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TOPOLOGICAL SIMPLICITY, COMMENSURATOR SUPER-RIGIDITY AND NON-LINEARITIES OF KAC–MOODY GROUPS

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with Appendix by
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Abstract. We provide new arguments to see topological Kac–Moody groups as generalized semisimple groups over local fields: they are products of topologically simple groups and their Iwahori subgroups are the normalizers of the pro-p Sylow subgroups. We use a dynamical characterization of parabolic subgroups to prove that some countable Kac–Moody groups with Fuchsian buildings are not linear. We show for this that the linearity of a countable Kac–Moody group implies the existence of a closed embedding of the corresponding topological group in a non-Archimedean simple Lie group, thanks to a commensurator super-rigidity theorem proved in the Appendix by P. Bonvin.

Introduction

This paper contains two kinds of results, according to which definition of Kac–Moody groups is adopted. The main goal is to prove non-linearities of countable Kac–Moody groups as defined by generators and relations by J. Tits [T2], but our strategy, based on continuous extensions of abstract group homomorphisms, leads to prove structure results on topological Kac–Moody groups [RR]. Our first theorem says that infinitely many countable Kac–Moody groups are not linear (4.C.1).

Theorem. *Let Λ be a generic countable Kac–Moody group over a finite field with right-angled Fuchsian buildings. Then Λ is not linear over any field. Moreover for each prime number p, there are infinitely many such non-linear groups defined over a finite field of characteristic p.*

The second theorem says that topological Kac–Moody groups can be seen as generalizations of semisimple groups over local fields. The statement below roughly sums up 2.A.1 and 1.B.2.

Theorem. (i) *A topological Kac–Moody group over a finite field is the direct product of topologically simple groups, with one factor for each connected component of its Dynkin diagram.*

(ii) *The Iwahori subgroups, i.e. the chamber fixators for the natural action on the building of the group, are characterized as the normalizers of the pro-p Sylow subgroups.*

The motivation for both results is an analogy with well-known classical cases corresponding to the affine type in Kac–Moody theory. Affine Kac–Moody Lie algebras are central extensions of split semisimple Lie algebras over Laurent polynomials [K]. For Kac–Moody groups over finite fields, the affine case corresponds to $\{0;\infty\}$-arithmetic subgroups of simply connected split semisimple groups $\mathbf{G}$ over function fields. It has already been observed that general Kac–Moody groups over finite fields provide generalizations of these arithmetic groups: there is a natural diagonal action of such a group Λ on the product of two isomorphic buildings, and when the groundfield is large enough, Λ is a lattice of the product of (the automorphism groups of) the buildings [CaG], [R1]. In the affine case, when $\Lambda = \mathbf{G}(\mathbf{F}_q[t,t^{-1}])$, the two buildings are the Bruhat–Tits buildings of $\mathbf{G}$ over the completions $\mathbf{F}_q(\!(t)\!)$ and $\mathbf{F}_q(\!(t^{-1})\!)$.

In the general case, most buildings are obviously new, e.g. because their Weyl groups are neither finite nor affine. Still, on the group side, one could imagine a situation where a well-known discrete group acts on exotic geometries. A first step to disprove this is to prove that for a countable Kac–Moody group over $\mathbf{F}_q$, the only possible linearity is over a field of the same characteristic p [R4]. Our first theorem above shows that there exist Kac–Moody groups over finite fields which are not linear, even in equal characteristic. The affine example of arithmetic groups shows that the answer to the linearity question cannot be stated in the equal characteristic case as for inequal characteristics, and suggests that the former case is harder than the latter.

This is indeed the case, but the proof is fruitful since it gives structure results for topological Kac–Moody groups. Such a group is defined in [RR] as the closure of the non-discrete action of a countable Kac–Moody group on only one of the two buildings. In [loc. cit.] it was proved that these groups satisfy the axioms of a sharp refinement of Tits systems and that the parahoric subgroups (i.e. the spherical facet fixators) are virtually pro-p groups. These results are *a posteriori* not surprising in view of the affine case: then the topological Kac–Moody groups are of the form $\mathbf{G}(\mathbf{F}_q(\!(t)\!))$

with **G** as above. Our second theorem makes deeper the analogy between topological Kac–Moody groups and semisimple groups over local fields of positive characteristic (see also 1.B.1).

Here is the connection between the two results. The general strategy of the proof consists in strengthening the linearity assumption for a countable Kac–Moody group to obtain a closed embedding of the corresponding topological Kac–Moody group into a simple non-Archimedean Lie group. The main tool is a general commensurator super-rigidity, stated by M. Burger [Bu] according to ideas of G. Margulis [M1] and proved in the Appendix by P. Bonvin. Thanks to it, we prove the following dichotomy (3.A).

Theorem. *Let Λ be a countable Kac–Moody group generated by its root groups, with connected Dynkin diagram and large enough finite groundfield. Then either Λ is not linear, or the corresponding topological group is a closed subgroup of a simple non-Archimedean Lie group G, with equivariant embedding of the vertices of the Kac–Moody building to the vertices of the Bruhat–Tits building of G.*

To prove this result we need further structure results about closures of Levi factors: they have a Tits sub-system, are virtually products of topologically simple factors and their buildings naturally appear in the building of the ambient Kac–Moody group (2.B.1). The dichotomy of this theorem is used to obtain complete non-linearities by proving that some topological Kac–Moody groups are not closed subgroups of Lie groups. We concentrate at this stage on Kac–Moody groups with right-angled Fuchsian buildings. We define generalized parabolic subgroups as boundary point stabilizers (as for symmetric spaces or Bruhat–Tits buildings). The last argument comes from dynamics in algebraic groups: according to G. Prasad, to each suitable element of a non-Archimedean semisimple group is attached a proper parabolic subgroup [Pr]. These results and their analogues for groups with right-angled Fuchsian buildings enable us to exploit the incompatibility between the geometries of CAT(-1) Kac–Moody and of Euclidean Bruhat–Tits buildings (4.C.1).

Now that we know that some Kac–Moody groups are non-linear, it makes sense to ask whether some of them are non-residually finite, or even simple. For this question, the work by M. Burger and Sh. Mozes [BuMo2] on lattices of products of trees may be relevant. Moreover the work by R. Pink on compact subgroups of non-Archimedean Lie groups [P] may lead to a complete answer to the linearity question of countable Kac–Moody groups, purely in terms of the Dynkin diagram (and of the size of the finite groundfield).

This article is organized as follows. Sect. 1 provides references on Kac–Moody groups as defined by Tits, and recalls facts on topological Kac–Moody groups, as well as the analogy with algebraic groups. The new result here is the intrinsic characterization of Iwahori subgroups. In sect. 2, we prove the topological simplicity theorem, based on Tits system and pro-p group arguments. In sect. 3, we show that the linearity of a countable Kac–Moody group leads to a closed embedding of the corresponding topological group in an algebraic one. In sect. 4, we define and analyze generalized parabolic subgroups of topological Kac–Moody groups with right-angled Fuchsian buildings. We prove a decomposition involving a generalized unipotent radical, and then study these groups dynamically. We conclude with the proof of the non-linearity theorem, followed by a geometric explanation in terms of compactifications of buildings. P. Bonvin wrote an appendix to this paper, providing the proof of the commensurator super-rigidity theorem used in sect. 3.

Conventions. 1) The word «building» refers to the standard combinatorial notion [Bro], [Ro], as well as to the metric realization of it [D]. The latter realization of a building, due to M. Davis, is called here its CAT(0)-*realization*.

2) If a group G acts on a set X, the pointwise stabilizer of a subset Y is called its *fixator* and is denoted by $\mathrm{Fix}_G(Y)$; the usual global stabilizer is denoted by $\mathrm{Stab}_G(Y)$. The notation $G\,|_Y$ means the group obtained from G by factoring out the kernel of the action on a G-stable subset Y in X.

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1 Structure Theorem

We review some properties of topological Kac–Moody groups, keeping in mind the analogy with semisimple groups over local fields of positive characteristic. We quote results of combinatorial nature, to be used to prove the topological simplicity theorem, and we prove some results on pro-p Sylow subgroups.

1.A Topological Kac–Moody groups. Topological Kac–Moody groups were introduced in [RR, 1.B]. Their geometric definition uses the buildings naturally associated to the countable Kac–Moody groups defined over finite fields.

1.A.1 According to Tits [T2, 3.6], a *split Kac–Moody group* is defined by generators and Steinberg relations once a Kac–Moody root datum and a groundfield $\mathbf{K}$ are given. A *generalized Cartan matrix* [K] is the main ingredient of a Kac–Moody root datum, the other part determining the maximal split torus of the group. More precisely, what Tits defines by generators and relations is a functor on rings. When the generalized Cartan matrix is a Cartan matrix in the usual sense (which we call *of finite type*), the functor coincides over fields with the functor of points of a Chevalley–Demazure group scheme.

An *almost split Kac–Moody group* is the group of fixed points of a Galois action on a split group [R2, §11]. Let Λ be an almost split Kac–Moody group. Any such group satisfies the axioms of a *twin root datum* [R2, §12]. This is a group combinatorics sharply refining the structure of a BN-pair, and the associated geometry is a pair of twinned buildings, conventionally one for each sign [T5]. We denote by X (resp. X_-) the positive (resp. negative) building of Λ. The group Λ acts diagonally on $X \times X_-$ in a natural way. We choose a pair of opposite chambers R (in X) and R_- (in X_-), which defines a pair of opposite apartments A (in X) and A_- (in X_-) [A, I.2]. We call R the *standard positive chamber* and A the *standard positive apartment*.

Examples. 1) The group $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$ is a Kac–Moody group of affine type. The generalized Cartan matrix is $\widetilde{A}_{n-1}$ and the groundfield is $\mathbf{K}$. More generally, values of Chevalley schemes over rings of Laurent polynomials are Kac–Moody of affine type [A, I.1 example 3].

2) The group $\mathrm{SU}_3(\mathbf{F}_q[t,t^{-1}])$ is an almost split Kac–Moody group of rank one. The so-obtained geometry is a semi-homogeneous twin tree of valencies $1+q$ and $1+q^3$ [R3, 3.5].

Definition. (i) *We call Γ the fixator of the negative chamber R_-, i.e.* $\Gamma = \mathrm{Fix}_\Lambda(R_-)$.

(ii) *We call Ω the fixator of the standard positive chamber R, i.e.* $\Omega = \mathrm{Fix}_\Lambda(R)$.

Remark. In the above quoted references, the groups Λ, Γ and Ω are denoted by G, B_- and B, respectively. The reason is that the group Ω

(resp. Γ) is the Borel subgroup of the positive (resp. negative) Tits system of the twin root datum of Λ [T5], [R2, §7].

EXAMPLE. For $\Lambda = \mathrm{SL}_n(\mathbf{K}[t, t^{-1}])$, a natural choice of R and R_- defines the group Ω as the subgroup of $\mathrm{SL}_n(\mathbf{K}[t])$ which reduces to upper triangular matrices modulo t, and Γ is the subgroup of $\mathrm{SL}_n(\mathbf{K}[t^{-1}])$ which reduces to lower triangular matrices modulo t^{-1}.

In this article, we are not interested in Tits group functors whose generalized Cartan matrices are of finite type, i.e. whose values over finite fields are finite groups of Lie type. Kac–Moody groups of affine type are seen as a guideline to generalize classical results about algebraic and arithmetic groups. We are mainly interested in Kac–Moody groups which do not admit any natural matrix interpretation, and we want to understand to what extent these new groups can be compared to the obviously linear ones.

1.A.2 We now define topological groups – see [RR, 1.B] for a wider framework.

ASSUMPTION. *In this article, Λ is a countable Kac–Moody group over the finite field $\mathbf{F}_q$ of characteristic p with q elements, i.e. Λ is the group of rational points of an almost split Kac–Moody group with infinite Weyl group W.*

The kernel of the Λ-action on X is the finite center $Z(\Lambda)$, and the group $\Lambda/Z(\Lambda)$ still enjoys the structure of twin root datum [RR, Lemma 1.B.1]. Another consequence of the finiteness of the groundfield is that the full automorphism group $\mathrm{Aut}(X)$ is an uncountable totally disconnected locally compact group.

DEFINITION. (i) *We call* topological Kac–Moody group (associated to Λ) *the closure in* $\mathrm{Aut}(X)$ *of the group $\Lambda/Z(\Lambda)$. We denote it by $\overline{\Lambda}$.*

(ii) *We call* parahoric subgroup (associated to F) *the fixator in $\overline{\Lambda}$ of a given spherical facet F. We denote it by $\overline{\Lambda}_F$, and when the facet is a chamber, we call it an* Iwahori subgroup.

REMARKS. 1) The group $\overline{\Lambda}$ is so to speak a completion of the group Λ. (Recall that the Λ-action on the single building X is not discrete since the stabilizers are parabolic subgroups with infinite «unipotent radical» [R2, Theorem 6.2.2].) There exist other (algebraic) definitions of Kac–Moody groups; they are also presented as completions of the groups Λ [T3, §1.5]. The relation between the groups $\overline{\Lambda}$ and the latter groups is not clear (see, however [R5, Question 36]).

2) Parahoric subgroups can equivalently be defined as the fixators in $\overline{\Lambda}$ of the facets in the CAT(0)-realization of the building X, because the class of facets represented in it is exactly the class of spherical facets [D].

EXAMPLE. For $\Lambda = \mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$, X is the Bruhat–Tits building of $\mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))$, that is a Euclidean building of type $\widetilde{A}_{n-1}$. If $\mu_n(\mathbf{F}_q)$ denotes the n-th roots of unity in $\mathbf{F}_q$, the image $\Lambda/Z(\Lambda)$ of $\mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])$ under the action on X is $\mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])/\mu_n(\mathbf{F}_q)$ and the completion $\overline{\Lambda}$ is $\mathrm{PSL}_n(\mathbf{F}_q(\!(t)\!)) = \mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))/\mu_n(\mathbf{F}_q)$.

1.B Refined Tits system and virtual pro-p-ness of parahoric subgroups. The reference for this subsection is [RR, 1.C].

1.B.1 Let us state [RR, Theorem 1.C.2] showing that topological Kac–Moody groups generalize semisimple groups over local fields of positive characteristic.

