2.40 and Proposition 2.25 we have the following characterizations of a semi-simple Lie group, which will be used in the lecture by J. Maubon (see Proposition 6.3).

Proposition 3.2. — *A connected Lie group G is* semi-simple *if and only if G does not contain non-trivial connected normal abelian Lie subgroups.*

In particular the center Z_G of a semi-simple Lie group is discrete. We have the following refinement for the simple Lie groups.

Proposition 3.3. — *A normal subgroup A of a (connected) simple Lie group G which is not equal to G belongs to the center Z of G.*

Proof. — Let A_o be subset of A defined as follows: $a \in A_o$ if there exists a continuous curve $c(t)$ in A with $c(0) = e$ and $c(1) = a$. Obviously A_o is a path-connected subgroup of G, so according to Theorem 2.23 A_o is a Lie subgroup of G. If $c(t)$ is a continuous curve in A, $gc(t)g^{-1}$ is also a continuous curve in A for all $g \in G$, and then A_o is a normal subgroup of G. From Proposition 2.25 we know that the Lie algebra of A_o is an ideal of $\mathfrak{g}$, hence is equal to $\{0\}$ since $\mathfrak{g}$ is simple and $A \neq G$. We have proved that $A_o = \{e\}$, which means that every continuous curve in A is constant. For every $a \in A$ and every continous curve $\gamma(t)$ in G, the continuous curve $\gamma(t)a\gamma(t)^{-1}$ in A must be constant. It proves that A belongs to the center of G. □

We now come back to the exact sequence (2.22).

Lemma 3.4. — *If* $\mathfrak{g}$ *is a semi-simple Lie algebra, the vector space of derivations* $\mathrm{Der}(\mathfrak{g})$ *is equal to* $\mathrm{ad}(\mathfrak{g})$.

Proof. — Let D be a derivation of $\mathfrak{g}$. Since $B_\mathfrak{g}$ is non-degenerate there exists a unique $X_D \in \mathfrak{g}$ such that $\mathrm{Tr}(D\mathrm{ad}(Y)) = B_\mathfrak{g}(X_D, Y)$, for all $Y \in \mathfrak{g}$. Now we compute

$$\begin{aligned} B_\mathfrak{g}([X_D, Y], Z]) = B_\mathfrak{g}(X_D, [Y, Z]) &= \mathrm{Tr}(D\mathrm{ad}([Y, Z])) \\ &= \mathrm{Tr}(D[\mathrm{ad}(Y), \mathrm{ad}(Z)]) \\ &= \mathrm{Tr}([D, \mathrm{ad}(Y)]\mathrm{ad}(Z)) \qquad (1) \\ &= \mathrm{Tr}(\mathrm{ad}(DY)\mathrm{ad}(Z)) \qquad (2) \\ &= B_\mathfrak{g}(DY, Z). \end{aligned}$$

(1) is a general fact about the trace: $\mathrm{Tr}(A[B, C]) = \mathrm{Tr}([A, B]C)$ for any $A, B, C \in \mathfrak{gl}(\mathfrak{g})$. (2) uses the definition of a derivation (see (2.21)). Using now the non-degeneracy of $B_\mathfrak{g}$ we get $D = \mathrm{ad}(X_D)$. □

The equality of Lie algebras $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ tells us that the adjoint group is equal to the identity component of the automorphism group: $\mathrm{Ad}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_o$.

Lemma 3.5. — *If* G *is a (connected) semi-simple Lie group, its center* Z_G *is discrete and the adjoint group* $\mathrm{Ad}(\mathfrak{g})$ *has zero center.*

Proof. — The center $Z(G)$ is discrete because the semi-simple Lie algebra $\mathfrak{g}$ has zero center. Let $\mathrm{Ad}(g)$ be an element of the center of $\mathrm{Ad}(\mathfrak{g})$: we have

$$\begin{aligned} \mathrm{Ad}(\exp_G(X)) = \mathrm{Ad}(g)\mathrm{Ad}(\exp_G(X))\mathrm{Ad}(g)^{-1} &= \mathrm{Ad}(g\exp_G(X)g^{-1}) \\ &= \mathrm{Ad}(\exp_G(\mathrm{Ad}(g)X)) \end{aligned}$$

for any $X \in \mathfrak{g}$. So $\exp_G(-X)\exp_G(\mathrm{Ad}(g)X) \in Z(G)$, $\forall\, X \in \mathfrak{g}$. But since $Z(G)$ is discrete it implies that $\exp_G(X) = \exp_G(\mathrm{Ad}(g)X))$, $\forall\, X \in \mathfrak{g} : g$ commutes with any element of $\exp_G(\mathfrak{g})$. Since $\exp_G(\mathfrak{g})$ generates G, we have finally that $g \in Z(G)$ and so $\mathrm{Ad}(g) = 1$. □

The important point here is that a (connected) semi-simple Lie group is a central extension by a discrete subgroup of a quasi-algebraic group. The Lie group $\mathrm{Aut}(\mathfrak{g})$ is defined by a finite number of polynomial identities in

$\mathrm{GL}(\mathfrak{g})$: it is an algebraic group. And $\mathrm{Ad}(\mathfrak{g})$ is a connected component of $\mathrm{Aut}(\mathfrak{g})$: it is a quasi-algebraic group. There is an important case where the Lie algebra structure imposes some restriction on the center.

***Theorem 3.6* (Weyl).** — *Let G be a connected Lie group such that $B_{\mathfrak{g}}$ is* negative definite. *Then G is a* compact *semi-simple Lie group and the center Z_G is finite.*

There are many proofs, for example [**2**][Section II.6], [**1**][Section 3.9]. Here we only stress that the condition "$B_{\mathfrak{g}}$ is negative definite" imposes that $\mathrm{Aut}(\mathfrak{g})$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, hence $\mathrm{Ad}(\mathfrak{g})$ is compact. Now if we consider the exact sequence $0 \to Z_G \to G \to \mathrm{Ad}(\mathfrak{g}) \to 0$ we see that G is compact if and only if Z_G is finite.

Definition 3.7. — *A real Lie algebra is* compact *if its Killing form is negative definite.*

3.1. Cartan decomposition for subgroups of $\mathrm{GL}(\mathbb{R}^n)$. — Let Sym_n be the vector subspace of $\mathfrak{gl}(\mathbb{R}^n)$ formed by the symmetric endomorphisms, and let Sym_n^+ be the open subspace of Sym_n formed by the positive definite symmetric endomorphisms. Consider the exponential $\mathrm{e} : \mathfrak{gl}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$. We compute its differential.

Lemma 3.8. — *For any $X \in \mathfrak{gl}(\mathbb{R}^n)$, the tangent map $\mathbf{T}_X\mathrm{e} : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ is equal to $\mathrm{e}^X \left(\frac{1-\mathrm{e}^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)$. In particular, $\mathbf{T}_X\mathrm{e}$ is a singular map if and only if the adjoint map $\mathrm{ad}(X) : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ has a non-zero eigenvalue belonging to $2i\pi\mathbb{Z}$.*

Proof. — Consider the smooth functions $F(s,t) = \mathrm{e}^{s(X+tY)}$, and $f(s) = \frac{\partial F}{\partial t}(s,0)$: we have $f(0) = 0$ and $f(1) = \mathbf{T}_X\mathrm{e}(Y)$. If we differentiate F first with respect to t, and after with respect to s, we find that f satisfies the differential equation $f'(s) = Y\mathrm{e}^{sX} + Xf(s)$ which is equivalent to

$$(\mathrm{e}^{-sX}f)' = \mathrm{e}^{-sX}Y\mathrm{e}^{-sX} = \mathrm{e}^{-s\,\mathrm{ad}(X)}Y.$$

Finally we find $f(1) = \mathrm{e}^X(\int_0^1 \mathrm{e}^{-s\,\mathrm{ad}(X)}ds)Y$. □

It is an easy exercise to show that the exponential map realizes a one-to-one map from Sym_n onto Sym_n^+. The last Lemma tells us that $\mathbf{T}_X\mathrm{e}$ is not singular for every $X \in \mathrm{Sym}_n$. So we have proved the

Lemma 3.9. — *The exponential map* $A \mapsto \mathrm{e}^A$ *realizes a smooth diffeomorphism from* Sym_n *onto* Sym_n^+.