Theorem. *Let Λ be an almost split Kac–Moody group over $\mathbf{F}_q$ and $\overline{\Lambda}$ be its associated topological group. Let $R \subset A$ be the standard chamber and apartment of the building X associated to Λ. We denote by $\mathcal{B}$ the standard Iwahori subgroup $\overline{\Lambda}_R$ and by W_R the Coxeter group associated to A, generated by reflections along the panels of R.*

(i) *The topological Kac–Moody group $\overline{\Lambda}$ enjoys the structure of a refined Tits system with abstract Borel subgroup $\mathcal{B}$ and Weyl group W_R, which is also the Weyl group of Λ.*

(ii) *Any parahoric subgroup $\overline{\Lambda}_F$ is a semidirect product $M_F \ltimes \widehat{U}_F$ where M_F is a finite reductive group of Lie type over $\mathbf{F}_q$ and $\widehat{U}_F$ is a pro-p group. In particular, any parahoric subgroup is virtually a pro-p group.* □

REMARKS. 1) The group Λ is *strongly transitive* on the building X, i.e. transitive on the pairs of chambers at given W-distance from one another [Ro, §5]. This implies that $\overline{\Lambda}$ is strongly transitive on X too, and that X is also the building associated to the above Tits system of $\overline{\Lambda}$.

2) Refined Tits system were defined by V. Kac and D. Peterson [KP]. For twin root data, there are two relevant standard Borel subgroups playing symmetric roles. For refined Tits systems, only one conjugacy class of Borel subgroups is introduced. The latter structure is abstractly implied by the former one [R2, 1.6], but it applies to a strictly wider class of groups, e.g. $\mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))$ satisfies the axioms of a refined Tits system while it doesn't admit a twin root datum structure.

Let us also briefly mention how the groups M_F and $\widehat{U}_F$ are defined geometrically. We first note that in the CAT(0)-realization of buildings only spherical facets appear. We denote by $\mathrm{St}(F)$ the *star* of a facet F, that is the set of chambers whose closure contains F. Theorem 6.2.2 of [R2] applies and we have a Levi decomposition $\Lambda_F = \mathrm{Stab}_\Lambda(F) = M_F \ltimes U_F$, where M_F is a Kac–Moody subgroup for a Cartan submatrix of finite type and U_F fixes pointwise $\mathrm{St}(F)$. The group $\widehat{U}_F$ is the closure of U_F in $\overline{\Lambda}$, hence it fixes $\mathrm{St}(F)$ too. Moreover by [loc. cit., Proposition 6.2.3], $\mathrm{St}(F)$ is a geometric realization of the finite building attached to M_F, and the action by M_F is the standard one. Therefore the image of the surjective homomorphism $\pi_F : \overline{\Lambda}_F \to M_F$ associated to $\overline{\Lambda}_F = M_F \ltimes \widehat{U}_F$ gives the local action of $\overline{\Lambda}_F$ on $\mathrm{St}(F)$.

These facts are analogues of classical results in Bruhat–Tits theory [BruT, Proposition 5.1.32]. Namely, any facet in the Bruhat–Tits building of a semisimple group $\mathbf{G}$ over a local field k defines an integral structure over the valuation ring $\mathcal{O}$ of k. The reduction of the $\mathcal{O}$-structure modulo the maximal ideal is a semisimple group over the residue field, whose building is the star of the facet. The integral points of the $\mathcal{O}$-structure act on it via the natural action of the reduction. The splitting $\overline{\Lambda}_F = M_F \ltimes \widehat{U}_F$ of a parahoric subgroup as a semidirect product is specific to the case of valuated fields in equal characteristic, and in the case of locally compact fields this only occurs in characteristic p.

Example. Let v be a vertex in the Bruhat–Tits building of $\mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))$. Then its fixator is isomorphic to $\mathrm{SL}_n(\mathbf{F}_q[[t]])$ and its star is isomorphic to the building of $\mathrm{SL}_n(\mathbf{F}_q)$. The subgroup $\widehat{U}_v$ is the first congruence subgroup of $\mathrm{SL}_n(\mathbf{F}_q[[t]])$, i.e. the matrices reducing to the identity modulo t. The above reduction corresponds concretely to taking the quotient by $\widehat{U}_v$, and the Iwahori subgroups fixing a chamber in $\mathrm{St}(v)$ are the subgroups reducing to a Borel subgroup of $\mathrm{SL}_n(\mathbf{F}_q)$, e.g. reducing to the upper triangular matrices.

In our case, the Bruhat decomposition and the rule to multiply double classes [Bou, IV.2] have topological consequences.

Corollary. *A topological Kac–Moody group is compactly generated.*

Proof. Let $\overline{\Lambda}$ be such a group and let $\mathcal{B} = \overline{\Lambda}_R$ be the standard Iwahori subgroup. Then $\overline{\Lambda}$ is generated by $\mathcal{B}$ and by the compact double classes $\mathcal{B}s\mathcal{B}$, when s runs over the finite set of reflections along the panels of R. □

1.B.2 Pro-p Sylow subgroups of totally disconnected groups are defined in [S, I.1.4] for instance. The following proposition is suggested by classical results on pro-p subgroups of non-Archimedean Lie groups, e.g. [PlR, Theorem 3.10].

PROPOSITION. (i) *Given any chamber R, the group $\widehat{U}_R$ of the decomposition $\overline{\Lambda}_R = M_R \ltimes \widehat{U}_R$ is the unique pro-p Sylow subgroup of the Iwahori subgroup $\overline{\Lambda}_R$.*

(ii) *Let K be a pro-p subgroup of $\overline{\Lambda}$. Then there is a chamber R such that K lies in the pro-p Sylow subgroup $\widehat{U}_R$ of $\overline{\Lambda}_R$.*

(iii) *The pro-p Sylow subgroups of $\overline{\Lambda}$ are precisely the subgroups $\widehat{U}_R$ when R ranges over the chambers of the building X; they are all conjugate.*

(iv) *The Iwahori subgroups of $\overline{\Lambda}$ are intrinsically characterized as the normalizers of the pro-p Sylow subgroups of $\overline{\Lambda}$.*

Proof. (i) By quasi-splitness of an almost split Kac–Moody group Λ over $\mathbf{F}_q$ [R2, 13.2], the Levi factor M_R of a chamber fixator in Λ is the $\mathbf{F}_q$-points of a torus. Therefore its order is prime to p and we conclude by the decomposition $\overline{\Lambda}_R = M_R \ltimes \widehat{U}_R$.

(ii) Let K be a pro-p subgroup of $\overline{\Lambda}$. Since it is compact, it fixes a spherical facet F [R2, 4.6] and by Theorem 1.B.1 (ii) we can write $K \subset \overline{\Lambda}_F = M_F \ltimes \widehat{U}_F$. Let us look at the local action $\pi_F : \overline{\Lambda}_F \to M_F$ (1.B.1). By the Bruhat decomposition in split BN-pairs, the p-Sylow subgroups of finite reductive groups of Lie type are the unipotent radicals of the Borel subgroups [Bo+, B Corollary 3.5], so the p-subgroup $\pi_F(K)$ of M_F is contained in some Borel subgroup of M_F, hence fixes a chamber R in $\mathrm{St}(F)$. Since $\widehat{U}_F$ fixes pointwise $\mathrm{St}(F)$, the whole subgroup K fixes R and we conclude by (i).

(iii) The first assertion follows immediately from (ii), and the second one follows from the transitivity of Λ on the chambers of X.

(iv) According to (iii), it is enough to show that we have $\mathcal{B} = N_{\overline{\Lambda}}(\widehat{U}_R)$ for the Iwahori subgroup $\mathcal{B} = \overline{\Lambda}_R$ fixing the standard positive chamber R. By 1.B.1 (ii), we know that $\mathcal{B}$ lies in $N_{\overline{\Lambda}}(\widehat{U}_R)$. By [Bou, IV.2.5] the normalizer $N_{\overline{\Lambda}}(\widehat{U}_R)$ is an abstract parabolic subgroup of the Tits system of 1.B.1 (i) with abstract Borel subgroup $\mathcal{B}$. If $N_{\overline{\Lambda}}(\widehat{U}_R)$ were bigger than $\mathcal{B}$, it would contain a reflection conjugating a positive root group to a negative one. But this is in contradiction with axiom (RT3) of refined Tits systems [KP], so we have $\mathcal{B} = N_{\overline{\Lambda}}(\widehat{U}_R)$. □

1.C Lattices and generalized arithmeticity. We briefly discuss existence and generalized arithmeticity of lattices in topological Kac–Moody groups.

1.C.1 We keep the almost split Kac–Moody group Λ over $\mathbf{F}_q$. The Weyl group W is infinite, and we denote by $W(t)$ its growth series $\sum_{w\in W} t^{\ell(w)}$.

Theorem. *Assume that $W(1/q) < \infty$. Then Λ is a lattice of $X \times X_-$ for its diagonal action, and for any point $x_- \in X_-$ the subgroup $\Lambda_{x_-} = \mathrm{Fix}_\Lambda(x_-)$ is a lattice of X. These lattices are not uniform.* □

This result was independently proved in [CaG] and in [R1]. In the case of $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$, the lattices of the form Λ_{x_-} are all commensurable to the arithmetic lattice $\mathrm{SL}_n(\mathbf{F}_q[t^{-1}])$ in $\mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))$. Recall that by Margulis commensurator criterion [Z, Theorem 6.2.5] , a lattice in a semisimple Lie group G is arithmetic if and only if its commensurator is dense in G. Taking this characterization as a definition for general situations, [RR, Lemma 1.B.3 (ii)] says that the groups Λ_{x_-} are arithmetic lattices of $\overline{\Lambda}$ by the very definition of this topological group. Here is the statement, whose proof is based on refined Tits sytem arguments.

Lemma. *For any $x_- \in X_-$, the group Λ lies in the commensurator $\mathrm{Comm}_{\overline{\Lambda}}(\Lambda_{x_-})$.* □

Remark. Some lattices may have big enough commensurators to be arithmetic in $\mathrm{Aut}(X)$, meaning that the commensurators are dense in $\mathrm{Aut}(X)$. This is the case for the Nagao lattice $\mathrm{SL}_2(\mathbf{F}_q[t^{-1}])$ in the full automorphism group of the $q+1$-regular tree [Moz]. The proof can be formalized and extended to exotic trees admitting a Moufang twinning [AR].

1.C.2 A way to produce lattices in automorphism groups of cell-complexes is to take fundamental groups of complexes of groups [BrH, III.$\mathcal{C}$], the point being then to recognize the covering space. A positive result is the following – see [Bour3]: let R be a regular right-angled r-gon in the hyperbolic plane $\mathbf{H}^2$ and let $\underline{q} = \{q_i\}_{1\le i\le r}$ be a sequence of integers ≥ 2. (When $\underline{q}$ is constant, we replace $\underline{q}$ by its value q.) Then there exists a unique *right-angled Fuchsian building* $I_{r,1+\underline{q}}$ with apartments isomorphic to the tiling of $\mathbf{H}^2$ by R, and such that the link at any vertex of type $\{i; i+1\}$ is the complete bipartite graph of parameters $(1+q_i, 1+q_{i+1})$. The so-obtained lattices are uniform and abstractly defined by $\Gamma_{r,1+\underline{q}} = \langle \{\gamma_i\}_{i\in\mathbf{Z}/r\mathbf{Z}} : \gamma_i^{q_i+1} = 1 \text{ and } [\gamma_i, \gamma_{i+1}] = 1\rangle$. This uniqueness is a key argument to prove [RR, Proposition 5.C]

Proposition. *For any prime power q, there exists a non-uniquely defined Kac–Moody group Λ over $\mathbf{F}_q$ whose building is $I_{r,1+q}$. Moreover we can choose Λ such that its natural image in $\mathrm{Aut}(I_{r,1+q})$ contains the uniform lattice $\Gamma_{r,1+q}$.* □

This result says that the buildings $I_{r,1+q}$ are relevant to both Kac–Moody theory and generalized hyperbolic geometry since they carry a natural CAT(-1)-metric. They provide a well-understood infinite family of exotic Kac–Moody buildings (indexed by $r \geq 5$ when q is fixed). The corresponding countable groups Λ are actually typical groups to which our non-linearity result applies (4.C). We study these buildings more carefully in 4.A. Finally, combining [R4, Theorem 4.6] and [Bour1] leads to a somewhat surprising situation, with coexistence of lattices with sharply different properties [RR, Corollary 5.C].

Corollary. *Whenever q is large enough and r is even and large enough with respect to q, the topological group $\overline{\Lambda}$ associated to the above Λ contains both uniform Gromov-hyperbolic lattices embedding convex-cocompactly into real hyperbolic spaces, and non-uniform Kac–Moody lattices which are not linear in characteristic $\neq p$, containing infinite groups of exponent dividing p^2.* □

2 Topological simplicity theorem

We prove that topological Kac–Moody groups are direct products of topologically simple groups, as well as some results on Levi factors and homomorphisms to non-Archimedean groups.

2.A Topological simplicity. Here is a further argument supporting the analogy with semisimple algebraic groups over local fields of positive characteristic.

2.A.1 As for the simplicity of classical groups, we need to assume the groundfield to be large enough, because in our proof we need simplicity of some rank-one finite groups of Lie type.

Theorem. *Let Λ be a countable Kac–Moody group which is almost split over the finite field $\mathbf{F}_q$, with $q > 4$. We assume that Λ is generated by its root groups.*

(i) *If the Dynkin diagram of Λ is connected, the associated topological Kac–Moody group $\overline{\Lambda}$ is topologically simple.*

(ii) *For an arbitrary Dynkin diagram, the group $\overline{\Lambda}$ is a direct product of topologically simple groups, each factor being the topological Kac–Moody group associated to a connected component of the Dynkin diagram.*

Proof. (ii) The building of a Kac–Moody group Λ is defined as a gluing $X = (\Lambda \times A)/\sim$, where A is the model for an apartment, i.e. the CAT(0)-realization of the Coxeter complex of the Weyl group W [D]. The Λ-action comes from factorizing the map $\Lambda \times \Lambda \times A \to \Lambda \times A$ which sends (λ', λ, x) to $(\lambda'\lambda, x)$ [R2, §4]. The rule by which the Coxeter diagram of W is deduced from the Dynkin diagram [T2, 3.1] implies that irreducible factors correspond to connected components of the diagram. The model A is then the direct product of the models for the Coxeter complexes of the irreducible factors of W. By the defining relations of Λ [T2, 3.6], a root group indexed by a root in the subsystem associated to a given connected component of the Dynkin diagram centralizes a root group arising from another connected component. By the relationship between buildings and Tits systems [Ro, §5], the Kac–Moody subgroup defined by a given connected component acts trivially on a factor of the building X arising from any other connected component. Therefore proving (ii) is reduced to proving (i).

(i) Let $\mathcal{B}=\overline{\Lambda}_R$ be the standard Iwahori subgroup. By Theorem 1.B.1 (i), it is the Borel subgroup of a Tits system with the same Coxeter system as the one for Λ. By the Kac–Moody analogue of Lang's theorem [R2, 13.2.2], the Levi factor M_R in $\mathcal{B} = M_R \ltimes \widehat{U}_R$ is a maximally split maximal $\mathbf{F}_q$-torus T of Λ, i.e. the rational points of a finite $\mathbf{F}_q$-torus.