Let $\mathrm{O}(\mathbb{R}^n)$ be the orthogonal group : $k \in \mathrm{O}(\mathbb{R}^n) \Longleftrightarrow {}^t kk = Id$. Every $g \in \mathrm{GL}(\mathbb{R}^n)$ decomposes in a unique manner as $g = kp$ where $k \in \mathrm{O}(\mathbb{R}^n)$ and $p \in \mathrm{Sym}_n^+$ is the square root of ${}^t gg$. The map $(k, p) \mapsto kp$ defines a smooth diffeomorphism from $\mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n^+$ onto $\mathrm{GL}(\mathbb{R}^n)$. If we use Lemma 3.9, we get the following

Proposition 3.10 **(Cartan decomposition)**. — *The map*

$$\begin{array}{rcl} \mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n & \longrightarrow & \mathrm{GL}(\mathbb{R}^n) \\ (k, X) & \longmapsto & k\mathrm{e}^X \end{array} \tag{3.25}$$

is a smooth diffeomorphism.

We will now extend the Cartan decomposition to an algebraic[(4)] subgroup G of $\mathrm{GL}(\mathbb{R}^n)$ which is stable under the *transpose map*. In other terms G is stable under the automorphism $\Theta_o : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ defined by

$$\Theta_o(g) = {}^t g^{-1}. \tag{3.26}$$

The classical groups like $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $\mathrm{Sp}(\mathbb{R}^{2n})$ fall into this category. The Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ of G is stable under the transpose map, so we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{o}(n, \mathbb{R})$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}_n$.

Lemma 3.11. — *Let* $X \in \mathrm{Sym}_n$ *such that* $\mathrm{e}^X \in G$. *Then* $\mathrm{e}^{tX} \in G$ *for every* $t \in \mathbb{R}$ *: in other words* $X \in \mathfrak{p}$.

Proof. — The element e^X can be diagonalized : there exist $g \in \mathrm{GL}(\mathbb{R}^n)$ and a sequence of real numbers $\lambda_1, \ldots, \lambda_n$ such that $\mathrm{e}^{tX} = g\,\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})g^{-1}$ for all $t \in \mathbb{R}$ (here $\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})$ is a diagonal matrix). From the hypothesis we have that $\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})$ belongs to the algebraic group $g^{-1}Gg$ when $t \in \mathbb{Z}$. Now it is an easy fact that for any polynomial in n variables P, if $\phi(t) = P(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n}) = 0$ for all $t \in \mathbb{Z}$, then ϕ is identically equal to 0. So we have proved that $\mathrm{e}^{tX} \in G$ for every $t \in \mathbb{R}$ whenever $\mathrm{e}^X \in G$. □

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(4) i.e. defined by a finite number of polynomial equalities.

Consider the Cartan decomposition $g = k\mathrm{e}^X$ of an element $g \in G$. Since G is stable under the transpose map $\mathrm{e}^{2X} = {}^t g g \in G$. From Lemma 3.11 we get that $X \in \mathfrak{p}$ and $k \in G \cap \mathrm{O}(\mathbb{R}^n)$. Finally, if we restrict the diffeomorphism 3.25 to the submanifold $(G \cap \mathrm{O}(\mathbb{R}^n)) \times \mathfrak{p} \subset \mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n$ we get a diffeomorphism

$$(G \cap \mathrm{O}(\mathbb{R}^n)) \times \mathfrak{p} \xrightarrow{\sim} G. \tag{3.27}$$

Let K be the connected Lie subgroup of G associated with the subalgebra $\mathfrak{k}$: K is equal to the identity component of the compact Lie group $G \cap \mathrm{O}(\mathbb{R}^n)$ hence K is compact. If we restrict the diffeomorphism (3.27) to the identity component G^o of G we get the diffeomorphism

$$K \times \mathfrak{p} \xrightarrow{\sim} G^o. \tag{3.28}$$

3.2. Cartan involutions. — We start again with the situation of a closed subgroup G of $\mathrm{GL}(\mathbb{R}^n)$ stable under the transpose map $A \mapsto {}^t A$. Then the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ of G is also stable under the transpose map.

Proposition 3.12. — *If the Lie algebra $\mathfrak{g}$ has a center reduced to* 0*, then $\mathfrak{g}$ is semi-simple. In particular, the bilinear map $(X, Y) \mapsto B_{\mathfrak{g}}(X, {}^t Y)$ defines a scalar product on $\mathfrak{g}$. Moreover if we consider the transpose map $D \mapsto {}^{\mathrm{t}} D$ on $\mathfrak{gl}(\mathfrak{g})$ defined by this scalar product, we have* $\mathrm{ad}({}^t X) = {}^{\mathrm{t}}\mathrm{ad}(X)$ *for all $X \in \mathfrak{g}$.*

Proof. — Consider the scalar product on $\mathfrak{g}$ defined by $(X, Y)_{\mathfrak{g}} := \mathrm{Tr}({}^t X Y)$ where Tr is the canonical trace on $\mathfrak{gl}(\mathbb{R}^n)$. With the help of $(-,-)_{\mathfrak{g}}$, we have a transpose map $D \mapsto {}^{\mathrm{T}} D$ on $\mathfrak{gl}(\mathfrak{g})$: $(D(X), Y)_{\mathfrak{g}} = (X, {}^{\mathrm{T}} D(Y))_{\mathfrak{g}}$ for all $X, Y \in \mathfrak{g}$ and $D \in \mathfrak{gl}(\mathfrak{g})$. A small computation shows that ${}^{\mathrm{T}}\mathrm{ad}(X) = \mathrm{ad}({}^t X)$, and then $B_{\mathfrak{g}}(X, {}^t Y) = \mathrm{Tr}'(\mathrm{ad}(X)\, {}^{\mathrm{T}}\mathrm{ad}(Y))$ defines a symmetric bilinear map on $\mathfrak{g} \times \mathfrak{g}$ (here Tr' is the trace map on $\mathfrak{gl}(\mathfrak{g})$). If $\mathfrak{g}$ has zero center then $B_{\mathfrak{g}}(X, {}^t X) > 0$ if $X \neq 0$. Let $D \mapsto {}^{\mathrm{t}} D$ be the transpose map on $\mathfrak{gl}(\mathfrak{g})$ defined by this scalar product. We have

$$B_{\mathfrak{g}}(\mathrm{ad}(X)Y, {}^t Z) = -B_{\mathfrak{g}}(Y, [X, {}^t Z]) = B_{\mathfrak{g}}(Y, {}^t[{}^t X, Z]) = B_{\mathfrak{g}}(Y, {}^t(\mathrm{ad}({}^t X) Z)),$$

for all $X, Y, Z \in \mathfrak{g}$: in other terms $\mathrm{ad}({}^t X) = {}^{\mathrm{t}}\mathrm{ad}(X)$. □

Definition 3.13. — *A linear map $\tau : \mathfrak{g} \to \mathfrak{g}$ on a Lie algebra is an* involution *if τ is an* automorphism *of the Lie algebra $\mathfrak{g}$ and $\tau^2 = 1$.*

When τ is an involution of $\mathfrak{g}$, we define the bilinear map

$$B^\tau(X,Y) := -B_{\mathfrak{g}}(X,\tau(Y)) \tag{3.29}$$

which is *symmetric*. We have the decomposition

$$\mathfrak{g} = \mathfrak{g}_1^\tau \oplus \mathfrak{g}_{-1}^\tau \tag{3.30}$$

where $\mathfrak{g}_{\pm 1}^\tau = \{X \in \mathfrak{g} \mid \tau(X) = \pm X\}$. Since $\tau \in \mathrm{Aut}(\mathfrak{g})$ we have

$$[\mathfrak{g}_\varepsilon^\tau, \mathfrak{g}_{\varepsilon'}^\tau] \subset \mathfrak{g}_{\varepsilon\varepsilon'}^\tau \quad \text{for all} \quad \varepsilon, \varepsilon' \in \{1,-1\}, \tag{3.31}$$

and

$$B_{\mathfrak{g}}(X,Y) = 0 \quad \text{for all} \quad X \in \mathfrak{g}_1^\tau,\ Y \in \mathfrak{g}_{-1}^\tau. \tag{3.32}$$

The subspace[(5)] $\mathfrak{g}^\tau$ is a sub-algebra of $\mathfrak{g}$, $\mathfrak{g}_{-1}^\tau$ is a module for $\mathfrak{g}^\tau$ through the adjoint action, and the subspace $\mathfrak{g}^\tau$ and $\mathfrak{g}_{-1}^\tau$ are *orthogonal* with respect to B^τ.

Definition 3.14. — *An involution θ on a Lie algebra $\mathfrak{g}$ is a* Cartan involution *if the symmetric bilinear map B^θ defines a scalar product on $\mathfrak{g}$.*

Note that the existence of a Cartan involution implies the semi-simplicity of the Lie algebra.

EXAMPLE : $\theta_o(X) = -{}^tX$ is an involution on the Lie algebra $\mathfrak{gl}(\mathbb{R}^n)$. We prove in Proposition 3.12 that if a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ is stable under the transpose map and has zero center, then the linear map θ_o restricted to $\mathfrak{g}$ is a Cartan involution. It is the case, for example, of the subalgebras $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{o}(p,q)$.

In the other direction, if a semi-simple Lie algebra $\mathfrak{g}$ is equipped with a Cartan involution θ, a small computation shows that

$$^t\mathrm{ad}(X) = -\mathrm{ad}(\theta(X)), \quad X \in \mathfrak{g},$$

where $A \mapsto {}^tA$ is the transpose map on $\mathfrak{gl}(\mathfrak{g})$ defined by the scalar product B^θ. So the subalgebra $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, which is isomorphic to $\mathfrak{g}$, is stable

(5) From now on, we will just denote by $\mathfrak{g}^\tau$ the subalgebra $\mathfrak{g}_1^\tau$.

under the transpose map. Conclusion : for a real Lie algebra $\mathfrak{g}$ with zero center, the following statements are equivalent :

• $\mathfrak{g}$ can be realized as a subalgebra of matrices stable under the transpose map,

• $\mathfrak{g}$ is a semi-simple Lie algebra equipped with a Cartan involution.

In the next section, we will see that any real semi-simple Lie algebra has a Cartan involution.

3.3. Compact real forms. — We have seen the notion of *complexification* of a real Lie algebra. In the other direction, a complex Lie algebra $\mathfrak{h}$ can be considered as a *real* Lie algebra and we then denote it by $\mathfrak{h}^{\mathbb{R}}$. The behaviour of the Killing form with respect to this operation is

$$B_{\mathfrak{h}^{\mathbb{R}}}(X,Y) = 2\,\mathrm{Re}(B_{\mathfrak{h}}(X,Y)) \quad \text{for all } X,Y \in \mathfrak{h}. \tag{3.33}$$

For a complex Lie algebra $\mathfrak{h}$, we speak of *anti-linear involutions* : these are the involutions of $\mathfrak{h}^{\mathbb{R}}$ which anti-commute with the complex multiplication. If τ is an anti-linear involution of $\mathfrak{h}$ then $\mathfrak{h}^{\tau}_{-1} = i\mathfrak{h}^{\tau}$, i.e.

$$\mathfrak{h} = \mathfrak{h}^{\tau} \oplus i\mathfrak{h}^{\tau}. \tag{3.34}$$

Definition 3.15. — *A* real form *of a complex Lie algebra* $\mathfrak{h}$ *is a real subalgebra* $\mathfrak{a} \subset \mathfrak{h}^{\mathbb{R}}$ *such that* $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$*, i.e.* $\mathfrak{a}_{\mathbb{C}} \simeq \mathfrak{h}$*. A* compact real form *of a complex Lie algebra is a* real form *which is a* compact *Lie algebra (see Def. 3.7).*

For any real form $\mathfrak{a}$ of $\mathfrak{h}$, there exists a unique anti-linear involution τ such that $\mathfrak{h}^{\tau} = \mathfrak{a}$. Equation (3.34) tells us that $\tau \mapsto \mathfrak{h}^{\tau}$ is a one-to-one correspondence between the *anti-linear involutions* of $\mathfrak{h}$ and the *real forms* of $\mathfrak{h}$. If $\mathfrak{a}$ is a real form of a complex Lie algebra $\mathfrak{h}$, we have like in (2.24) that

$$B_{\mathfrak{a}}(X,Y) = B_{\mathfrak{h}}(X,Y) \quad \text{for all } X,Y \in \mathfrak{a}. \tag{3.35}$$

In particular $B_{\mathfrak{h}}$ takes real values on $\mathfrak{a} \times \mathfrak{a}$.

Lemma 3.16. — *Let θ be an anti-linear involution of a complex Lie algebra $\mathfrak{h}$. θ is a Cartan involution of the real Lie algebra $\mathfrak{h}^{\mathbb{R}}$ if and only if $\mathfrak{h}^{\theta}$ is a compact real form of $\mathfrak{h}$.*

Proof. — Consider the decomposition $\mathfrak{h} = \mathfrak{h}^\theta \oplus i\mathfrak{h}^\theta$ and $X = a + ib$ with $a, b \in \mathfrak{h}^\theta$. We have

$$\begin{aligned} B_{\mathfrak{h}^{\mathbb{R}}}(X, \theta(X)) &= 2(B_{\mathfrak{h}}(a,a) + B_{\mathfrak{h}}(b,b)) \qquad (1) \\ &= 2(B_{\mathfrak{h}^\theta}(a,a) + B_{\mathfrak{h}^\theta}(b,b)) \qquad (2). \end{aligned}$$

Relations (1) and (2) are consequences of (3.33) and (3.35). So we see that $-B^\theta_{\mathfrak{h}^{\mathbb{R}}}$ is positive definite on $\mathfrak{h}^{\mathbb{R}}$ if and only if the Killing form $B_{\mathfrak{h}^\theta}$ is negative definite. □

EXAMPLE : the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is a real form of $\mathfrak{sl}(n, \mathbb{C})$. The complex Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ has other real forms like

- $\mathfrak{su}(n) = \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid {}^t\overline{X} + X = 0\}$,
- $\mathfrak{su}(p, q) = \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid {}^t\overline{X}\mathrm{I}_{p,q} + \mathrm{I}_{p,q}X = 0\}$, where $\mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$.

Here the anti-linear involutions are respectively $\sigma(X) = \overline{X}$, $\sigma_a(X) = -{}^t\overline{X}$, and $\sigma_b(X) = -\mathrm{I}_{p,q}{}^t\overline{X}\mathrm{I}_{p,q}$. Among the real forms $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(n)$, $\mathfrak{su}(p, q)$ of $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{su}(n)$ is the only one which is compact.

Let $\mathfrak{g}$ be a real Lie algebra, and let σ be the anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ associated with the real form $\mathfrak{g}$. We have a one-to-one correspondence

$$\tau \mapsto \mathfrak{u}(\tau) := (\mathfrak{g}_{\mathbb{C}})^{\tau\circ\sigma} \tag{3.36}$$

between the set of involutions of $\mathfrak{g}$ and the set of real forms of $\mathfrak{g}_{\mathbb{C}}$ which are σ-stable. If τ is an involution of $\mathfrak{g}$, we consider its $\mathbb{C}$-linear extension to $\mathfrak{g}_{\mathbb{C}}$ (that we still denote by τ). The composite $\tau \circ \sigma = \sigma \circ \tau$ is then an anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ which commutes with σ: hence the real form $\mathfrak{u}(\tau) := (\mathfrak{g}_{\mathbb{C}})^{\tau\circ\sigma}$ is stable under σ. If $\mathfrak{a}$ is a real form on $\mathfrak{g}_{\mathbb{C}}$ defined by an anti-linear involution ρ which commutes with σ, then $\sigma \circ \rho$ is a $\mathbb{C}$-linear involution on $\mathfrak{g}_{\mathbb{C}}$ which commutes with σ: it is then the complexification of an involution τ on $\mathfrak{g}$, and we have $\mathfrak{a} = \mathfrak{u}(\tau)$.

Proposition 3.17. — *Let $\mathfrak{g}$ be a real semi-simple Lie algebra. Let τ be an involution of $\mathfrak{g}$ and let $\mathfrak{u}(\tau)$ be the real form of $\mathfrak{g}_{\mathbb{C}}$ defined by (3.36). The following statements are equivalent:*

- *τ is a Cartan involution of $\mathfrak{g}$,*
- *$\mathfrak{u}(\tau)$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ (which is σ-stable).*

Proof. — If $\mathfrak{g} = \mathfrak{g}^\tau \oplus \mathfrak{g}^\tau_{-1}$ is the decomposition related to the eigenspaces of τ then $\mathfrak{u}(\tau) = \mathfrak{g}^\tau \oplus i\,\mathfrak{g}^\tau_{-1}$. Take $X = a + ib \in \mathfrak{u}(\tau)$ with $a \in \mathfrak{g}^\tau$ and $b \in \mathfrak{g}^\tau_{-1}$. We have

$$\begin{aligned} B_{\mathfrak{u}(\tau)}(X, X) &= B_{\mathfrak{g}_\mathbb{C}}(X, X) \qquad (1) \\ &= B_\mathfrak{g}(a, a) - B_\mathfrak{g}(b, b) \quad (2) \\ &= -B^\tau_\mathfrak{g}(\tilde{X}, \tilde{X}), \end{aligned}$$

where $\tilde{X} = a + b \in \mathfrak{g}$. (1) is due to (3.35). In (2) we use (2.24) and the fact that $\mathfrak{g}^\tau$ and $\mathfrak{g}^\tau_{-1}$ are $B_\mathfrak{g}$-orthogonal. Then we see that $B_{\mathfrak{u}(\tau)}$ is negative definite if and only if $B^\tau_\mathfrak{g}$ is positive definite. □

Now we give a way to prove that a real semi-simple Lie algebra $\mathfrak{g}$ has a Cartan involution. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$ and let σ the anti-linear involution of $\mathfrak{g}_\mathbb{C}$ corresponding to the real form $\mathfrak{g}$. We know from Proposition 3.17 that it is equivalent to seek for the σ-stable compact real forms of $\mathfrak{g}_\mathbb{C}$. We use first the following fundamental fact.