Let I be the indexing set of the simple roots of Λ and let G_i be the standard semisimple Levi factor of type $i \in I$. The group G_i is a finite almost simple group of Lie type generated by the root groups attached to the simple root a_i and its opposite [R2, 6.2]. By our assumption they generate Λ as an abstract group, hence $\overline{\Lambda}$ as a topological group. Note that G_i has no non-trivial abelian quotient, and it has no quotient isomorphic to a p-group either – see Remark 2 below.

We isolate the remaining arguments in a lemma also used to prove 2.B.1 (iv).

Lemma. *Let G be a topological group acting continuously and strongly transitively by type-preserving automorphisms on a building X with irreducible Weyl group. We denote by $\mathcal{B}$ a chamber fixator and we assume that* *it is an abelian-by-pro-p extension. We also assume that G is topologically generated by finite subgroups admitting no non-trivial quotient isomorphic to any abelian or p-group. Then any proper closed normal subgroup of G acts trivially on X.*

This proves the theorem by choosing G to be $\overline{\Lambda}$, $\mathcal{B}$ to be $\overline{\Lambda}_R$ and by taking $\{G_i\}_{i\in I}$ as generating family of subgroups, since by definition $\overline{\Lambda}$ acts faithfully on X. Note that $\overline{\Lambda}_R = T \ltimes \widehat{U}_R$, where T is the maximally split maximal $\mathbf{F}_q$-torus attached to A. □

Let us now prove the lemma.

Proof. By [Ro, §5], G admits an irreducible Tits system with $\mathcal{B}$ as abstract Borel subgroup. Assume we are given a closed normal subgroup H in G. Then by [Bou, IV.2.7], we have either $H.\mathcal{B} = \mathcal{B}$ or $H.\mathcal{B} = G$. The first case implies that H lies in $\mathcal{B}$, and since H is normal in G,

$$H \subset \bigcap_{g\in G} g\,\mathrm{Fix}_G(R)g^{-1} = \mathrm{Fix}_G\Big(\bigcup_{g\in G} gR\Big) = \mathrm{Fix}_G\Big(\bigcup_{g\in G} g\overline{R}\Big) = \mathrm{Fix}_G(X)\,,$$

because G is type-preserving and transitive on the chambers of the building X.

From now on, we assume that H doesn't act trivially on X. By the previous point, this implies that we have the identification of compact groups: $G/H \simeq \mathcal{B}/(\mathcal{B}\cap H)$. We denote by $1 \to \widehat{U} \to \mathcal{B} \to T \to 1$ the extension of the assumption, and we consider the homomorphism $\widehat{U} \to \mathcal{B}/(\mathcal{B}\cap H)$ sending u to $u(\mathcal{B}\cap H)$. Its kernel is $\widehat{U}\cap H$, so we have an injection $\widehat{U}/(\widehat{U}\cap H) \hookrightarrow \mathcal{B}/(\mathcal{B}\cap H)$, and since $\widehat{U}$ is normal in $\mathcal{B}$, so is $\widehat{U}/(\widehat{U}\cap H)$ in $\mathcal{B}/(\mathcal{B}\cap H)$. Then we consider the composition of surjective homomorphisms $\mathcal{B} \to \frac{\mathcal{B}}{(\mathcal{B}\cap H)} \to \frac{\mathcal{B}/(\mathcal{B}\cap H)}{\widehat{U}/(\widehat{U}\cap H)}$. Its kernel contains $\widehat{U}$, which shows that $\frac{\mathcal{B}/(\mathcal{B}\cap H)}{\widehat{U}/(\widehat{U}\cap H)}$ is a quotient of the abelian group T. Therefore if we denote by $\{G_i\}_{i\in I}$ the generating family of subgroups of the assumptions, the image of each G_i under $G \to \frac{G}{H} \simeq \frac{\mathcal{B}}{(\mathcal{B}\cap H)} \to \frac{\mathcal{B}/(\mathcal{B}\cap H)}{\widehat{U}/(\widehat{U}\cap H)}$ is trivial.

In other words, the image of G_i under $G \to G/H \simeq \mathcal{B}/(\mathcal{B}\cap H)$ is a finite subgroup of the group $\widehat{U}/(\widehat{U}\cap H)$. But the group $\widehat{U}/(\widehat{U}\cap H)$ is a pro-p group since it is the quotient of a pro-p group by a closed normal subgroup [DiSMS, 1.11]. Therefore, once again in view of the possible images of G_i, we must have $G_i/(G_i\cap H) = \{1\}$, i.e. G_i lies in H. Since the groups G_i topologically generate G, we have $H = G$. □

Example. In order to see which classical result is generalized here, we take $\Lambda = \mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])$. Let $\mu_n(\mathbf{F}_q)$ be the n-th roots of unity in $\mathbf{F}_q$. The image of $\mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])$ for the action on its positive Euclidean building is $\mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])/\mu_n(\mathbf{F}_q)$. Though $\mathrm{SL}_n(\mathbf{F}_q[t,t^{-1}])$ does not come from a simply connected Kac–Moody root datum, it is generated by standard rank-one Levi factors. The lemma says that the closed normal subgroups of $\mathrm{SL}_n(\mathbf{F}_q(\!(t)\!))$ are central, i.e. in $\mu_n(\mathbf{F}_q)$.

REMARKS. 1) An argument in the proof is that the $\mathbf{F}_q$-almost simple finite Levi factors of the panel fixators generate the countable group Λ and don't admit any quotient isomorphic to an abelian or a p-group. The same proof works if the panel fixators are replaced by a generating family of facet fixators with the same property on quotients. For $\mathrm{SL}_3(\mathbf{F}_2(\!(t)\!))$ the Levi factors of the panels fixators are solvable isomorphic to $\mathrm{SL}_2(\mathbf{F}_2)$, but the proof works with the Levi factors for the three standard vertices, because these groups are isomorphic to $\mathrm{SL}_3(\mathbf{F}_2)$.

2) Assuming $q > 4$ is a way to have this property on quotients for any facet fixator since non-simplicity of rational points of adjoint $\mathbf{F}_q$-simple groups occurs only for $q \leq 4$ [Car, 11.1.2 and 14.4.1]. Indeed, let G be the group generated by the p-Sylow subgroups of the rational points of an almost simple $\mathbf{F}_q$-group with $q > 4$, and let $H = G/K$ be a quotient. By simplicity, $K/Z(G)$ equals $\{1\}$ or $G/Z(G)$. The first case implies that H is equal to the non-abelian simple group $G/Z(G)$. In the second case we have $G = K \cdot Z(G)$, showing that H is a quotient of $Z(G)$, which is a group of order prime to p since it lies in a finite torus over $\mathbf{F}_q$. If U denotes a p-Sylow subgroup of G then $U/(U \cap K)$ is trivial in H, implying that K contains the p-Sylow subgroups of G, hence equals G.

2.A.2 In [R4] it is proved that the group Λ cannot be linear in characteristic $\neq p$ (e.g. in characteristic 0): no homomorphism $\Lambda \to \mathrm{GL}_N(k)$ hence no homomorphism $\overline{\Lambda} \to \mathrm{GL}_N(k)$ can be injective. Combined with the topological simplicity of $\overline{\Lambda}$, this leads to

COROLLARY. *Let k be a local field of characteristic $\neq p$ and let $\mathbf{G}$ be a linear algebraic group over k. Then any continuous homomorphism $\varphi : \overline{\Lambda} \to \mathbf{G}(k)$ has trivial image.* □

2.B Closures of Levi factors and maps to non-Archimedean groups. We investigate the structure of closures of Levi factors. This enables us to prove a result on continuous homomorphisms from topological Kac–Moody groups to Lie groups over non-Archimedean fields.

2.B.1 We keep our Kac–Moody group Λ and the inclusion of the chamber R in the apartment A. We denote by $\{a_i\}_{i\in I}$ the finite family of simple roots, which we see as half-spaces of A whose intersection equals R. Let us now choose a subset J of I. We can thus introduce the *standard parabolic subgroup* Λ_J in Λ, which is the union $\Lambda_J = \bigsqcup_{w\in W_J} \Omega w \Omega$ indexed by the Coxeter group generated by the reflections along the walls ∂a_i for $i \in J$. By [R2, Theorem 6.2.2], the group Λ_J admits a Levi decomposition $\Lambda_J = M_J \ltimes U_J$, where the Levi factor M_J is the Kac–Moody subgroup generated by the maximal split torus T and the root groups indexed by the roots $w.a_i$ for $w \in W_J$ and $i \in J$.

Definition. (i) *We denote by $\overline{M}_J$ the closure of M_J in $\overline{\Lambda}$, and by $\overline{G}_J$ the topological group generated by the root groups $U_{\pm a_i}$ when i ranges over J.*

(ii) *The intersection of roots $\bigcap_{i\in J} a_i$ is called the* inessential chamber *of $\overline{M}_J$ in X, and is denoted by R_J.*

(iii) *We denote by $\mathcal{B}_J$ the stabilizer of R_J in $\overline{M}_J$.*

(iv) *The union of closures $\bigcup_{g\in M_J} g.\overline{R}_J$ is called the* inessential building *of $\overline{M}_J$ in X, and is denoted by X_J.*

What supports the terminology is the following

Proposition. (i) *The closure group $\overline{M}_J$ admits a natural refined Tits system with Weyl group W_J and abstract Borel subgroup $\mathcal{B}_J$.*

(ii) *The subgroup $\mathcal{B}_J$ fixes pointwise $\overline{R}_J$ and admits a decomposition $\mathcal{B}_J = T \ltimes \widehat{U_J}$, where T is the maximally split maximal $\mathbf{F}_q$-torus attached to A and $\widehat{U_J}$ is a pro-p group.*

(iii) *The space X_J is a geometric realization of the building of $\overline{M}_J$ arising from the above Tits system structure.*

(iv) *The group $\overline{G}_J$ is of finite index in $\overline{M}_J$ and when $q > 4$ it admits a direct product decomposition into topologically simple factors.*

Proof. (ii) By [R2, 6.2], the group M_J has a twin root datum structure with positive Borel subgroup $\Omega \cap M_J$ (which we denote by Ω_J), negative Borel subgroup $\Gamma \cap M_J$ (which we denote by Γ_J), and Weyl group W_J. Moreover the group Ω_J is generated by T and the root groups U_a where a is a positive root of the form $w.a_i$ with $i \in J$ and $w \in W_J$. By the proof of Corollary 1 in [MooP, 5.7], such a root contains R_J, so the corresponding group U_a fixes $\overline{R}_J$ pointwise. Since T fixes pointwise the whole apartment A and $\mathcal{B}_J = \overline{\Omega_J}$, we proved the first assertion. We have $T \subset \Omega_J$ and $\mathcal{B}_J \subset \mathcal{B}$ since $\mathcal{B}_J$ fixes R: this proves the second point by setting $\widehat{U_J} = \mathcal{B}_J \cap \widehat{U}_R$.

The Borel subgroup Ω_J of the positive Tits system of M_J is the fixator of R_J, so X_J is a geometric realization of the corresponding building and $\overline{M}_J$ is strongly transitive on X_J. This proves (iii) and the fact that $\overline{M}_J$ admits a natural Tits system with Weyl group W_J and abstract Borel subgroup $\mathcal{B}_J$. We are now in position to apply the same arguments as in [RR, 1.C] to prove that we actually have a refined Tits system, which proves (i).

(iv) The first assertion follows from the fact that we have $\overline{M}_J = \overline{G}_J \cdot T$. The same argument as for Theorem 2.A.1 (ii) implies that we are reduced to considering the groups corresponding to the connected components of the Dynkin subdiagram obtained from the one of Λ by removing the vertices of type $\notin J$ (and the edges containing one of them). Such groups commute with one another, and they are topologically simple by Lemma 2.A.1. $\square$

REMARKS. 1) Any proper submatrix of an irreducible affine generalized Cartan matrix is of finite type [K, Theorem 4.8]. Consequently, the Levi factors of the parabolic subgroups of Λ are all finite groups of Lie type in this case.

2) Conversely, if the submatrix of type J of the generalized Cartan matrix of Λ is not spherical (i.e. of finite type), then $\overline{M}_J$ contains an infinite topologically simple subgroup. Such a group comes from a non-spherical connected component of the Dynkin subdiagram given by J.

2.B.2 The virtual topological simplicity of closures of Levi factors enables us to prove the following result about actions of Kac–Moody groups on Euclidean buildings.

PROPOSITION. *Let Λ be a countable Kac–Moody group which is almost split over the finite field $\mathbf{F}_q$ with $q > 4$, which is generated by its root groups and which has connected Dynkin diagram. We assume we are given a continuous group homomorphism $\mu : \overline{\Lambda} \to \mathbf{G}(k)$, where $\mathbf{G}$ is a semisimple group defined over a non-Archimedean local field k. We denote by Δ the Bruhat–Tits building of $\mathbf{G}$ over k, and for each point x in the Kac–Moody building X, we denote by $\overline{\Lambda}_x$ the fixator $\mathrm{Fix}_{\overline{\Lambda}}(x)$.*

(i) *For any point x in the Kac–Moody building X, the set of fixed points $\Delta^{\mu(\overline{\Lambda}_x)}$ of $\mu(\overline{\Lambda}_x)$ in Δ is a non-empty closed convex union of facets in the Euclidean building Δ.*
(ii) *If the image $\mu(\overline{\Lambda})$ is non-trivial, then for any pair of distinct vertices $v \neq v'$ in X, the sets of fixed points $\Delta^{\mu(\overline{\Lambda}_v)}$ and $\Delta^{\mu(\overline{\Lambda}_{v'})}$ are disjoint.*

REMARK. As W is infinite, (ii) implies that $\mu(\overline{\Lambda})$ is unbounded in $\mathbf{G}(k)$ since it has no global fixed point in Δ.

Proof. (i) Let $x \in X$. By the very definition of the topology on $\mathrm{Aut}(X)$, the group $\overline{\Lambda}_x$ is compact, and so is its continuous image $\mu(\overline{\Lambda}_x)$. By the Bruhat–Tits fixed point lemma [Bro, VI.4], the set of $\mu(\overline{\Lambda}_x)$-fixed points in Δ is non-empty. The $\mathbf{G}(k)$-action on Δ is simplicial, and since $\overline{\Lambda}$ is topologically simple (2.A.1) and μ is continuous, the $\mu(\overline{\Lambda})$-action is type-preserving. Therefore each time a subgroup of $\mu(\overline{\Lambda})$ fixes a point, it fixes the closure of the facet containing it. The convexity of $\Delta^{\mu(\overline{\Lambda}_x)}$ comes from the intrinsic definition of a geodesic segment in Δ [Bro, VI.3A], and from the fact that $\mathbf{G}(k)$ acts isometrically on Δ.