Theorem 3.18. — *Any complex semi-simple Lie algebra has a compact real form.*

A proof can be found in [**3**][Section 7.1]. The existence of a σ-stable compact real form is given by the following

Lemma 3.19. — *Let $\tau : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ be an anti-linear involution corresponding to a compact real form of $\mathfrak{g}_\mathbb{C}$. There exists $\phi \in \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ such that the anti-linear involution $\phi\tau\phi^{-1}$ commutes with σ. Hence $\phi\tau\phi^{-1}|_\mathfrak{g}$ is a Cartan involution of to $\mathfrak{g}$.*

Proof. — The complex vector space $\mathfrak{g}_\mathbb{C}$ is equipped with the hermitian metric : $(X, Y) \to B_{\mathfrak{g}_\mathbb{C}}(X, \tau(Y))$. It easy to check that $\tau\sigma$ belongs to the intersection

(3.37)

$$\mathrm{Aut}(\mathfrak{g}_\mathbb{C}) \cap \{\text{hermitian endomorphisms}\} = \{\phi \in \mathrm{Aut}(\mathfrak{g}_\mathbb{C}) \mid \tau\phi\tau = \phi^{-1}\}$$

The map $\rho = (\tau\sigma)^2$ is positive definite. Following Lemma 3.11, the one parameter subgroup $r \in \mathbb{R} \mapsto \rho^r$ belongs to the identity component $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})_o$ (since $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ is an algebraic subgroup of $\mathrm{GL}((\mathfrak{g}_\mathbb{C})^\mathbb{R})$). We leave as an exercise to check that ρ^r commutes with $\tau\sigma$ for all $r \in \mathbb{R}$.

Since $\tau\rho^r\tau = \rho^{-r}$ (see (3.37)) it is easy to see that $\rho^r\tau\rho^{-r}$ commutes with σ if $r = \frac{-1}{4}$. □

3.4. Cartan decomposition on the group level. — Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$. So we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}^\theta_{-1}$ is a $\mathfrak{k}$-module. Let K be the connected Lie subgroup of G associated with $\mathfrak{k}$. This section is devoted to the proof of the following

Theorem 3.20. — *(a) K is a closed subgroup of G*

(b) the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k\exp_G(X)$ is a diffeomorphism onto

(c) K contains the center Z of G

(d) K is compact if and only if Z is finite

(e) there exists a Lie group automorphism Θ of G, with $\Theta^2 = 1$ and with differential θ

(f) the subgroup of G fixed by Θ is K.

Proof. — The Lie group $\widehat{G} = \mathrm{Ad}(\mathfrak{g})$ which is equal to the image of G by the adjoint action is the identity component of $\mathrm{Aut}(\mathfrak{g})$. The Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ which is equal to the subspace of derivations $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is stable under the transpose map $A \mapsto {}^tA$ on $\mathfrak{gl}(\mathfrak{g})$ associated with the scalar product B_θ on $\mathfrak{g}$ (since $-{}^t\mathrm{ad}(X) = \mathrm{ad}(\theta(X))$). Since $\widehat{G}$ is generated by the elements $\mathrm{e}^{\mathrm{ad}(X)}$, $X \in \mathfrak{g}$, $\widehat{G}$ is stable under the group morphism $A \mapsto {}^tA^{-1}$. We have $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} \oplus \widehat{\mathfrak{p}}$ where $\widehat{\mathfrak{k}} = \{A \in \widehat{\mathfrak{g}} \mid {}^tA = -A\}$ and $\widehat{\mathfrak{p}} = \{A \in \widehat{\mathfrak{g}} \mid {}^tA = A\}$. We have of course $\widehat{\mathfrak{g}} = \mathrm{ad}(\mathfrak{g})$, $\widehat{\mathfrak{k}} = \mathrm{ad}(\mathfrak{k})$ and $\widehat{\mathfrak{p}} = \mathrm{ad}(\mathfrak{p})$. Let $\widehat{K}$ be the compact Lie group equal to $\widehat{G} \cap \mathrm{O}(\mathfrak{g})$: its Lie algebra is $\widehat{\mathfrak{k}}$. Since $\mathrm{Aut}(\mathfrak{g})$ is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$, (3.28) applies here and gives the diffeomorphism

$$\begin{aligned} \widehat{K} \times \widehat{\mathfrak{p}} &\longrightarrow \widehat{G} \\ (k, A) &\longmapsto k\mathrm{e}^A. \end{aligned} \tag{3.38}$$

We consider the *closed* Lie subgroup

$$K := \mathrm{Ad}^{-1}(\widehat{K})$$

of G : its Lie algebra is $\mathfrak{k}$. By definition K contains the center $Z = \mathrm{Ad}^{-1}(\mathrm{Id})$ of G. If we take the pull-back of (3.38) through $\mathrm{Ad} : G \to \widehat{G}$

we get the diffeomorphism

$$\begin{array}{rcl} K \times \mathfrak{p} & \longrightarrow & G \\ (k, X) & \longmapsto & k \exp_G(X), \end{array} \tag{3.39}$$

which proves that K is connected since G is connected : hence K is the connected Lie subgroup of G associated with the Lie subalgebra $\mathfrak{k}$. Finally Z belongs to K and $K/Z \simeq \widehat{K}$ is compact: the points (a), (b), (c) and (d) are proved.

Let $\Theta : G \to G$ defined by $\Theta(k \exp_G(X)) = k \exp_G(-X)$ for $k \in K$ and $X \in \mathfrak{p}$. We have obviously $\Theta^2 = 1$ and $\mathrm{Ad}(\Theta(g)) = {}^t\mathrm{Ad}(g)^{-1}$. If we take g_1, g_2 in G we see that

$$\begin{array}{rcl} \mathrm{Ad}(\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1}) & = & \left({}^t(\mathrm{Ad}(g_1)\mathrm{Ad}(g_2))^{-1}\right)\left({}^t\mathrm{Ad}(g_2)\right)\left({}^t\mathrm{Ad}(g_1)\right) \\ & = & 1. \end{array}$$

So $\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1} \in Z$ for every g_1, g_2 in G. Since G is connected and Z is discrete it gives $\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1} = 1$: (e) and (f) are proved. □

4. Invariant connections

A connection ∇ on the tangent bundle $\mathbf{T}M$ of a manifold M is a differential linear operator

$$\nabla : \Gamma(\mathbf{T}M) \longrightarrow \Gamma(\mathbf{T}^*M \otimes \mathbf{T}M) \tag{4.40}$$

satisfying Leibnitz's rule: $\nabla(fs) = df \otimes s + f\nabla s$ for every $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(\mathbf{T}M)$. Here $\Gamma(-)$ denotes the space of sections of the corresponding bundle. The contraction of ∇s by $v \in \Gamma(\mathbf{T}M)$ is a vector field on M denoted $\nabla_v s$.

The *torsion* of a connection ∇ on $\mathbf{T}M$ is the $(2,1)$-tensor T^∇ defined by

$$T^\nabla(u, v) = \nabla_u v - \nabla_v u - [u, v], \tag{4.41}$$

for all vector fields u, v on M. The *curvature* of a connection ∇ on $\mathbf{T}M$ is the $(3,1)$-tensor R^∇ defined by

$$R^\nabla(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}, \tag{4.42}$$

for all vector fields u, v on M. Here $R^\nabla(u,v)$ is a differential operator acting on $\Gamma(\mathbf{T}M)$ which commutes with the multiplication by functions on M : so it is defined by the action of an element of $\Gamma(\mathrm{End}(\mathbf{T}M))$. For convenience we denote by $R^\nabla(u,v) \in \Gamma(\mathrm{End}(\mathbf{T}M))$ this element. We can specialize the curvature tensor R^∇ at each $m \in M$: $R^\nabla_m(U,V) \in \mathrm{End}(\mathbf{T}_m M)$ for each $U, V \in \mathbf{T}_m M$.

4.1. Connections invariant under a group action. — Suppose now that a Lie group G acts smoothly on a manifold M. The corresponding action of G on the vector spaces $\mathcal{C}^\infty(M)$, $\Gamma(\mathbf{T}M)$ and $\Gamma(\mathbf{T}^*M)$ is

$$\underline{g} \cdot f(m) = f(g^{-1}m), \quad m \in M,$$

$$\underline{g} \cdot s(m) = \mathbf{T}_{g^{-1}m} g(s(g^{-1}m)), \quad m \in M,$$

and

$$\underline{g} \cdot \xi(m) = \xi(g^{-1}m) \circ \mathbf{T}_m g^{-1}, \quad m \in M,$$

for every $f \in \mathcal{C}^\infty(M)$, $s \in \Gamma(\mathbf{T}M))$, $\xi \in \Gamma(\mathbf{T}^*M)$ and $g \in G$. Here we denote by $\mathbf{T}_n g$ the differential at $n \in M$ of the smooth map $m \mapsto gm$. Note that the G-action is compatible with the canonical bracket $\langle -, - \rangle$: $\Gamma(\mathbf{T}^*M) \times \Gamma(\mathbf{T}M) \to \mathcal{C}^\infty(M)$: $\langle \underline{g} \cdot \xi, \underline{g} \cdot s \rangle = \underline{g} \cdot \langle \xi, s \rangle$. We still denote by $\underline{g}$ the action of $g \in G$ on $\Gamma(\mathbf{T}^*M \otimes \mathbf{T}M)$.

Definition 4.1. — *A connection ∇ on the tangent bundle $\mathbf{T}M$ is G-invariant if*

$$\underline{g} \nabla \underline{g}^{-1} = \nabla, \quad \text{for every } g \in G. \tag{4.43}$$

This condition is equivalent to asking that $\nabla_{\underline{g} \cdot v}(\underline{g} \cdot s) = \underline{g} \cdot (\nabla_v s)$ for every vector fields s, v on M and $g \in G$.

For every $X \in \mathfrak{g}$, the differential of $t \to \underline{\exp_G(tX)}$ at $t = 0$ defines linear operators on $\mathcal{C}^\infty(M)$, $\Gamma(\mathbf{T}M)$ and $\Gamma(\mathbf{T}^*M)$, all denoted by $\mathcal{L}(X)$. For $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(M)$ we have $\mathcal{L}(X)f = X_M(f)$ and $\mathcal{L}(X)s = [X_M, s]$ where X_M is the vector field on M defined in Section 2.4. The map $X \mapsto \mathcal{L}(X)$ is a Lie algebra morphism :

$$[\mathcal{L}(X), \mathcal{L}(Y)] = \mathcal{L}([X,Y]), \quad \text{for all} \quad X, Y \in \mathfrak{g}. \tag{4.44}$$

Definition 4.2. — *The* moment *of a G-invariant connection ∇ on $\mathbf{T}M$ is the linear endomorphism of $\Gamma(\mathbf{T}M)$ defined by*

$$\Lambda(X) = \mathcal{L}(X) - \nabla_{X_M}, \quad X \in \mathfrak{g}. \tag{4.45}$$

Since $\Lambda(X)$, $X \in \mathfrak{g}$, commutes with the multiplication by functions on M, we can and we will see $\Lambda(X)$ as an element of $\Gamma(\mathrm{End}(\mathbf{T}M))$. The invariance condition (4.43) tells us that the map $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ is G-equivariant:

$$\Lambda(\mathrm{Ad}(g)Y) = \underline{g}\Lambda(Y)\underline{g}^{-1}, \quad \text{for every} \quad (g, Y) \in G \times \mathfrak{g}. \tag{4.46}$$

If we differentiate (4.46) at $g = 1$, we get

$$\Lambda([X, Y]) = [\mathcal{L}(X), \Lambda(Y)], \quad \text{for every } X, Y \in \mathfrak{g}. \tag{4.47}$$

We end this section by computing the values of the torsion and of the curvature on vector fields generated by the G-action. A direct computation gives

$$T^\nabla(X_M, Y_M) = [X, Y]_M - \Lambda(X)Y_M + \Lambda(Y)X_M. \tag{4.48}$$

for every $X, Y \in \mathfrak{g}$. Now using (4.44) and (4.47) we have for the curvature

$$R^\nabla(X_M, Y_M) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]), \tag{4.49}$$

for every $X, Y \in \mathfrak{g}$.

4.2. Invariant Levi-Civita connections. — Suppose now that the manifold M carries a Riemannian structure invariant under the Lie group G. The scalar product of two vector fields u, v will be denoted by (u, v). The invariance condition says that the equality

$$\underline{g} \cdot (u, v) = (\underline{g} \cdot u, \underline{g} \cdot v) \tag{4.50}$$

holds in $\mathcal{C}^\infty(M)$ for $u, v \in \Gamma(\mathbf{T}M)$ and $g \in G$. If we differentiate (4.50) at $g = e$ we get

$$X_M(u, v) = ([X_M, u], v) + (u, [X_M, v]). \tag{4.51}$$

Let ∇^{LC} be the Levi-Civita connection on M relatively to the Riemannian metric: it is the unique torsion free connection which preserves the Riemannian metric. Since the Riemannian metric is G-invariant, the connection $\underline{g}\nabla^{\mathrm{LC}}\underline{g}^{-1}$ preserves also the Riemannian metric and is torsion

free for every $g \in G$. Hence ∇^{LC} is a G-invariant connection. Recall that for $u, v \in \Gamma(\mathbf{T}M)$ the vector field $\nabla^{\mathrm{LC}}_u v$ is defined by the relations
(4.52)
$$2(\nabla^{\mathrm{LC}}_u v, w) = ([u,v],w) - ([v,w],u) + ([w,u],v) + u(v,w) + v(u,w) - w(u,v).$$

If we take $u = X_M$ and $v = Y_M$ in the former relation we find with the help of (4.51) that

$$2(\nabla^{\mathrm{LC}}_{X_M} Y_M, w) = ([X,Y]_M, w) - w(X_M, Y_M). \tag{4.53}$$

So we have proved the

Proposition 4.3. — *For any $X, Y \in \mathfrak{g}$ we have*
$$\nabla^{\mathrm{LC}}_{X_M} Y_M = \frac{1}{2}\Big([X,Y]_M - \overrightarrow{\mathrm{grad}}(X_M, Y_M)\Big).$$

5. Invariant connections on homogeneous spaces

The main references for this section are [**2**] and [**4**].

5.1. Existence of invariant connections. — We work here with the homogeneous space $M = G/H$ where H is a closed subgroup with Lie algebra $\mathfrak{h}$ of a Lie group G. We denote by $\pi : G \to M$ the quotient map. The quotient vector space $\mathfrak{g}/\mathfrak{h}$ is equipped with the H-action induced by the adjoint action. We consider the space $G \times \mathfrak{g}/\mathfrak{h}$ with the following H-action: $h \cdot (g, \overline{X}) = (gh^{-1}, \overline{\mathrm{Ad}(h)X})$. This action is proper and free so the quotient $G \times_H \mathfrak{g}/\mathfrak{h}$ is a smooth manifold: the class of $(g, \overline{X})$ in $G \times_H \mathfrak{g}/\mathfrak{h}$ is denoted by $[g, \overline{X}]$. We use here the following G-equivariant isomorphism

$$\begin{aligned} G \times_H \mathfrak{g}/\mathfrak{h} &\longrightarrow \mathbf{T}M \\ [g, \overline{X}] &\longmapsto \frac{d}{dt}\pi(g \exp_G(tX))|_{t=0}. \end{aligned} \tag{5.54}$$

Using (5.54) we have

$$\begin{aligned} \Gamma(\mathbf{T}M) &\xrightarrow{\sim} (\mathcal{C}^\infty(G) \otimes \mathfrak{g}/\mathfrak{h})^H \\ s &\mapsto \widetilde{s} \end{aligned} \tag{5.55}$$

and

$$\begin{array}{lccc} (5.56) & \Gamma(\mathrm{End}(\mathbf{T}M)) & \xrightarrow{\sim} & (\mathcal{C}^\infty(G)\otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h}))^H \\ & A & \mapsto & \widetilde{A}. \end{array}$$

For example, the vector fields X_M, $X \in \mathfrak{g}$, give rise through the isomorphism (5.55) to the functions $\widetilde{X_M}(g) = -\mathrm{Ad}(g)^{-1}X \bmod \mathfrak{g}/\mathfrak{h}$.

Let ∇ be a G-invariant connection on the tangent bundle $\mathbf{T}M$, and let $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ be the corresponding G-equivariant map defined by (4.45). Let $\widetilde{\Lambda} : \mathfrak{g} \to (\mathcal{C}^\infty(G)\otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h}))^H$ be the map Λ through the identifications (5.56). The mapping $\widetilde{\Lambda}$ is G-equivariant and each $\widetilde{\Lambda}(X), X \in \mathfrak{g}$ is a H-equivariant map from G to $\mathrm{End}(\mathfrak{g}/\mathfrak{h})$:

$$\begin{array}{lrcl} (5.57) & \widetilde{\Lambda}(\mathrm{Ad}(g)X)(g') & = & \widetilde{\Lambda}(X)(g^{-1}g') \\ & \widetilde{\Lambda}(X)(gh^{-1}) & = & \mathrm{Ad}(h)\circ\widetilde{\Lambda}(X)(g)\circ \mathrm{Ad}(h)^{-1} \end{array}$$

for every $g, g' \in G$, $h \in H$ and $X \in \mathfrak{g}$.