(ii) A type of vertex in the CAT(0)-realization of a building defines a subdiagram in the Dynkin diagram of Λ which is spherical and maximal for this property [R2, 4.3]. Therefore the fixator of a vertex v in the building X is a maximal spherical parabolic subgroup of the Tits system of 1.B.1 with Borel subgroup the Iwahori subgroup $\mathcal{B}$. Now let v' be another vertex in X. By [Bou, IV.2.5], the group generated by $\overline{\Lambda}_v$ and $\overline{\Lambda}_{v'}$ is a parabolic subgroup of the latter Tits system, which cannot be spherical since it is strictly bigger than $\overline{\Lambda}_v$. By the second remark in 2.B.1, the closed subgroup $\overline{\langle \overline{\Lambda}_v, \overline{\Lambda}_{v'} \rangle}$ generated by $\overline{\Lambda}_v$ and $\overline{\Lambda}_{v'}$ contains an infinite topologically simple group H; up to conjugacy, this group is a factor of Proposition 2.B.1 (iv).

We assume now that there exist two vertices $v \neq v'$ in X such that $\Delta^{\mu(\overline{\Lambda}_v)} \cap \Delta^{\mu(\overline{\Lambda}_{v'})}$ contains a point $y \in \Delta$. Then the image of $\overline{\langle \overline{\Lambda}_v, \overline{\Lambda}_{v'} \rangle}$ by μ lies in the compact fixator of y in $\mathbf{G}(k)$. Restricting μ to the topologically simple group H, and composing with an embedding of k-algebraic groups $\mathbf{G} \subset \mathrm{GL}_m$, we are in position to apply the lemma below: $\mu(H)$ is trivial since it is bounded. But then the kernel of μ is non-trivial hence equal to $\overline{\Lambda}$ by topological simplicity (2.A.1). Finally $\mu(\overline{\Lambda}) \neq \{1\}$ implies $\Delta^{\mu(\overline{\Lambda}_v)} \cap \Delta^{\mu(\overline{\Lambda}_{v'})} = \varnothing$ whenever $v \neq v'$. □

We finally prove the following quite general and probably well-known lemma.

LEMMA. *Let H be an infinite topological group all of whose proper closed normal subgroups are finite. Let k be a non-Archimedean local field and let $\mu : H \to \mathrm{GL}_m(k)$ be a continuous homomorphism for some $m \geq 1$. Then $\mu(H)$ is either trivial or unbounded.*

Proof. We denote by $\mathcal{O}$ the valuation ring and choose a uniformizer ϖ. Let us assume that $\mu(H)$ is bounded. After conjugation, we may – and shall – assume that $\mu(H)$ lies in $\mathrm{GL}_m(\mathcal{O})$ [PlR, 1.12]. For each integer $N \geq 1$, the group $\mathrm{GL}_m(\mathcal{O})$ (denoted by K) has an open finite index congruence subgroup $K(N) = \ker(\mathrm{GL}_m(\mathcal{O}) \to \mathrm{GL}_m(\mathcal{O}/\varpi^N))$. For each $N \geq 1$, $\mu^{-1}(K(N))$ is a closed normal finite index subgroup of H. Since H is infinite, so is $\mu^{-1}(K(N))$, and by the hypothesis on closed normal subgroups of H we have $H = \mu^{-1}(K(N))$. Since $\bigcap_{N\geq 1} K(N) = \{1\}$, we finally have $\mu(H) = \{1\}$. □

3 Embedding Theorem

We use the commensurator super-rigidity to prove that the linearity of a countable Kac–Moody group implies that the corresponding topological group is a closed subgroup of a non-Archimedean Lie group.

3.A Embedding theorem and non-amenability. We first state the result. Then we discuss its hypotheses, mostly the one involving amenability.

Theorem. *Let Λ be an almost split Kac–Moody group over the finite field $\mathbf{F}_q$ of characteristic p with $q > 4$ elements, with infinite Weyl group W and buildings X and X_-. Let $\overline{\Lambda}$ be the corresponding Kac–Moody topological group. We make the following assumptions:*

(TS) *the group $\overline{\Lambda}$ is topologically simple;*

(NA) *the group $\overline{\Lambda}$ is not amenable;*

(LT) *the group Λ is a lattice of $X \times X_-$ for its diagonal action.*

Then, if Λ is linear over a field of characteristic p, there exists:

- *a local field k of characteristic p and a connected adjoint k-simple group $\mathbf{G}$,*
- *a topological embedding $\mu : \overline{\Lambda} \to \mathbf{G}(k)$ with Hausdorff unbounded and Zariski dense image,*
- *and a μ-equivariant embedding $\iota : \mathcal{V}_X \to \mathcal{V}_\Delta$ from the set of vertices of the Kac–Moody building X of Λ to the set of vertices of the Bruhat–Tits building Δ of $\mathbf{G}(k)$.*

When $q > 4$ and Λ is generated by its root groups, condition (TS) is equivalent to the connectedness of the Dynkin diagram of Λ (2.A.1). Condition (LT) is equivalent to Γ being a lattice of $\overline{\Lambda}$, which holds whenever q is big enough (1.C.1). Finally, condition (NA) is fulfilled for q large enough, too.

Lemma. *Let Λ be an almost split Kac–Moody group over the finite field $\mathbf{F}_q$ of characteristic p with $q > 4$ elements, with infinite Weyl group W and buildings X and X_-. Let $\overline{\Lambda}$ be the corresponding Kac–Moody topological group. Then, if q is large enough, neither Λ for the discrete topology nor $\overline{\Lambda}$ for its topology of closed subgroup of* $\mathrm{Aut}(X)$ *is amenable.*

Proof. Let us start with preliminary remarks. The group Λ is chamber-transitive on X, so it is non-compact, and $\overline{\Lambda}$ is cocompact in $\mathrm{Aut}(X)$. Therefore $\overline{\Lambda}$ is amenable (resp. Kazhdan) if and only if $\mathrm{Aut}(X)$ is [Z, 4.1.11] (resp. [M1, III.2.12]). Since amenable groups with property (T) are compact [Z, 7.1.9], condition (NA) is satisfied whenever $\mathrm{Aut}(X)$ has (T). Another case when (NA) is fulfilled is when the building X admits a CAT(-1) metric. Indeed, amenability of $\mathrm{Aut}(X)$ would imply the existence of an $\mathrm{Aut}(X)$-invariant probability measure on $\partial_\infty X$. Since $\mathrm{Aut}(X)$ is non-compact, the relevant Furstenberg lemma in this context [BuMo1, Lemma 2.3] would imply that the support of this measure consists of at most two points. This would imply the stability of a boundary point or a geodesic, in contradiction with the chamber-transitivity of the Λ-action on X.

We henceforth assume that (LT) holds, so that amenability of Λ, of $\overline{\Lambda}\times\overline{\Lambda}$ and of $\overline{\Lambda}$ are equivalent. Since the Weyl group W of Λ is infinite, its Coxeter diagram has at least one non-spherical connected component. As for the proof of Theorem 2.A.1 and since a closed subgroup of an amenable group is amenable, we are reduced to considering the case when the Coxeter diagram of W is connected. If W is of affine type, then we use the above remarks: if W is infinite dihedral then the building X is a tree hence CAT(-1), and in any other case [DyJ] implies that $\mathrm{Aut}(X)$ has property (T) when q is large enough. Otherwise, [MV, Corollary 2] shows that W virtually surjects onto a non-abelian free group, which implies that W is non-amenable; then the group N, which lifts W modulo the finite $\mathbf{F}_q$-torus T, is not amenable, and neither is Λ which contains it. □

The remainder of this section is devoted to the proof of the embedding theorem.

3.B Semisimple Zariski closure and injectivity. The linearity assumption says that there is a field $\mathbf{F}$ of characteristic p and an injective group homomorphism $\eta : \Lambda \hookrightarrow \mathrm{GL}_N(\mathbf{F})$ to a general linear group over $\mathbf{F}$. We choose an algebraic closure $\overline{\mathbf{F}}$ of $\mathbf{F}$, and denote by $\mathbf{H}$ the Zariski closure $\overline{\eta(\Lambda)}^Z$ of the image of η in GL_N. Let $\mathcal{R}\mathbf{H}^\circ$ be the radical of the identity component $\mathbf{H}^\circ$ of $\mathbf{H}$ in the Zariski topology. The group $\mathbf{H}^\circ/\mathcal{R}\mathbf{H}^\circ$ is connected normal and of finite index in $\mathbf{H}/\mathcal{R}\mathbf{H}^\circ$, hence it is its (semisimple) identity component. We denote by $\mathbf{L}$ the quotient of $\mathbf{H}/\mathcal{R}\mathbf{H}^\circ$ by its finite center $\mathbf{Z}(\mathbf{H}/\mathcal{R}\mathbf{H}^\circ)$, and by q the natural quotient map $\mathbf{H} \to \mathbf{L}$. Note that $\mathbf{L}^\circ$ is adjoint semisimple and that the identity component of $\ker(q)$ is solvable. We consider the composed homomorphism,

$$\varphi : \Lambda \xrightarrow{\eta} \mathbf{H} \xrightarrow{q} \mathbf{L}\,.$$

Let $\mathbf{L} \hookrightarrow \mathrm{GL}_M$ be an embedding of linear algebraic groups. Since Λ is finitely generated, there is a finitely generated field extension $\mathbf{E}$ of $\mathbf{Z}/p\mathbf{Z}$ in $\overline{\mathbf{F}}$ such that $\varphi(\Lambda)$ lies in $\mathrm{GL}_M(\mathbf{E})$. The group $\varphi(\Lambda)$ is Zariski dense in $\mathbf{L}$, so the group $\mathbf{L}$ is defined over $\mathbf{E}$ [Bo, AG 14.4], and $\varphi(\Lambda)$ lies in $\mathbf{L}(\mathbf{E})$.

Lemma. *The group homomorphism $\varphi : \Lambda \to \mathbf{L}(\mathbf{E})$ is injective.*

Proof. Let us assume that the kernel of φ is non-trivial, so that its closure is a non-trivial closed normal subgroup of $\overline{\Lambda}$. By topological simplicity (2.A.1) we have $\overline{\ker(\varphi)} = \overline{\Lambda}$. But since $\varphi = q \circ \eta$ and since η is injective, we have $\ker(\varphi) = \eta^{-1}(\eta(\Lambda) \cap \ker(q)) \simeq \eta(\Lambda) \cap \ker(q)$. This would imply that $\ker(\varphi)$ is virtually solvable, hence amenable for the discrete topology [Z, 4.1.7]. Then $\overline{\ker(\varphi)} = \overline{\Lambda}$ and [Z, 4.1.13] would imply that $\overline{\Lambda}$ is amenable, which is excluded by (NA). □

3.C Unbounded image and continuous extension. Our next goal is to check that we are in position to use commensurator super-rigidity.

Proposition. (i) *There exist an infinite order element γ in the lattice Γ and a field embedding $\sigma : \mathbf{E} \to k$ into a local field of characteristic p such that $\sigma(\varphi(\gamma))$ is semisimple with an eigenvalue of absolute value > 1 in the adjoint representation of $\mathbf{L}(k)$.*

(ii) *There is a connected adjoint k-simple group $\mathbf{G}$ and an injective continuous group homomorphism $\mu : \overline{\Lambda} \to \mathbf{G}(k)$. The map μ coincides on a finite index subgroup of Λ with the composition of φ with the projection onto a k-simple factor of $\mathbf{L}^\circ$; its image is Zariski dense and Hausdorff unbounded.*

Proof. (i) Let us fix a reflection s in a wall H_s containing a panel of the standard chamber R, and let us denote by a_s the simple root bordered by H_s. The condition (TS) implies that the Weyl group W of Λ (or $\overline{\Lambda}$) is indecomposable (of non-spherical type). Therefore by [H, Proposition 8.1, p. 309] there is a reflection r in a wall H_r such that rs has infinite order, implying that H_r and H_s don't meet in the interior of the Tits cone of W [R2, 5.2].

We call $-b$ the negative root bordered by H_r. If $-a_s \cap -b = \varnothing$ then by definition $\{-a_s; -b\}$ is a non-prenilpotent pair of roots [T2, 3.2]; otherwise $\{-a_s; -s.b\}$ is. In any case the group generated by the corresponding root groups is isomorphic to $\mathbf{F}_q * \mathbf{F}_q$ [T4, Proposition 5]. Therefore it contains an element $\tilde{\gamma} \in \Gamma$ of infinite order, and by injectivity $\varphi(\tilde{\gamma})$ has infinite order too. Since the field $\mathbf{E}$ has characteristic $p > 0$, a suitable power p^r kills the unipotent part of the Jordan decomposition of $\varphi(\tilde{\gamma})$. Let us set $\gamma = \tilde{\gamma}^{(p^r)}$, so that $\varphi(\gamma)$ is semisimple. It has an eigenvalue λ of infinite multiplicative order, so by [T1, Lemma 4.1] there is a local field k endowed with a valuation v and a field embedding $\sigma : \mathbf{E}[\lambda] \to k$ such that $v(\sigma(\lambda)) \neq 0$. Up to replacing γ by γ^{-1}, this proves (i).

(ii) By (i) the composed map $(\mathbf{L}(\sigma) \circ \varphi) : \Lambda \to \mathbf{L}(\mathbf{E}[\lambda]) \to \mathbf{L}(k)$, which for short we still denote by φ, is such that Γ has unbounded image in $\mathbf{L}(k)$. Let us introduce the preimage $\Lambda^\circ = \varphi^{-1}(\mathbf{L}^\circ)$. It is a normal finite index subgroup of Λ. We also set $\Gamma^\circ = \Gamma \cap \Lambda^\circ$. Since Γ° is of finite index in Γ, $\varphi(\Gamma^\circ)$ is not relatively compact in $\mathbf{L}^\circ(k)$. The connected adjoint semisimple k-group $\mathbf{L}^\circ$ decomposes as a direct product of adjoint connected k-simple factors. One of them, which we denote by $\mathbf{G}$, is such that the projection of $\varphi(\Gamma^\circ)$ is not relatively compact. The abstract group homomorphism we consider now, and which we denote by $\varphi|_{\Lambda^\circ}$, is obtained by composing with the projection onto $\mathbf{G}$. Therefore we obtain $\varphi|_{\Lambda^\circ} : \Lambda^\circ \to \mathbf{G}(k)$ such that $\varphi(\Gamma^\circ)$ is unbounded in $\mathbf{G}(k)$. We also have $\overline{\varphi(\Lambda^\circ)}^Z = \mathbf{G}$.