Definition 5.1. — *Let* $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ *be the map defined by* $\lambda(X) = \widetilde{\Lambda}(X)(e)$.

From (5.57), we see that λ is H-equivariant and determines completely Λ :

$$(5.58) \qquad \widetilde{\Lambda}(X)(g) = \lambda(\mathrm{Ad}(g)^{-1}X).$$

So we have proved that the G-invariant connection ∇ is uniquely determined by the mapping $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$.

Proposition 5.2. — (a) *The linear map* $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ *is* H-*equivariant, and when restricted to* $\mathfrak{h}$, λ *is equal to the adjoint action.*

(b) *A linear map* λ *satisfying the conditions of* (a) *determines a unique* G-*invariant connection on* $\mathbf{T}(G/H)$.

Proof. — We have $\Lambda(X) = \mathcal{L}(X) - \nabla_{X_M}$. So if $X_M(m) = 0$ [(6)], we have $\Lambda(X)_m = \mathcal{L}(X)_m$ as endomorphisms of $\mathbf{T}_mM$. When $m = \overline{e} \in M$, $X_M(\overline{e}) = 0$ if and only if $X \in \mathfrak{h}$, and then the endomorphism $\mathcal{L}(X)_{\overline{e}}$ of

(6) $X_M(m) = 0$ if and only if m is fixed by the 1-parameter subgroup $\exp_G(\mathbb{R}X)$.

$\mathbf{T}_{\overline{e}}M = \mathfrak{g}/\mathfrak{h}$ is equal to $\mathrm{ad}(X)$. So $\lambda(X) = \mathrm{ad}(X)$ for all $X \in \mathfrak{h}$. The first point is proved.

Let $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ be a linear map satisfying the conditions (a), and let $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ be the corresponding G-equivariant map defined by λ : for $\overline{g} \in M$ and $X \in \mathfrak{g}$ the map $\Lambda(X)_{\overline{g}}$ is

$$\begin{array}{ccc} \mathbf{T}_{\overline{g}}M & \longrightarrow & \mathbf{T}_{\overline{g}}M \\ [\mathrm{g,Y}] & \longmapsto & [g, \lambda(g^{-1}X)Y]. \end{array}$$

By definition we have $\Lambda(X)_{\overline{g}} = \mathcal{L}(X)_{\overline{g}}$ when $X_M(\overline{g}) = 0$. Finally we define a G-invariant connection ∇ on $\mathbf{T}M$ by posing for any vector field v, s on M and $m \in M$:

$$(\nabla_v s)|_m = (\mathcal{L}(X)s)|_m - \Lambda(X)_m(s|_m),$$

where $X \in \mathfrak{g}$ is chosen so that $X_M(m) = s|_m$. □

Counter-example : Consider the homogeneous space[(7)] $M = \mathrm{SL}(2,\mathbb{R})/H$ where

$$H = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0\}.$$

We are going to prove that *the tangent bundle* $\mathbf{T}M$ *does not carry a* G*-invariant connection*. Consider the basis (e, f, g) of $\mathfrak{sl}(2,\mathbb{R})$, where

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have $[e, f] = 2e$, $[g, f] = -2g$, and $[e, g] = -f$. Since the Lie algebra of H is $\mathfrak{h} := \mathbb{R}f \oplus \mathbb{R}g$, we use the identifications $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{h} \cong \mathbb{R}e$ and $\mathrm{End}(\mathfrak{sl}(2,\mathbb{R})/\mathfrak{h}) \cong \mathbb{R}$. For the induced adjoint action of $\mathfrak{h}$ on $\mathbb{R}e$ we have : $\widehat{\mathrm{ad}}(f) = -2$ and $\widehat{\mathrm{ad}}(g) = 0$. We are interested in a map $\lambda : \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}$ satisfying

- λ is H-equivariant, i.e. $\lambda([X, Y]) = 0$ whenever $X \in \mathfrak{h}$.
- $\lambda(X) = \widehat{\mathrm{ad}}(X)$ for $X \in \mathfrak{h}$.

Theses conditions can not be fulfilled since the first point gives $\lambda(f) = \lambda([g, e]) = 0$, and with the second point we have $\lambda(f) = \widehat{\mathrm{ad}}(f) = -2$.

(7) The manifold M is diffeomorphic to the circle.

The previous example shows that some homogeneous spaces do not have an invariant connection. For the remaining of Section 5 we work with the following

Assumption 5.3. — *The subalgebra $\mathfrak{h}$ has a H-invariant supplementary subspace $\mathfrak{m}$ in $\mathfrak{g}$.*

In [**4**] the homogeneous spaces G/H are called of *reductive type* when the assumption 5.3 is satisfied. This hypothesis guarantees the existence of invariant connections as we will see now.

Let $X \mapsto X_{\mathfrak{m}}$ denotes the H-equivariant projection onto $\mathfrak{m}$ relatively to $\mathfrak{h}$. This projection induces an H-equivariant isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$. Then a G-invariant connection on $\mathbf{T}(G/H)$ is determined uniquely by a linear H-equivariant mapping $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ which extends the adjoint action $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. So λ is completely determined by its restriction

$$\lambda|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$$

We now define a family of invariant connections ∇^a, $a \in \mathbb{R}$, when G/H is a homogeneous space of *reductive type*.

Definition 5.4. — *Let G/H be a homogeneous space of* reductive type. *For any $a \in \mathbb{R}$, we define a H-equivariant mapping $\lambda^a : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ by $\lambda^a(X) = \mathrm{ad}(X)$ for $X \in \mathfrak{h}$ and*

$$\lambda^a(X)Y = a[X,Y]_{\mathfrak{m}} \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

We denote by ∇^a the G-invariant connection associated with λ^a.

The connection ∇^0 is called the *canonical* connection. Note that the connections $\nabla^a, a \in \mathbb{R}$, are distinct except when the bracket $[-,-]_{\mathfrak{m}} = 0$ is identically equal to 0.

We finish this section by looking to the torsion free invariant connections.

Proposition 5.5. — *Let ∇ be a G-invariant connection on $\mathbf{T}(G/H)$ and let $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ be the associated H-equivariant map. The connection ∇ is* torsion free *if and only if we have*

$$[X,Y]_{\mathfrak{m}} = \lambda(X)Y - \lambda(Y)X \quad \text{for all} \quad X, Y \in \mathfrak{m}. \tag{5.59}$$

Condition (5.59) is equivalent to asking that

$$\lambda(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + b(X,Y), \tag{5.60}$$

where $b : \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ *is a* symmetric *bilinear map.*

Proof. — The vector fields X_M, $X\in\mathfrak{g}$, generate the tangent space of $M = G/H$, hence the connection is torsion free if and only if $T^\nabla(X_M,Y_M)=0$ for every $X,Y\in\mathfrak{g}$. Following (4.48) the condition is

$$[X,Y]_M = \Lambda(X)Y_M - \Lambda(Y)X_M \quad \text{for all} \quad X,Y\in\mathfrak{g}. \tag{5.61}$$

A small computation shows that the function $\widetilde{X_M} : G\to\mathfrak{m}$ associated with the vector field X_M via the isomorphism (5.55) is defined by $\widetilde{X_M}(g) = -[\mathrm{Ad}(g)^{-1}X]_{\mathfrak{m}}$. For the function $\widetilde{\lambda(X)Y_M} : G\to\mathfrak{m}$ we have

$$\widetilde{\lambda(X)Y_M}(g) = -\lambda(\mathrm{Ad}(g)^{-1}X)[\mathrm{Ad}(g)^{-1}Y]_{\mathfrak{m}}, \quad \text{for all} \quad X,Y\in\mathfrak{g}.$$

So condition (5.61) is equivalent to

$$[X,Y]_{\mathfrak{m}} = \lambda(X)Y_{\mathfrak{m}} - \lambda(Y)X_{\mathfrak{m}} \quad \text{for all} \quad X,Y\in\mathfrak{g}. \tag{5.62}$$

It is now easy to see that (5.62) is equivalent to (5.59) and (5.60). □

Corollary 5.6. — *Let $a\in\mathbb{R}$ and let ∇^a be the G-invariant connection introduced in Definition 5.4. By Proposition 5.5, we see that*

- *if the bracket $[-,-]_{\mathfrak{m}}$ is identically equal to 0 : $\nabla^a=\nabla^0$ is torsion free.*
- *if the bracket $[-,-]_{\mathfrak{m}}$ is not equal to 0, ∇^a is torsion free if and only if $a=\frac{1}{2}$.*

5.2. Geodesics on a homogeneous space. — Let ∇ be a G-invariant connection on $M=G/H$ associated with a H-equivariant map $\lambda:\mathfrak{m}\to\mathrm{End}(\mathfrak{m})$. A smooth curve $\gamma: I\to M$ is a *geodesic* relatively to ∇ if

$$\nabla_{\gamma'}(\gamma') = 0. \tag{5.63}$$

The last condition can be understood locally as follows. Let $t_0\in I$ and let $\mathcal{U}\subset M$ be a neighborhood of $\gamma(t_0)$: if $\mathcal{U}$ is small enough there exists a vector field v on $\mathcal{U}$ such that $v(\gamma(t)) = \gamma'(t)$ for $t \in I$ closed to t_0. Then for t near t_0, condition (5.63) is equivalent to

$$\nabla_v v|_{\gamma(t)} = 0. \tag{5.64}$$

Proposition 5.7. — *For $X \in \mathfrak{m}$, we consider the curve $\gamma_X(t) = \pi(\exp_G(tX))$ on G/H, where $\pi : G \to G/H$ denotes the canonical projection and $\exp_G$ is the exponential map of the Lie group G. The curve γ_X is a geodesic for the connection ∇, if and only if $\lambda(X)X = 0$.*

Proof. — The vector field X_M, which is defined on M, satisfies $X_M(\gamma_X(t)) = \gamma'_X(t)$ for $t \in \mathbb{R}$. Since $\nabla_{X_M} X_M = \Lambda(X)X_M$ we get

$$\nabla_{X_M} X_M|_{\gamma_X(t)} = [\gamma_X(t), \lambda(X)X] \quad \text{in} \quad \mathbf{T}M \simeq G \times_H \mathfrak{m},$$

so the conclusion follows. □

Corollary 5.8. — *Let ∇^a be the connection on G/H introduced in Def. 5.4. Then*

- *the maximal geodesics are the curves $\gamma(t) = \pi(g \exp_G(tX))$, where $g \in G$ and $X \in \mathfrak{m}$.*
- *the exponential mapping $\exp_{\overline{e}} : \mathfrak{m} \to G/H$ is defined by $\exp_{\overline{e}}(X) = \pi(\exp_G(X))$.*

5.3. Levi-Civita connection on a homogeneous space. — We suppose now that one has an $\mathrm{Ad}(H)$-invariant scalar product on the supplementary subspace $\mathfrak{m}$ of $\mathfrak{h}$, which we just denote by $(-,-)$.

We define a G-invariant Riemannian metric $(-,-)_M$ on $M = G/H$ as follows. Using the identification $G \times_H \mathfrak{m} \simeq \mathbf{T}M$, we take $(v,w)_M = (X,Y)$ for the tangent vectors $v = [g,X]$ and $w = [g,Y]$ of $\mathbf{T}_{\overline{g}}M$. Let ∇^{LC} be the Levi-Civita connection on M corresponding to this Riemannian metric. Since the Riemannian metric is G-invariant, the connection ∇^{LC} is G-invariant (see Section 4.2). Let $\lambda^{\mathrm{LC}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ be the H-equivariant map associated with the connection ∇^{LC}. Since ∇^{LC} preserves the metric we have

$$\lambda^{\mathrm{LC}}(X) \in \mathrm{so}(\mathfrak{m}) \quad \text{for every} \quad X \in \mathfrak{g}. \tag{5.65}$$

Here $\mathrm{so}(\mathfrak{m})$ denotes the Lie algebra of the orthogonal group $\mathrm{SO}(\mathfrak{m})$.

Proposition 5.9. — *The map* λ^{LC} *is determined by the following conditions:* $\lambda^{\mathrm{LC}}(X) = \mathrm{ad}(X)$ *for* $X \in \mathfrak{h}$ *and* $\lambda^{\mathrm{LC}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + b^{\mathrm{LC}}(X,Y)$ *for* $X, Y \in \mathfrak{m}$*, where* $b^{\mathrm{LC}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ *is the* symmetric *bilinear map defined by*

$$2(b^{\mathrm{LC}}(X,Y), Z) = ([Z,X]_{\mathfrak{m}}, Y) + ([Z,Y]_{\mathfrak{m}}, X) \quad X, Y, Z \in \mathfrak{m}. \tag{5.66}$$

Proof. — We use the decomposition (5.60) together with the fact that $(\lambda^{\mathrm{LC}}(X)Y, Z) = -(Y, \lambda^{\mathrm{LC}}(X)Z)$ for $X, Y, Z \in \mathfrak{m}$. It gives

$$(b^{\mathrm{LC}}(X,Y), Z) + (b^{\mathrm{LC}}(Z,X), Y) = \frac{-1}{2}\Big(([X,Y]_{\mathfrak{m}}, Z) + ([X,Z]_{\mathfrak{m}}, Y)\Big). \tag{5.67}$$

Now if we interchange X, Y, Z with Z, X, Y and after in Y, Z, X, we get two other equalities. If we sum them with alternative sign, on the left-hand side we get the term $2(b^{\mathrm{LC}}(X,Y), Z)$, while on the right-hand side we get $-([X,Z]_{\mathfrak{m}}, Y) - ([Y,Z]_{\mathfrak{m}}, X)$. □

Example. Suppose that G is a compact Lie group and H is a closed subgroup. Let $(-,-)_{\mathfrak{g}}$ be a G-invariant scalar product on $\mathfrak{g}$. We take $\mathfrak{m}$ as the orthogonal subspace of $\mathfrak{h}$. We take on G/H the G-invariant Riemannian metric coming from the scalar product $(-,-)_{\mathfrak{g}}$ restricted to $\mathfrak{m}$. In this situation we see that the bilinear map b^{LC} vanishes. So, the Levi-Civita connection on G/H is equal to the connection $\nabla^{1/2}$ (see Definition 5.4). Then we know after Corollary 5.8 that the geodesics on G/H are of the form $\gamma(t) = \pi(g \exp_G(tX))$ with $X \in \mathfrak{m}$.

5.4. Levi-Civita connection on symmetric spaces of the non-compact type. — We come back to the situation of Section 3.4. Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $\Theta : G \to G$ be an involution of G such that $\theta = d\Theta$ is a Cartan involution of $\mathfrak{g}$. On the Lie algebra level we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the closed connected subgroup $K = G^{\Theta}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. We denote by $X \mapsto X_{\mathfrak{k}}$ and $X \mapsto X_{\mathfrak{p}}$ the projections such that $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ for $X \in \mathfrak{g}$.

We consider here the homogeneous space $M = G/K$. Since $\mathrm{Ad}(K)$ is compact, the vector subspace $\mathfrak{p} \simeq \mathbf{T}_{\bar{e}}M$ carries $\mathrm{Ad}(K)$-invariant scalar products that induce G-invariant Riemannian metrics on M. One of them is of particular interest : the Killing form $B_{\mathfrak{g}}$.

Proposition 5.10. — *The Levi-Civita connection* ∇^{LC} *on* G/K *associated with* any $\mathrm{Ad}(K)$*-invariant scalar product on* $\mathfrak{p}$ *coincides with the canonical connection* ∇^0 *(see Definition 5.4).*

Proof. — Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, we have $[X,Y]_{\mathfrak{p}} = 0$ when $X, Y \in \mathfrak{p}$. By Proposition 5.9, we have then $\lambda^{\mathrm{LC}}(X) = \mathrm{ad}(X_{\mathfrak{k}})$ for $X \in \mathfrak{p}$, which means that $\nabla^{\mathrm{LC}} = \nabla^0$. □

In this setting Corollary 5.8 gives

Corollary 5.11. — • *All the maximal geodesics on* G/K *are defined over* $\mathbb{R}$*: the Riemannian manifold* G/K *is complete.*

• *the exponential mapping* $\exp_{\bar e} : \mathfrak{p} \to G/K$ *is defined by* $\exp_{\bar e}(X) = \pi(\exp_G(X))$.

We will now compute the curvature tensor R^{LC} of ∇^{LC}. By definition R^{LC} is a 2-form on M with values in $\mathrm{End}(\mathbf{T}M)$. We take $X, Y \in \mathfrak{g}$ and look at $R^{\mathrm{LC}}(X_M, Y_M) \in \Gamma(\mathrm{End}(\mathbf{T}M))$ or equivalently at the function $\widetilde{R^{\mathrm{LC}}(X_M, Y_M)} : G \to \mathrm{End}(\mathfrak{p})$: (4.49) gives

$$\begin{aligned}
\widetilde{R^{\mathrm{LC}}(X_M, Y_M)}(g) &= -[\lambda^{\mathrm{LC}}(g^{-1}X), \lambda^{\mathrm{LC}}(g^{-1}X)] + \lambda^{\mathrm{LC}}([g^{-1}X, g^{-1}Y]) \\
&= -[\mathrm{ad}((g^{-1}X)_{\mathfrak{k}}), \mathrm{ad}((g^{-1}X)_{\mathfrak{k}})] + \mathrm{ad}([g^{-1}X, g^{-1}Y]_{\mathfrak{k}}) \\
&= \mathrm{ad}([(g^{-1}X)_{\mathfrak{p}}, (g^{-1}Y)_{\mathfrak{p}}]).
\end{aligned}$$

At the point $\bar e \in M$, the curvature tensor R^{LC} specializes in a map $R^{\mathrm{LC}}_{\bar e} : \mathfrak{p} \times \mathfrak{p} \to \mathrm{End}(\mathfrak{p})$.