By Lemma 1.C.1 the group Λ is contained in the commensurator $\mathrm{Comm}_{\overline{\Lambda}}(\Gamma)$. Since Γ° is of finite index in Γ, we have $\mathrm{Comm}_{\overline{\Lambda}}(\Gamma) = \mathrm{Comm}_{\overline{\Lambda}}(\Gamma^\circ)$, so we are in position to apply the commensurator superrigidity theorem of the Appendix in order to extend $\varphi|_{\Lambda^\circ}$ to a continuous homomorphism $\mu : \overline{\Lambda^\circ} \to \mathbf{G}(k)$, where $\overline{\Lambda^\circ}$ denotes the closure of Λ° in $\mathrm{Aut}(X)$. The non-trivial closed subgroup $\overline{\Lambda^\circ}$ is normal in $\overline{\Lambda}$, hence it is $\overline{\Lambda}$ by topological simplicity (2.A.1). Therefore there is a map $\mu : \overline{\Lambda} \to \mathbf{G}(k)$ which coincides with the abstract group homomorphism φ on Λ°. By topological simplicity of $\overline{\Lambda}$, μ is either injective or trivial. By Zariski density of the image, the only possible case is that μ be injective. Summing up, we have obtained an injective continuous group homomorphism $\mu : \overline{\Lambda} \hookrightarrow \mathbf{G}(k)$ such that $\mu(\Gamma^\circ)$ is unbounded in $\mathbf{G}(k)$ and $\overline{\mu(\Lambda^\circ)}^Z = \mathbf{G}$. □

3.D Embedding of vertices and closed image. We can finally conclude in view of the following lemma.

LEMMA. (i) *There is a μ-equivariant injective unbounded map $\iota : \mathcal{V}_X \hookrightarrow \mathcal{V}_\Delta$*

from the vertices of the Kac–Moody building X into the vertices of the Bruhat–Tits building Δ of $\mathbf{G}(k)$.

(ii) *The continuous homomorphism μ sends closed subsets of $\overline{\Lambda}$ to closed subsets of $\mathbf{G}(k)$.*

First, let us briefly recall some facts (see [Ro, §5] for the general connection between Tits systems and buildings, 1.B for our specific case). We keep the inclusion $R \subset A$ of the standard chamber in the standard apartment. Let W_R be the quotient of the stabilizer $N_A = \mathrm{Stab}_{\overline{\Lambda}}(A)$ by the fixator $\Omega_A = \mathrm{Fix}_{\overline{\Lambda}}(A)$, which is the Weyl group of the building X (and of the groups Λ and $\overline{\Lambda}$). It is generated by the reflections along the panels of R, and is simply transitive on the chambers of A. We denote by $\mathcal{B}$ the standard Iwahori subgroup $\overline{\Lambda}_R$. By Theorem 1.B.1, $\mathcal{B}$ is the Borel subgroup of a Tits system in $\overline{\Lambda}$ with Weyl group W_R. The Tits system structure implies a Bruhat decomposition [Bou, IV.2.3]: $\overline{\Lambda} = \bigsqcup_{w \in W_R} \mathcal{B} w \mathcal{B}$.

Proof. (i) By 2.B.2 (i) we choose, for each vertex v in the closure of the chamber R, a $\mu(\overline{\Lambda}_v)$-fixed vertex $\iota(v) \in \Delta$. We can extend μ-equivariantly this choice $\overline{R} \cap \mathcal{V}_X \to \mathcal{V}_\Delta$ to obtain a map $\iota : \mathcal{V}_X \to \mathcal{V}_\Delta$, where $\iota(v)$ is a $\mu(\overline{\Lambda}_v)$-fixed vertex in Δ for each vertex v in X. By 2.B.2 (ii), the sets of fixed points $\Delta^{\mu(\overline{\Lambda}_v)}$ are mutually disjoint when v ranges over $\mathcal{V}_X$, so ι is injective. By discreteness of the vertices in Δ, the unboundedness of ι follows from its injectivity because $\mathcal{V}_X$ is infinite (since so is W).

(ii) Let F be a closed subset of $\overline{\Lambda}$; we must show that $\overline{\mu(F)} \subset \mu(F)$. Let $g = \lim_{n\to\infty} \mu(h_n)$ be in $\overline{\mu(F)}$, with $h_n \in F$ for each $n \geq 1$. It is enough to show that $\{h_n\}_{n\geq 1}$ has a converging subsequence. By the Bruhat decomposition $\overline{\Lambda} = \bigsqcup_{w \in W_R} \mathcal{B} w \mathcal{B}$, we can write $h_n = k_n w_n k'_n$ with $k_n, k'_n \in \mathcal{B}$ and $w_n \in N_A$. Since by compactness of $\mathcal{B}$ the sequences $\{k_n\}_{n\geq 1}$ and $\{k'_n\}_{n\geq 1}$ admit cluster values, we are reduced to the situation where $g = \lim_{n\to\infty} \mu(w_n)$ with $w_n \in N_A$ for each $n \geq 1$.

Let us assume that the union of chambers $\bigcup_{n\geq 1} w_n.R$ is unbounded in A. Then there is an injective subsequence of chambers $\{w_{n_j}.R\}_{j\geq 1}$. Let us fix a vertex $v \in \overline{R}$. Since its stabilizer in W_R is finite, possibly after extracting again a subsequence, we get an injective sequence of vertices $\{w_{n_j}.v\}_{j\geq 1}$. But $\mu(w_{n_j}).\iota(v) = \iota(w_{n_j}.v)$ where $\iota : \mathcal{V}_X \hookrightarrow \mathcal{V}_\Delta$ is the μ-equivariant embedding of vertices of (i). Since $g = \lim_{n\to\infty} \mu(w_n)$, the continuity of the $\mathbf{G}(k)$-action on Δ implies: $\lim_{j\to\infty} \mu(w_{n_j}).\iota(v) = g.\iota(v)$. By discreteness of the vertices in Δ, the sequence $\{\iota(w_{n_j}.v)\}_{j\geq 1}$ hence the sequence $\{w_{n_j}.v\}_{j\geq 1}$ is eventually constant: a contradiction.

We henceforth know that the sequence $\{w_n.R\}_{n\geq 1}$ is bounded, hence takes finitely many values. So there is a subsequence $\{w_{n_j}\}_{j\geq 1}$ and $w \in N_A$ such that $\{w_{n_j}\}_{j\geq 1}$ is constant equal to w modulo Ω_A. This proves the lemma, possibly after extracting a converging subsequence in the compact subgroup Ω_A of $\mathcal{B}$. □

4 Some Concrete Non-linear Examples

We prove that most countable Kac–Moody groups with right-angled Fuchsian buildings are not linear over any field. This requires to settle structure results for boundary point stabilizers and generalized unipotent radicals. Dynamical arguments due to Prasad play a crucial role. The geometry of compactifications of buildings sheds some light on ideas of the proof.

4.A Groups with right-angled Fuchsian buildings. Let R be a regular hyperbolic right-angled r-gon, so that $r \geq 5$. To obtain a countable Kac–Moody group with a building covered by chambers isometric to R, we need to lift the Coxeter diagram of the corresponding Fuchsian Weyl group W_R to a Dynkin diagram. The Coxeter diagram of the latter group is connected and all its edges are labelled by ∞, so according to the rule [T2, 3.1] infinitely many Dynkin diagrams are suitable. Henceforth, Λ denotes a Kac–Moody group over $\mathbf{F}_q$ whose positive building X is isomorphic to some $I_{r,1+q}$ (1.C.2). We choose a standard positive chamber R in a standard positive apartment $A \simeq \mathbf{H}^2$ (1.A.1). We denote by d the natural CAT(-1) distance on X and by ℓ the length of any edge. We fix a numbering $\{E_i\}_{i\in\mathbf{Z}/r\mathbf{Z}}$ by $\mathbf{Z}/r\mathbf{Z}$ of the edges of R, and we denote by a_i the simple root containing R whose wall contains E_i. Note that since all wall intersections are orthogonal, all the edges in a given wall L have the same type, which we also call the *type* of L [RR, 4.A].

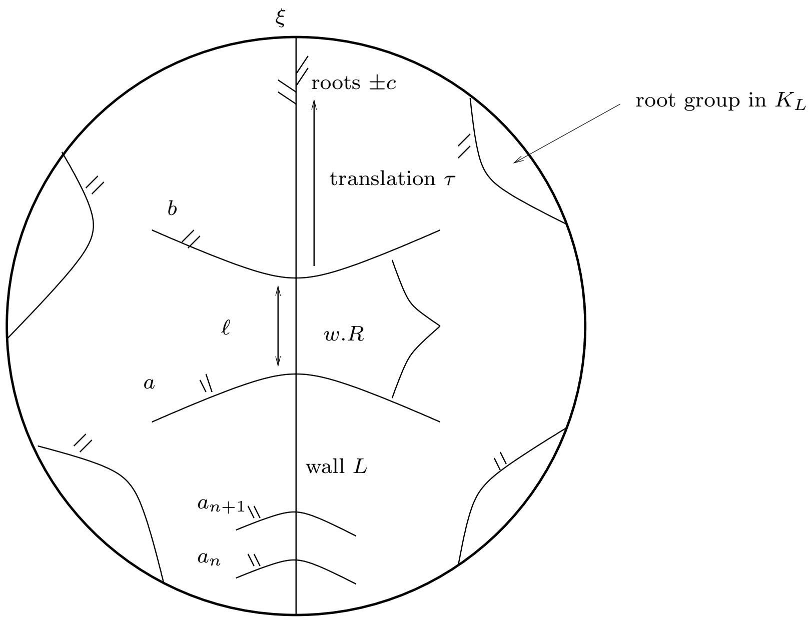
4.A.1 A *geodesic ray* in a geodesic CAT(-1) metric space (X,d) is an isometry $r : [0;\infty) \to X$. The *Busemann function* of r is the function $f_r : X \to \mathbf{R}$ defined by $f_r(x) = \lim_{t\to\infty}(d(x,r(t)) - t)$. The *horosphere* (resp. *horoball*) associated to r is the level set $\{f_r = 0\}$ (resp. $\{f_r \leq 0\}$), which we denote by $H(r)$ (resp. $Hb(r)$). Let $\xi \in \partial_\infty X$ be a boundary point of $X \simeq I_{r,1+q}$, i.e. an asymptotic class of geodesic ray [GH, §7].

Definition. (i) *We call* parabolic subgroup attached to ξ *the stabilizer* $\mathrm{Stab}_{\overline{\Lambda}}(\xi)$. *We denote it by* P_ξ.

(ii) *We call* horospheric subgroup attached to ξ *the subgroup of* P_ξ *stabilizing each horosphere centered at* ξ. *We denote it by* D_ξ.

Remark. The terminology mimicks the geometric definition of proper parabolic subgroups in semisimple Lie groups, but there are differences. There are two kinds of boundary points, according to whether the point is the end of a wall or not. A point of the first kind is called *singular*; otherwise, it is called *regular*. This provides two kinds of parabolic subgroups which are both amenable by [BuMo1, Proposition 1.6]. The only classical case when all proper parabolic subgroups are amenable is when they are minimal, i.e. when the split rank equals one. But then all proper parabolics are conjugate.

We are now interested in singular boundary points. Let L be a wall and let a and b be two roots such that a contains b, and whose walls ∂a and ∂b intersect transversally L. The reflection along ∂a (resp. ∂b) is denoted by r_a (resp. r_b) and $r_b.r_a$ is a hyperbolic translation along L with attracting point ξ contained in a, which we denote by τ. We assume henceforth that the strip $a \cap (-b)$ of A doesn't contain any other wall intersecting L. Then τ is a *two-step hyperbolic translation*, i.e. a translation of (minimal) length 2ℓ. We set $a_0 = a$, $a_1 = b$, $a_{2j} = \tau^j.a$, $a_{2j+1} = \tau^j.b$ for $j \in \mathbf{Z}$, so that a_k contains a_{k+1}, and for each $n \in \mathbf{Z}$ we denote by v_n the vertex $\partial a_n \cap L$. The image of the geodesic ray $r : [0;\infty) \to X$ such that $r(k\ell) = v_{n+k}$ for each $k \geq 0$, is $[v_n\xi) = a_n \cap L$.



Definition. (i) *We denote by* V_n *the closed subgroup generated by the root groups* U_{a_k} *such that* $k \leq n$. *We denote* $\bigcup_{n \in \mathbf{Z}} V_n$ *by* $V_{\xi,A}$ *and call it a* generalized unipotent radical.

(ii) *We denote by* K_L *the fixator of the wall* L *in* $\overline{\Lambda}$.

(iii) *We denote by* $M_{L,A}$ *the finite reductive group over* $\mathbf{F}_q$ *generated by the torus* T *and the two opposite root groups* $U_{\pm c}$ *such that* $\partial c = L$.

Remark. For each $n \geq 1$ the group V_n is compact since it fixes the half-space a_n of A.

Two half-walls define the same *germ* if they intersect along a half-wall. Two half-walls in the same germ clearly define the same boundary point, and the converse is true because any two disjoint closed edges are at distance $\geq \ell$, so we can talk of the germ of a singular boundary point. Since any element of $\overline{\Lambda}$ sends a wall onto a wall, we have the following characterizations.

Lemma. *The group* P_ξ *is the stabilizer of the germ of* ξ, *and* D_ξ *is the fixator of this germ, meaning that there is a half-wall in it which is fixed under* D_ξ. □

4.A.2 Thanks to the Moufang property and the language of horoballs, we can say more about the groups V_n. Let us denote by E the intersection of the wall L with the strip $a \cap (-b)$. By the previously assumed minimality of $a \cap (-b)$, E is reduced to the edge of a chamber $w.R$. Transforming the objects above by $w^{-1} \in W_R$, we may – and shall – assume that we are in the case where L is the wall ∂a_i (where i the type of E), and either $a = a_{i-1}$ and $b = -a_{i+1}$, or $a = a_{i+1}$ and $b = -a_{i-1}$. These two situations are completely analogous, and we assume that $a = a_{i-1}$ and $b = -a_{i+1}$. We denote by r_j the reflection in the edge E_j of $\overline{R}$, we set $J = \{i-1; i+1\}$, $W_J = \langle r_{i-1}, r_{i+1} \rangle$ and we use notions and notation of 2.B.1. Then $V_{\xi,A}$ is a subgroup of the topologically simple group $\overline{G}_J$, and the inessential building X_J is a combinatorial tree. Its vertices (resp. edges) are the $\overline{G}_J$-transforms of a line ∂a_{i-1} or ∂a_{i+1} (resp. of the strip $a_{i-1} \cap a_{i+1}$). The root groups in $\overline{G}_J$ are those indexed by the roots $a = w.a_j$ with $w \in W_J$ and $j \in J$. They are automorphisms of the tree X_J fulfilling the Moufang condition [Ro, §6.4]. Selecting in each strip of X_J the $\overline{G}_J$-transform of the edge E_i of R provides a bijection between the inessential tree X_J and the tree-wall attached to E_i. Recall that a *tree-wall* is a class of the equivalence relation on edges for which two edges are equivalent if they are contained in the same wall of some apartment [Bour2, 2.4]. We henceforth adopt the tree-wall viewpoint when dealing with X_J, so that v_i are vertices and ξ is a boundary point of it. Moreover the geodesic rays, horospheres and horoballs, mentioned below, are those defined in the tree-wall X_J.