Proposition 5.12. — *For* $X, Y \in \mathfrak{p}$*, we have*

$$R^{\mathrm{LC}}_{\bar e}(X, Y) = \mathrm{ad}([X, Y]).$$

We will now compute the sectional curvature when the Riemannian metric on $M = G/K$ is induced by the scalar product on $\mathfrak{p}$ defined by the Killing form $B_{\mathfrak{g}}$. The sectional curvature is a real function κ defined on the Grassmannian $Gr_2(\mathbf{T}M)$ of 2-dimensional vector subspaces of $\mathbf{T}M$ (see e.g. [**2**]). If $S \subset \mathbf{T}_{\bar e}M$ is generated by two *orthogonal* vectors $X, Y \in \mathfrak{p}$ we have

$$\begin{aligned}\kappa(S) &= \frac{B_{\mathfrak{g}}(R^{\mathrm{LC}}_{\bar{e}}(X,Y)X,Y)}{\|X\|^2\|Y\|^2} \qquad [1]\\ &= \frac{B_{\mathfrak{g}}([[X,Y],X],Y)}{\|X\|^2\|Y\|^2} \qquad [2]\\ &= -\frac{\|[X,Y]\|^2}{\|X\|^2\|Y\|^2} \qquad [3].\end{aligned}$$

[1] is the definition of the sectional curvature. [2] is due to Proposition 5.12, and [3] follows from the $\mathfrak{g}$-invariance of the Killing form and also from the fact that $-B_{\mathfrak{g}}$ defines a scalar product on $\mathfrak{k}$.

Conclusion : The homogeneous manifold G/K, when equipped with the Riemannian metric induced by the Killing form, is a complete Riemannian manifold with negative sectional curvature.

5.5. Flats on symmetric spaces of the non-compact type. — Let M be a Riemmannian manifold and N a connected submanifold of M. The submanifold N is called *totally geodesic* if for each geodesic $\gamma : I \to M$ of M we have for $t_0 \in I$

$$\Big(\gamma(t_0) \in N \quad \text{and} \quad \gamma'(t_0) \in \mathbf{T}_{\gamma(t_0)}N\Big) \Longrightarrow \gamma(t) \in N \quad \text{for all} \quad t \in I.$$

We consider now the case of the symmetric space G/K equipped with the Levi-Civita connection ∇^0.

Theorem 5.13. — *The set of totally geodesic submanifolds of G/K containing $\bar{e}$ is in one-to-one correspondence with the subspaces*[(8)] $\mathfrak{s} \subset \mathfrak{p}$ *satisfying* $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$.

For a proof see [**2**][Section IV.7]. The correspondence works as follows. If S is a totally geodesic submanifold of G/K, one has $R^{\mathrm{LC}}_n(u,v)w \in T_nS$ for each $n \in S$ and $u, v, w \in \mathbf{T}_nS$. Then when $\bar{e} \in S$ one takes $\mathfrak{s} := \mathbf{T}_{\bar{e}}S$: the last condition becomes $[[u,v],w] \in \mathfrak{s}$ for $u, v, w \in \mathfrak{s}$.

In the other direction, if $\mathfrak{s}$ is a Lie triple system one sees that $\mathfrak{g}_{\mathfrak{s}} := [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$. Let $G_{\mathfrak{s}}$ be the connected Lie subgroup

(8) Such subspaces of $\mathfrak{p}$ are called Lie triple system.

of G associated with $\mathfrak{g}_\mathfrak{s}$. One can prove that the orbit $S := G_\mathfrak{s} \cdot \bar{e}$ is a closed submanifold of G/K which is totally geodesic.

We are interested now in the "flats" of G/K. These are the totally geodesic submanifolds with a curvature tensor that vanishes identically. If we use the last Theorem one sees that the set of flats in G/K passing through $\bar{e}$ is in one-to-one correspondence with the set of *abelian* subspaces of $\mathfrak{p}$.

We will conclude this section with the

Lemma 5.14. — *Let $\mathfrak{s}, \mathfrak{s}'$ be two maximal abelian subspaces of $\mathfrak{p}$. Then there exists $k_o \in K$ such that $\mathrm{Ad}(k_o)\mathfrak{s} = \mathfrak{s}'$. In particular the subspaces $\mathfrak{s}$ and $\mathfrak{s}'$ have the same dimension.*

Proof. — FIRST STEP. Let us show that for any maximal abelian subspace $\mathfrak{s}$ there exists $X \in \mathfrak{s}$ such that the stabilizer $\mathfrak{g}^X := \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$ satisfies $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{s}$. We look at the *commuting* family $\mathrm{ad}(X)$, $X \in \mathfrak{s}$, of linear maps on $\mathfrak{g}$. All these maps are *symmetric* relatively to the scalar product $B^\theta := -B_\mathfrak{g}(\cdot, \theta(\cdot))$, so they can be diagonalized all together : there exists a finite set $\alpha_1, \cdots, \alpha_r$ of non-zero linear maps from $\mathfrak{s}$ to $\mathbb{R}$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{i=1}^{r} \mathfrak{g}_{\alpha_i},$$

with $\mathfrak{g}_{\alpha_i} = \{X \in \mathfrak{g} \mid [Z, X] = \alpha_i(Z)X, \forall\, Z \in \mathfrak{s}\}$. Here the subspace $\mathfrak{s}$ belongs to $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid [Z, X] = 0, \forall\, Z \in \mathfrak{s}\}$. Since we have assumed that $\mathfrak{s}$ is maximal abelian in $\mathfrak{p}$ we have $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$. For any $X \in \mathfrak{s}$ we have obviously

$$\mathfrak{g}^X = \mathfrak{g}_0 \oplus \sum_{\alpha_i(X)=0} \mathfrak{g}_{\alpha_i}.$$

If we take $X \in \mathfrak{s}$ such that $\alpha_i(X) \neq 0$ for all i, then $\mathfrak{g}^X = \mathfrak{g}_0$, hence $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$.

SECOND STEP. Take $X \in \mathfrak{s}$ and $X' \in \mathfrak{s}'$ such that $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{s}$ and $\mathfrak{g}^{X'} \cap \mathfrak{p} = \mathfrak{s}'$. We define the function $f(k) = B_\mathfrak{g}(X', \mathrm{Ad}(k)X)$ for $k \in K$. Let k_0 be a critical point of f (such a point exists since $\mathrm{Ad}(K)$ is compact). If we differentiate f at k_o we get $B_\mathfrak{g}(X', [Y, \mathrm{Ad}(k_o)X]) = 0$, $\forall\ Y \in \mathfrak{k}$. Since $B_\mathfrak{g}$ is $\mathfrak{g}$-invariant we get $B_\mathfrak{g}([X', \mathrm{Ad}(k_o)X], Y) = 0$, $\forall\ Y \in \mathfrak{k}$, so

$[X', \mathrm{Ad}(k_o)X] = 0$. Since $\mathfrak{g}^{\mathrm{Ad}(k_o)X} \cap \mathfrak{p} = \mathrm{Ad}(k_o)(\mathfrak{g}^X \cap \mathfrak{p}) = \mathrm{Ad}(k_o)\mathfrak{s}$, the last equality gives $X' \in \mathrm{Ad}(k_o)\mathfrak{s}$. And since $\mathrm{Ad}(k_o)\mathfrak{s}$ is an abelian subspace of $\mathfrak{p}$ we have then

$$\begin{aligned} \mathrm{Ad}(k_o)\mathfrak{s} &\subset \mathfrak{g}^{X'} \cap \mathfrak{p} \\ &\subset \mathfrak{s}'. \end{aligned}$$

Finally since $\mathfrak{s}, \mathfrak{s}'$ are two maximal abelian subspaces, the last equality guarantees that $\mathrm{Ad}(k_o)\mathfrak{s} = \mathfrak{s}'$. □

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Paul-Emile Paradan, Institut de Mathématiques et Modélisation, Université Montpellier 2, Place Eugène Bataillon, 34095 MONTPELLIER Cedex France
E-mail : paradan@math.univ-montp2.fr