LEMMA. *We assume that all the root groups indexed by the roots $a = w.a_j$ with $w \in W_J$ and $j \in J$, and containing ξ, commute with one another.*

(i) *The group V_n acts trivially on the horoball defined by $[v_n\xi)$.*
(ii) *For each vertex v on the horosphere defined by $[v_n\xi)$, the group U_{a_n} is simply transitive on the edges containing v which are outside the corresponding horoball.*
(iii) *We have $\bigcap_{n\geq 1} V_n = \{1\}$.*
(iv) *Any $g \in V_{\xi,A}$ stabilizing $[v_n\xi) = a_n \cap L$ belongs to V_n.*

REMARK. The roots $a = w.a_j$ with $w \in W_J$ and $j \in J$ are the real roots of a rank 2 Kac–Moody root system, and the corresponding root groups generate a rank 2 countable Kac–Moody group with generalized Cartan matrix $\begin{pmatrix} 2 & A_{i-1,i+1} \\ A_{i-1,i+1} & 2 \end{pmatrix}$. As explained in 4.A, these off-diagonal coefficients are ≤ -1, and their product is ≥ 4. According to the explicit commutator relations due to J. Morita [Mor, §3 (6)], the group generated by the root groups U_a such that $\bar{a}$ contains ξ is abelian whenever the off-diagonal coefficients are both ≤ -2 (otherwise it may be metabelian), so the assumption made in the lemma is quite non-restrictive.

Proof. Let v be a vertex in X_J and let v_N be the projection of v on the geodesic $\{v_i\}_{i\in\mathbf{Z}}$ for some $N \in \mathbf{Z}$. By the Moufang property, there are uniquely defined $m \geq 0$ and $u_i \in U_{a_i}$ for $N - m < i \leq N$ such that $v = (u_N u_{N-1} \dots u_{N-m+1}).v_{N-m}$. Denoting by f_ρ the Busemann function of $\rho = [v_0\xi)$, we have $f_\rho(v) = (m-N)\ell$. The Moufang property thus provides a parametrization of the horoballs centered at ξ in X_J since $H([v_j\xi)) = \{(u_N u_{N-1} \dots u_{N-m+1}).v_{N-m} : N \in \mathbf{Z},\ m \geq 0,\ N-m = j,\ u_N \neq 1$ if $m \neq 0$, and $u_i \in U_{a_i}$ for $N - m < i \leq N\}$. The commutation of all the root groups U_{a_k}, along with the parametrization of the horoballs by means of root groups, proves (i) and (ii). Moreover if $h \in \bigcap_{n\geq 1} V_n$, then (i) implies that h fixes all the horoballs centered at ξ in X_J, so h belongs to the kernel of the $\overline{G}_J$-action on X_J. We have $h = 1$ by topological simplicity (2.A.1), which proves (iii).

(iv) By (i) any $g \in V_{\xi,A}$ stabilizes the horospheres centered at ξ, so if g stabilizes $[v_n\xi)$, it fixes it. By definition of $V_{\xi,A}$ as an increasing union of groups, it is enough to show that $\mathrm{Fix}_{V_N}([v_n\xi)) = V_n$ for each $N > n$. This follows from the fact that an element of $V_m \setminus V_m$ for $m > n$ sends the segment $[v_n; v_m]$ on a segment with origin v_m whose first edge is different from $[v_n; v_{n+1}]$. □

4.A.3 We can now state the main properties of the boundary point stabilizers and of their generalized unipotent radicals in the commutative case, keeping the previous notation.

Proposition. *We assume that all the root groups indexed by the roots $a = w.a_j$ with $w \in W_J$ and $j \in J$, and containing ξ, commute with one another.*

(i) *The group $V_{\xi,A}$ is closed, normalized by $\langle \tau \rangle$ but not by K_L. Each group V_n is abelian of exponent p, hence so is $V_{\xi,A}$.*

(ii) *The group K_L is normalized but not centralized by $\langle \tau \rangle$. It admits a semidirect product decomposition $M_{L,A} \ltimes \widehat{U}_L$, where $\widehat{U}_L$ is a pro-p group. In particular, K_L is virtually a pro-p group.*

(iii) *The following decompositions hold: $P_\xi = K_L \cdot \langle \tau \rangle \cdot V_{\xi,A}$ and $D_\xi = K_L \cdot V_{\xi,A}$, with trivial pairwise intersections of the factors $\langle \tau \rangle \cap K_L = \langle \tau \rangle \cap V_{\xi,A} = K_L \cap V_{\xi,A} = \{1\}$.*

Proof. (i) Let $u \in \overline{V}_{\xi,A}$. We write: $u = \lim_{j \to \infty} u_j$ with $u_j \in V_{\xi,A}$ for each $j \geq 1$, and we have $u \in P_\xi$. By Lemma 4.A.1, there is an $n \in \mathbf{Z}$ such that u sends the geodesic ray $[v_n \xi)$ to a geodesic ray contained in L and ending at ξ, hence this last geodesic ray is $[v_m, \xi)$. The point $u(v_n)$, which is equal to $u_j(v_n)$ for j large enough, if different from v_n, would not belong to L. This implies that $u \in V_n \subset V_{\xi,A}$, hence $V_{\xi,A}$ is closed.

We turn now to the group-theoretic properties of $V_{\xi,A}$. By assumption, all the groups U_{a_k} commute, so the continuous commutator map $[\,.\,,.\,]$ is trivial on a topologically generating set for each V_n. This proves the commutativity of each group V_n. Any of the commuting root groups U_{a_k} is isomorphic to $(\mathbf{F}_q, +)$ so replacing $[\,.\,,.\,]$ by $.^p$ shows that each V_n is of exponent p. By definition, $V_{\xi,A}$ is normalized but not centralized by τ. Pick $n \in \mathbf{Z}$ and a root a' containing L so that $U_{a'}$ lies in K_L and $-a' \cap -a_n = \varnothing$ (i.e. $\{a'; a_n\}$ is not prenilpotent). By [T4, Proposition 5], the free product $U_{a'} * U_{a_n}$ injects in Λ, so for any $u' \in U_{a'} \setminus \{1\}$ and $u \in U_{a_n} \setminus \{1\}$ the order of $[u, u']$ is infinite whereas it would divide p if K_L normalized $V_{\xi,A}$.

(ii) If Π is a panel in the wall L, we have $K_L g \overline{\Lambda}_\Pi = M_\Pi \ltimes \widehat{U}_\Pi$ (1.B.1) and $M_\Pi = M_{L,A}$ by the precise version of the Levi decomposition [R2, Theorem 6.2.2]. The group $M_{L,A}$ fixes L, so $M_{L,A}$ lies in K_L. Moreover the kernel $\widehat{U}_\Pi \cap K_L$ of the restricted map $K_L \to M_{L,A}$, which we denote by $\widehat{U}_L$, is a pro-p group, and we have $M_{L,A} \cap \widehat{U}_L = \{1\}$ by the same argument as for [RR, Lemma 1.C.5 (i)]. The group $K_L = \mathrm{Fix}_{\overline{\Lambda}}(L)$ is normalized by $\langle\tau\rangle$ because τ stabilizes L. Now we pick a root $a' \neq \pm c$ containing L, so that $U_{a'}$ lies in K_L. For a large enough integer M, the root $\tau^M.a'$ contains L but $a' \cap \tau^M.a'$ is a strip in A, implying that $\{a';\tau^M.a'\}$ is not prenilpotent. As for (i) the free product $U_{a'} * U_{\tau^M.a'}$ injects in Λ. Let us pick $u \in U_{a'} \setminus \{1\}$. Since τ lifts an element of the Weyl group (which naturally permutes the root groups [T5, 3.3, Axiom (RGD2)]), we see that $\tau^M u \tau^{-M}$ lies in $U_{\tau^M.a'} \setminus \{1\}$. This shows that the commutator $[\tau^M, u]$ lies in $(U_{\tau^M.a'} \setminus \{1\}).(U_{a'} \setminus \{1\})$. Therefore $[\tau^M, u]$ has infinite order whereas it would be trivial if K_L were centralized by τ.

(iii) Let $r : [0;\infty) \to X$ be the geodesic ray such that $r(n\ell) = v_n$ for each $n \geq 0$, and let $g \in P_\xi$. By Lemma 4.A.1, there are integers $N \geq 1$ and $t \in \mathbf{Z}$ such that $(g.r)(n\ell) = r((n+t)\ell)$ for $n \geq N$. Since the $\overline{\Lambda}$-action on X is type-preserving, t is an even number, say $2m$, and we have $(\tau^{-m}g.r)(n\ell) = r(n\ell)$ for each $n \geq N$. Thus $\tau^{-m}g$ fixes the geodesic ray $[v_N\xi)$, hence belongs to D_ξ, and we are reduced to decompose D_ξ.

Let $d \in D_\xi$, which fixes a geodesic ray $[v_N\xi)$ by Lemma 4.A.1. The link of the vertex v_N is complete bipartite, so there is a chamber R' whose closure contains both $d.[v_N;v_{N-1}]$ and an edge E' contained in the wall ∂a_N. By the Moufang property, there exists $u_N \in U_{a_N}$ such that $(u_N^{-1}d).R'$ is the chamber in A whose closure contains both $[v_N;v_{N-1}]$ and E'; in particular, $u_N^{-1}d$ fixes the geodesic ray $[v_{N-1}\xi)$. By a downwards induction, for each $m < N$ we pick $u_m \in U_{a_m}$ such that $u_m^{-1}u_{m+1}^{-1}...u_N^{-1}d$ fixes the geodesic ray $[v_{m-1}\xi)$. By compactness of V_N, the sequence $\{u_N...u_{m+1}u_m\}_{m<N}$ has a cluster value $u \in V_{\xi,A}$ such that $u^{-1}d$ fixes the geodesic L, hence belongs to K_L. Taking inverses, we proved $D_\xi = K_L \cdot V_{\xi,A}$, and along with the previous paragraph, $P_\xi = K_L \cdot \langle\tau\rangle \cdot V_{\xi,A}$ since τ normalizes $V_{\xi,A}$ and K_L.

The trivial intersection $\langle\tau\rangle \cap V_{\xi,A} = \{1\}$ follows from the fact that $\langle\tau\rangle \simeq \mathbf{Z}$ whereas any non-trivial element in $V_{\xi,A}$ has order p, and $\langle\tau\rangle \cap K_L = \{1\}$ follows from the fact that no non-trivial power τ^m fixes L. An element in $K_L \cap V_{\xi,A}$ lies in any $\mathrm{Fix}_{V_{\xi,A}}([v_n\xi))$ $(n \in \mathbf{Z})$, hence in any V_n by Lemma 4.A.2 (iv), so $K_L \cap V_{\xi,A} = \{1\}$ follows from (iii) in the same lemma. □

Remark. 1) Horoball arguments as in 4.A.2 show that each group V_n is isomorphic to $(\mathbf{F}_q[[t]],+)$ and that there is an isomorphism $V_{\xi,A} \simeq (\mathbf{F}_q((t)),+)$ under which conjugation by τ corresponds to multiplication by t^{-2} and the t-valuation corresponds to the index $_n$.

2) Let us denote by $-\xi$ the other end of L, so that $L = (-\xi\xi)$. By

definition, $D_\xi \cap D_{-\xi}$ stabilizes L and actually fixes it since D_ξ stabilizes the horospheres centered at ξ. Therefore we have $D_\xi \cap D_{-\xi} = K_L$.

4.B Dynamics and parabolics. Let us have a dynamical viewpoint on the above groups. The prototype for parabolics, used in 4.C.1, is Prasad's work in the algebraic group case [Pr].

4.B.1 A first consequence of the existence of many hyperbolic translations is the connection with Furstenberg boundaries – see [M1, VI.1.5] for a definition, where this notion is simply called a *boundary*. We denote by $\mathcal{M}^1(\partial_\infty X)$ the space of probability measures on $\partial_\infty X$; it is compact and metrizable for the weak-$*$ topology. This subsection owes its existence to discussions with M. Bourdon and Y. Guivarc'h.

LEMMA. *The asymptotic boundary $\partial_\infty X$ is a Furstenberg boundary for $\overline{\Lambda}$.*

Proof. Let us prove that the $\overline{\Lambda}$-space $\partial_\infty X$ is both minimal and strongly proximal [M1, VI.1].

Strong proximality. Let $\mu \in \mathcal{M}^1(\partial_\infty X)$. Since some unions of walls in X are trees (of valency ≥ 3), the set of singular points is uncountable. Therefore Λ contains a hyperbolic translation τ along a wall, whose repelling point is not one of the at most countably many atoms for μ. By dominated convergence, the sequence converges in $\mathcal{M}^1(\partial_\infty X)$ to the Dirac mass centered at the attracting point of τ.

Minimality. Let $\xi \in \partial_\infty X$. We write it $\xi = r(\infty)$ for a geodesic ray $r : [0;\infty) \to X$ with $r(0) \in R$. For each $n \geq 1$, $r(n)$ is in the closure of a chamber $g_n.R$ with $g_n \in \overline{\Lambda}$. By the Bruhat decomposition $\overline{\Lambda} = \bigsqcup_{w\in W_R} \mathcal{B}w\mathcal{B}$, we have $r(n) \in k_n w_n.R$, hence $k_n^{-1}.r(n) \in A$. Let us denote by r_n the geodesic ray in $A \simeq \mathbf{H}^2$ starting at $r(0)$ and passing through $k_n^{-1}.r(n)$. By compactness of $\partial_\infty \mathbf{H}^2 \simeq S^1$ and $\mathcal{B}$, there is an increasing sequence $\{n_j\}_{j\geq 1}$ such that $r_{n_j}(\infty)$ converges to some $\eta \in \partial_\infty A$ and k_{n_j} converges to some $k \in \mathcal{B}$ as $j \to \infty$. Thus in the $\overline{\Lambda}$-compactification $X \sqcup \partial_\infty X$, we have $\xi = \lim_{n\to\infty} r(n) = \lim_{j\to\infty} r(n_j) = \lim_{j\to\infty} k_{n_j}.(k_{n_j}^{-1}.r(n_j)) = k.\eta$, which proves that $\partial_\infty A$ is a complete set of representatives for the $\mathcal{B}$-action on $\partial_\infty X$. Since the action of the Weyl group W_R, a lattice of $\mathrm{PSL}_2(\mathbf{R})$, is minimal on $\partial_\infty A$, we proved the minimality of the $\overline{\Lambda}$-action on $\partial_\infty X$. □

REMARK. Note that the group $\overline{\Lambda}$ admits a Furstenberg boundary on which it doesn't act transitively, whereas any such boundary for a semisimple algebraic group is an equivariant image of the maximal flag variety [BuMo1, §5].

4.B.2 Iteration of hyperbolic translations along walls also leads to computations of limits of later use for the non-linearity theorem.

PROPOSITION. *Let τ in Λ be a hyperbolic translation along a wall L, with attracting point ξ. Let v be a vertex on the wall L.*

(i) *We have $\lim_{n\to\infty} \tau^n \overline{\Lambda}_v \tau^{-n} = D_\xi$ in the compact metrizable space $\mathcal{S}_{\overline{\Lambda}}$ of closed subgroups of $\overline{\Lambda}$, endowed with the Chabauty topology.*
(ii) *For any $u \in V_{\xi,A}$, we have $\lim_{n\to\infty} \tau^{-n} u \tau^n = 1$.*
(iii) *For any $g \in D_\xi$, the sequence $\{\tau^{-n} g \tau^n\}_{n\geq 1}$ is bounded in $\overline{\Lambda}$.*

REMARK. Point (i) about the Chabauty topology on closed subgroups is used in the final discussion 4.C.2. This topology is always compact, and when S is Hausdorff, locally compact and second countable, it is separable and metrizable [CEG, 3.1.2]. When S is locally compact, a sequence $\{A_n\}_{n\geq 1}$ of closed subsets converges in the Chabauty topology to a closed subset A if and only if

1. Any limit $x = \lim_{k\to\infty} x_{n(k)}$ for an increasing $\{n(k)\}_{k\geq 1}$ with $x_{n(k)} \in A_{n(k)}$ satisfies $x \in A$.
2. Any $x \in A$ is the limit of a sequence $\{x_n\}_{n\geq 1}$ with $x_n \in A_n$ for each $n \geq 1$.

This characterization of convergence implies that for a locally compact group G, the subset $\mathcal{S}_G$ of closed subgroups is closed, hence compact.

Proof. (i) By compactness of $\mathcal{S}_{\overline{\Lambda}}$ it is enough to show that any accumulation point D of $\{\tau^n \overline{\Lambda}_v \tau^{-n}\}_{n\geq 1}$ is equal to D_ξ. In one direction, the very definition of the Chabauty topology implies that D contains K_L and $V_{\xi,A}$, hence D_ξ by Proposition 4.A.3 (iii). Indeed, the group K_L lies in D since it fixes all the vertices in L, hence lies in all the conjugates $\tau^n \overline{\Lambda}_v \tau^{-n}$. The limit group D also contains $V_{\xi,A}$ since for each $m \in \mathbf{Z}$ there is $N \in \mathbf{N}$ such that $\tau^n.v \in a_m$, hence V_m lies in $\tau^n \overline{\Lambda}_v \tau^{-n}$ for any $n \geq N$.

We are thus reduced to proving that any accumulation point D lies in D_ξ. Since $\tau^n v$ converges to ξ and $\tau^n \overline{\Lambda}_v \tau^{-n} = \mathrm{Fix}_{\overline{\Lambda}}(\tau^n v)$, by continuity of the extension of isometries to the boundary, we first have $D < P_\xi$.

Let $g \in D$, which by the previous paragraph and Proposition 4.A.3 (iii) we write $g = u\tau^N k$ with $u \in V_{\xi,A}$, $N \in \mathbf{Z}$ and $k \in K_L$. We choose this order to forget the factor k when g acts on v. Since D is a limit group, we also have $g = \lim_{j\to\infty} \tau^{n_j} k_j \tau^{-n_j}$ for a sequence $\{k_j\}_{j\geq 1}$ in $\overline{\Lambda}_v$ and integers $n_j \to \infty$ as $j \to \infty$. Therefore there is an index $J \geq 1$ for which $j \geq J$ implies $(u\tau^N).v = (\tau^{n_j} k_j \tau^{-n_j}).v$. Since u stabilizes all the horospheres centered at ξ, there is a vertex $z \in L$ with $(u\tau^N).v$ and $\tau^N.v$

at same distance from z. We choose an integer j which is big enough to have $d(\tau^{n_j}.v, (u\tau^N).v) = (n_j - N)\delta$, where δ is the translation length of τ. But the group $\tau^{n_j}\overline{\Lambda}_v\tau^{-n_j}$ stabilizes the spheres centered at $\tau^{n_j}.v$, so that $d\big(\tau^{n_j}.v, (\tau^{n_j}k_j\tau^{-n_j}).v\big) = n_j\delta$. In order to have $(u\tau^N).v = (\tau^{n_j}k_j\tau^{-n_j}).v$, we must have $N = 0$, i.e. $g = uk$: this shows that D lies in D_ξ.

(ii) Let $u \in V_{\xi,A}$. Then $u \in V_m$ for some $m \in \mathbf{Z}$. For each $N \geq 1$, the sequence $\{\tau^{-n}u\tau^n\}_{n\geq N}$ lies in the compact group V_{m-N}, so that any cluster value of $\{\tau^{-n}u\tau^n\}_{n\geq 1}$ belongs to V_{m-N}. By Lemma 4.A.2 (iii), this shows that the only cluster value of the sequence $\{\tau^{-n}u\tau^n\}_{n\geq 1}$ in the compact subset V_m is the identity element, which proves (ii).

(iii) Let $g \in D_\xi$, which we write $g = ku$ with $k \in K_L$ and $u \in V_{\xi,A}$ by Proposition 4.A.3 (iii). Since $u \in V_m$ for some $m \in \mathbf{Z}$, we have $\tau^{-n}u\tau^n \in V_m$ for each $n \geq 1$. By Proposition 4.A.3 (i) K_L is normalized by τ, so we finally have $\tau^{-n}g\tau^n \in K_L \cdot V_m$ for each $n \geq 1$. □

4.C Non-linearity in equal characteristic. We finally state and prove the non-linearity theorem for some countable Kac–Moody groups with hyperbolic buildings. It applies to an infinite family of groups, the Weyl group of which being of arbitrarily large rank.

4.C.1 The previous dynamical results from 4.B and the embedding Theorem 3.A provide the main arguments to prove the result below.

Theorem. *Let Λ be a countable Kac–Moody group over a finite field $\mathbf{F}_q$ of characteristic p, and let r be an integer ≥ 5. We assume that the geometry of Λ is a twinned pair of right-angled Fuchsian buildings $I_{r,q+1}$ with $q \geq \max\{r-2; 5\}$, and that a generalized unipotent radical of $\overline{\Lambda}$ is abelian. Then Λ is not linear over any field.*

Remark. According to 4.A.2, the assumption of the commutativity of a generalized unipotent radical is mild, since it amounts to requiring that for some $i \in \mathbf{Z}/r\mathbf{Z}$, both negative coefficients $A_{i-1,i+1}$ and $A_{i+1,i-1}$ be ≤ -2 (their product must always be ≥ 4 to have $X \simeq I_{r,q+1}$ – see 4.A).

Proof. By [R4, Proposition 4.3], it is enough to disprove linearity in equal characteristic. Let us assume that there is an abstract injective homomorphism from Λ to a linear group in characteristic p, in order to obtain a contradiction. Up to replacing Λ by a finite index subgroup, we may – and shall – assume that Λ is generated by its root groups.

The proof of Lemma 3.A shows that (NA) holds because X is CAT(-1) and $\overline{\Lambda}$ is chamber-transitive. By [RR, 1.C.1 last remark] the growth series of the Weyl group is $W(t) = \frac{(1+t)^2}{(1-(r-2)t+t^2)} \in \mathbf{Z}[[t]]$, and $W\left(\frac{1}{q}\right)$ is finite if and only if $q \geq r-2$; so (LT) holds by 1.C.1. Theorem 2.A.1 implies that (TS) is satisfied because the Dynkin diagrams leading to the buildings $I_{r,q+1}$ are connected [R2, 13.3.2] and $q > 4$. Consequently, we can apply Theorem 3.A to get a closed embedding $\mu : \overline{\Lambda} \hookrightarrow \mathbf{G}(k)$ of topological groups.

Let L be a wall with end ξ such that $V_{\xi,A}$ is abelian. We pick in Λ a hyperbolic translation τ along L with attracting point ξ as in 4.A.1, and $u \in V_{\xi,A} \setminus \{1\}$. We set $B = (\mathrm{Ad} \circ \mu)(\tau)$ and $Y = (\mathrm{Ad} \circ \mu)(u)$, where Ad is the adjoint representation of $\mathbf{G}$. Then Lemma 4.B.2 (ii) implies that $\{B^{-i}YB^i\}_{i \geq 1}$ contains the identity element in its closure, so [M1, Lemma II.1.4] says that the element B has two eigenvalues with different absolute values. We can thus use [Pr, Lemma 2.4], which provides us parabolic subgroups: by [loc. cit. (i)] and 4.B.2 (iii), there is a proper parabolic k-subgroup $\mathbf{P}_{\mu(\tau)}$ whose k-points $P_{\mu(\tau)}$ contain $\mu(D_\xi)$. Replacing τ by τ^{-1}, the attracting point becomes the boundary point $-\xi$ such that $(-\xi\xi) = L$. By [loc. cit. (ii)], the corresponding parabolic subgroup $P_{\mu(\tau^{-1})}$ is opposite $P_{\mu(\tau)}$, so that $\mu(D_\xi) \cap \mu(D_{-\xi})$ lies in the Levi factor $M_{\mu(\tau)} = P_{\mu(\tau)} \cap P_{\mu(\tau^{-1})}$. By the second remark following Proposition 4.A.3, we have $K_L = D_\xi \cap D_{-\xi}$, so we finally obtain that $\mu(K_L)$ lies in $M_{\mu(\tau)}$.

The contradiction comes when we look at the image $\mu(V_{\xi,A})$. By [loc. cit. (i)] and 4.B.2 (ii), it lies in the unipotent radical $\mathcal{R}_u P_{\mu(\tau)}$. The decomposition $D_\xi = K_L \cdot V_{\xi,A}$ of Proposition 4.A.3 (iii) then implies $\mu(K_L) = \mu(D_\xi) \cap M_{\mu(\tau)}$ and $\mu(V_{\xi,A}) = \mu(D_\xi) \cap \mathcal{R}_u P_{\mu(\tau)}$. But according to Proposition 4.A.3 (i), the group $V_{\xi,A}$ is not normalized by K_L. □

4.C.2 Let us give a geometric flavour to the above proof by using the framework of group-theoretic compactifications of buildings [AnGR]. We keep the notation of the previous proof, choose a k-embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_r$ of algebraic groups and still call μ the composed closed embedding $\mu : \overline{\Lambda} \to \mathrm{GL}_r(k)$. Replacing τ by $\tau^{(p^r)}$ for a big enough integer r, and taking a finite extension which we still denote by k, we may – and shall – assume that $\mu(\tau)$, which we denote by t, is diagonal with respect to a basis $\{e_i\}_{1 \leq i \leq r}$ of k^r. We write: $t.e_i = u_i \varpi^{\nu_i(t)} e_i$ where ϖ is the uniformizer of k, $u_i \in \mathcal{O}^\times$ and $\nu_i(t) \in \mathbf{Z}$. Composing μ with a permutation matrix enables to assume that $\nu_1(t) \leq \nu_2(t) \leq \ldots \leq \nu_r(t)$. The basis $\{e_i\}_{1 \leq i \leq r}$ defines a maximal flat $F \simeq \mathbf{R}^{r-1}$ in the Bruhat–Tits building Δ of $\mathrm{GL}_r(k)$, whose vertices are the homothety classes of $\mathcal{O}$-lattices $[\bigoplus_i \varpi^{\nu_i} \mathcal{O} e_i]$ when $\underline{\nu} = \{\nu_i\}_{1 \leq i \leq r}$ ranges over $\mathbf{Z}^r$. We denote by o the origin $[\bigoplus_i \mathcal{O} e_i]$. The same use of [M1, Lemma II.1.4] as in 4.C.1 shows that there is $i \in \{1; 2; \ldots; r-1\}$ such that

$\nu_i(t) < \nu_{i+1}(t)$. Geometrically, this means that $\{t^n.o\}_{n\geq 1}$ is a sequence of vertices in the Weyl chamber $\{\nu_1 \leq \nu_2 \leq \ldots \leq \nu_r\}$ which goes to infinity, staying in the intersection of the fundamental walls indexed by the indices i for which $\nu_i(t) = \nu_{i+1}(t)$.

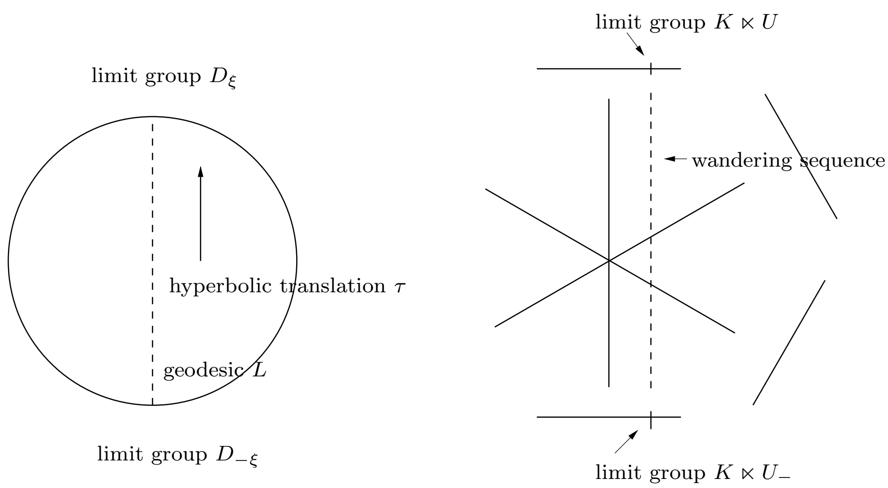
Sequences of points staying at given distance from some walls in a Weyl chamber while leaving the others typically converge in Furstenberg compactifications of symmetric spaces [Gu]. This is a hint to consider compactifications of Bruhat–Tits buildings in our context [AnGR]. We denote by $\mathcal{S}_G$ the space of closed subgroups of a locally compact metrizable group G, and we endow $\mathcal{S}_G$ with the compact metrizable Chabauty topology [CEG, 3.1.1]. The vertices $\mathcal{V}_\Delta$ in Δ are seen as the maximal compact subgroups in $\mathrm{SL}_r(k)$. Therefore $\mathcal{V}_\Delta$ lies in $\mathcal{S}_{\mathrm{SL}_r(k)}$ and we can sum up some results from [AnGR].

Theorem. *The above procedure leads to a $\mathrm{GL}_r(k)$-compactification of Δ where the boundary points are the following closed subgroups of $\mathrm{SL}_r(k)$. Start with a Levi decomposition $M \ltimes U$ of some proper parabolic subgroup and select K a maximal compact subgroup in M. Then $K \ltimes U$ is a limit group, and any limit group is of this form.* □

Remark. In higher rank (i.e. for $r \geq 3$), this compactification is not the one obtained by asymptotic classes of geodesic rays.

Now Proposition 4.B.2 (i) says that $\lim_{n\to\infty} \tau^n \overline{\Lambda}_v \tau^{-n} = D_\xi$ in $\mathcal{S}_{\overline{\Lambda}}$. It follows from the geometric characterization of convergence in the Chabauty topology (4.B.2), that μ induces an embedding $\mu : \mathcal{S}_{\overline{\Lambda}} \hookrightarrow \mathcal{S}_{\mathrm{SL}_r(k)}$. Applying μ to the above limit and using the theorem imply that $\mu(D_\xi)$ lies in $K \ltimes U$ and $\mu(D_{-\xi}) \subset K \ltimes U^-$, with U_- opposite U. This shows that $\mu(K_L)$ lies in K, an important step in the previous proof.

The comparison of hyperbolic and Euclidean apartments emphasizes a sharp difference between Fuchsian and affine root systems. In a Euclidean apartment, there is a finite number of parallelism classes of walls, whereas in the hyperbolic tiling there are arbitrarily large families of roots pairwise intersecting along strips. This explains why there are so many non-prenilpotent pairs of roots (hence free products $\mathbf{F}_q * \mathbf{F}_q$) in the latter case. This is used to prove that K_L doesn't normalize $V_{\xi,A}$ (4.A.3), a key fact for non-linearity. Another crucial difference is the dynamics of the Weyl groups on the boundaries of apartments: in the hyperbolic case, there are infinitely many hyperbolic translations with strong dynamics [GH, §8], whereas the



On the right-hand side: a compactified apartment for SL_3

finite index translation subgroup of a Euclidean Weyl group acts trivially on the boundary of a maximal flat. This makes the computation of limit groups easier in the former case (4.B.2), but the boundary of the Furstenberg compactification of a Bruhat–Tits building has a much richer group-theoretic structure since it contains compactifications of smaller Euclidean buildings [L, §14].

Appendix: Strong Boundaries and Commensurator Super-rigidity

BY P. BONVIN

Introduction

Let G be a locally compact second countable group, and let Γ be a lattice of G, i.e. a discrete subgroup such that G/Γ carries a finite G-invariant measure. The *commensurator* of Γ in G is the group $\mathrm{Comm}_G(\Gamma) := \{g \in G \mid \Gamma \cap g\Gamma g^{-1}$ has finite index in both Γ and $g\Gamma g^{-1}\}$. Our purpose is to use the recent double ergodicity theorem by V. Kaimanovich on Poisson boundaries, in order to show that G. Margulis' proof of the commensurator super-rigidity – as analyzed and generalized by N. A'Campo and M. Burger – extends to a quite general setting.

Theorem 1. *Let G be a locally compact second countable topological group, $\Gamma < G$ be a lattice and Λ be a subgroup of G with $\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma)$. Let k be a local field and H be a connected almost k-simple algebraic group. Assume $\pi : \Lambda \to H_k$ is a homomorphism such that $\pi(\Lambda)$ is Zariski dense in H and $\pi(\Gamma)$ is unbounded in the Hausdorff topology on H_k. Then π extends to a continuous homomorphism $\overline{\Lambda} \to H_k/Z(H_k)$, where $Z(H_k)$ is the center of H_k.*

This theorem is basically due to Margulis, who proved it in the case where G is a semisimple group over a locally compact field [M1, VII.5.4]. A deep idea in the proof is that the existence of the continuous extension follows from the existence of a Λ-equivariant map from the maximal Furstenberg boundary of G to a homogeneous space H_k/L_k, where L is a proper k-subgroup of H. A'Campo and Burger extended the result to the case where G is as above [A'CB], assuming the existence of a closed subgroup P playing the same measure-theoretic role as a minimal parabolic subgroup. This led Burger to state the above result assuming the existence of a substitute for a maximal Furstenberg boundary rather than a minimal parabolic subgroup [Bu]. The assumption that k be of characteristic 0, made so far, was removed too. M. Burger and N. Monod then constructed suitable boundaries for compactly generated groups (up to finite index, see [BuM, Theorem 6]), which implied the above result for a compactly generated group G [BuM, Remark 7]. The last step was made by V. Kaimanovich [Ka] (Theorem 2 below), who proved that the Poisson boundary for a nice measure on any locally compact second countable group is a strong hence a suitable boundary. Let us finish this historical summary by mentioning previous works: T.N. Venkataramana [V] first proved super-rigidity and arithmeticity theorems in arbitrary characteristics, and G.A. Margulis [M2] wrote an unpublished manuscript on equivariant generalized harmonic mappings leading to commensurator super-rigidity; at last Y. Shalom's representation-theoretic approach [Sh] also leads to a very general statement on super-rigidity of commensurators.

This note, which relies heavily on the proof given by A'Campo and Burger in [A'CB], shows how to use the previously quoted references to prove the above commensurator super-rigidity theorem. It is organized as follows. We first quote the results about boundaries of groups. Then we recall how the existence of the continuous extension is reduced to finding a Λ-equivariant map from a boundary of G to a non-trivial homogeneous space H_k/L_k. We finally sketch the steps to construct the required Λ-equivariant map, taking care of the fact that k is possibly of positive characteristic.

Poisson Boundaries and Strong Boundaries

Given a topological group G, a *Banach G-module* is a pair (π, E) where E is a Banach space and π is an isometric linear representation of G on E. The module (π, E) is *continuous* if the action map $G \times E \to E$ is continuous. A *coefficient G-module* is a Banach G-module (π, E) contragredient to some separable continuous Banach G-module, i. e. E is the dual of some separable Banach space $E^\flat$ and π consists of operators adjoint to a continuous action of G on $E^\flat$ (see [Mo, Chapter 1]). Denote by $\mathfrak{X}^{sep}$ the class of all separable coefficient modules.

Let G be a locally compact group, and (S, μ) be a Lebesgue space endowed with a measure class preserving action of G. Given any class of coefficient Banach modules $\mathfrak{X}$, the action of G on S is called *doubly* $\mathfrak{X}$-*ergodic* if for every coefficient G-module E in $\mathfrak{X}$, any weak-* measurable G-equivariant function $f : S \times S \to E$ (with respect to the diagonal action) is a.e. constant [Mo, 11.1.1].

Recall [Z, 4.3.1] that the G-action on S is called *amenable* if for every separable Banach space E and every measurable right cocycle $\alpha : S \times G \to \mathrm{Iso}(E)$ the following holds for α^*, the adjoint of the α-twisted action on $L^1(S, E)$: any α^*-invariant measurable field $\{A_s\}_{s \in S}$ of non-empty convex weak-* compact subsets A_s of the closed unit ball in E^* admits a measurable α^*-invariant section. We can now state V. Kaimanovich's result.

Theorem 2 [Ka]. *Let G be a locally compact σ-compact group. There exists a Lebesgue space (S, μ) endowed with a measure class preserving action of G such that*

(i) *The G-action on S is amenable.*
(ii) *The G-action on S is doubly $\mathfrak{X}^{sep}$-ergodic.*

Such a space (S, μ) is called a strong G-boundary [MoS, Def. 2.3].

The space S is a Poisson boundary for a suitable measure on G. As mentioned before, this theorem strengthens a result of Burger and Monod [BuM], who proved that any compactly generated locally compact group possesses a finite index open subgroup which has a strong boundary. Note that the double $\mathfrak{X}^{sep}$-ergodicity of the G-action on S implies that the Γ-action on S is doubly $\mathfrak{X}^{sep}$-ergodic [BuM, Prop. 3.2.4] and that G (as well as any finite index subgroup of Γ) acts ergodically on S and on $S \times S$.

Reduction to Finding a Suitable Equivariant Map

First, in view of the conclusion of the theorem, we may – and shall – assume until the end of the note that the group H is adjoint. Recall also that if $f : X \to Y$ is a measurable map from a Lebesgue space (X, λ) to a topological space Y, its *essential image* is the closed subset of Y defined by $\mathrm{Essv}(f) := \{y \in Y \mid \lambda(f^{-1}(V)) > 0 \text{ for any neighbourhood } V \text{ of } y\}$, and f is called *essentially constant* if $\mathrm{Essv}(f)$ reduces to a point. Here is the reduction theorem.

Theorem 3. *Let G be a locally compact second countable topological group, with strong G-boundary (S, μ). Let $\Gamma < G$ be a lattice and Λ be a subgroup of G with $\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma)$. Let k be a local field and H be a connected almost k-simple algebraic group. Assume that $\pi : \Lambda \to H_k$ is a homomorphism with Zariski dense image in H and that there exists a Λ-equivariant non-essentially constant measurable map $\theta : S \to H_k/L_k$, where L is a proper k-subgroup of H. Then there exists a continuous extension $\overline{\Lambda} \to H_k$ of π.*

The proof uses a simple and powerful ergodic argument [A'CB, Sect. 2.3], used many times in the full proof of super-rigidity. We will often deal with measurable maps $\Theta : B \to M$ where B is an ergodic Γ-space, M is a space with a continuous H_k-action and Θ is equivariant with respect to a group homomorphism $\Gamma \to H_k$. Then if M is a separable complete metrizable space and if the H_k-orbits are locally closed in M, there is a H_k-orbit O in M such that a conull subset of B is sent to O by Θ. This is to be combined with the fact that a k-algebraic action of a k-group G on a k-variety V induces a continuous action of G_k on V_k in the Hausdorff topology, and with the following crucial result, due to I. Bernstein and A. Zelevinski.

Theorem 4 [BZ, 6.15]**.** *Let k be a local field, V be a k-variety and G be a k-group acting k-algebraically on V. Then the orbits of G_k in V_k are locally closed.*

This theorem has a wide range of application because it implies local closedness of orbits in many spaces. Let $\mathcal{F}(S, W_k)$ be the space of classes of measurable maps from S to W_k, where W is a k-variety. We endow $\mathcal{F}(S, W_k)$ with the topology of convergence in measure on $\mathcal{F}(S, W_k)$. It is metrizable by a complete separable metric. Then, according to [A'CB, Lemma 6.7], the H_k-orbits in $\mathcal{F}(S, W_k)$ are locally closed. The proof there is given for a local field k of characteristic 0, but it goes through in the general case once the stabilizer $\mathrm{Stab}_H(w)$ of any k-rational point $w \in W_k$, only k-closed in general, is replaced by the k-subgroup $\overline{\mathrm{Stab}_{H_k}(w)}^Z$. The ergodic argument applied to the function space $\mathcal{F}(S, W_k)$ is a key point in the proof of the above reduction theorem, whose proof can now be sketched (see [A'CB, Sect. 7] for further details).

Proof. Since $\overline{\pi(\Lambda)}^Z = H$ and since θ is Λ-equivariant, we have $\overline{\mathrm{Essv}(\theta)}^Z = W$ where $W := H/L$. We define $\overline{\theta} : \overline{\Lambda} \to \mathcal{F}(S, W_k)$ by $\overline{\theta}(\lambda)(s) := \theta(\lambda s)$. It is Λ-equivariant and continuous. Since Λ acts ergodically on $\overline{\Lambda}$ and since the H_k-orbits in $\mathcal{F}(S, W_k)$ are locally closed, there is an H_k-orbit $O \subset \mathcal{F}(S, W_k)$ such that $\overline{\theta}(\lambda) \in O$ for almost all $\lambda \in \overline{\Lambda}$. One deduces then from the fact that O is open in $\overline{O}$ and $\overline{\theta}$ is continuous, that $\overline{\theta}(\overline{\Lambda}) \subset O$. In particular $O = H_k\theta$. Then it follows from $\overline{\mathrm{Essv}(\theta)}^Z = W$ that $\mathrm{Stab}_{H_k}(\theta)$ fixes pointwise W and thus is trivial since H is adjoint. Therefore the map $h : \overline{\Lambda} \to H_k$ defined by $\overline{\theta}(\lambda) = h(\lambda)\theta$ for any $\lambda \in \overline{\Lambda}$, is a continuous homomorphism. Since θ is Λ-equivariant, h is the desired extension of π.

Constructing the Required Equivariant Map

We now sketch the proof of the existence of a Λ-equivariant map as above under the hypotheses of Theorem 1. Since k is of arbitrary characteristic, the adjoint representation Ad of H needs no longer be irreducible. Still, we can choose $\rho : H \to \mathrm{GL}(V)$ a faithful rational representation of H, defined and irreducible over k, on a finite-dimensional k-vector space V. The induced map $\rho : H_k \to \mathrm{PGL}(V_k)$ is injective because H is adjoint, and by [M1, I.2.1.3] it is a closed embedding. We have a homomorphism $\rho \circ \pi : \Gamma \to \mathrm{PGL}(V_k)$, so that Γ acts by homeomorphisms on the compact metric space $\mathbf{P}V_k$. This induces a continuous action $\Gamma \times M^1(\mathbf{P}V_k) \to M^1(\mathbf{P}V_k)$, where $M^1(\mathbf{P}V_k)$ is the space of probability measures on $\mathbf{P}V_k$ endowed with the compactmetrizable weak-$*$ topology.

Proposition 1. *Let G be a locally compact group and (S, μ) be a Lebesgue space on which G acts amenably. Then, possibly after discarding an invariant null set in S, there exists a measurable Γ-equivariant map $\phi : S \to M^1(\mathbf{P}V_k)$.*

Proof. This follows immediately from Theorem 4.3.5 and Proposition 4.3.9 in [Z]. □

At this stage, we have a measurable map $\phi : S \to M^1(\mathbf{P}V_k)$ which is equivariant for the Γ-action only, and which goes to a space of probability measures. The next step provides a Γ-equivariant map to a homogeneous space H_k/L_k.

We denote by $\mathrm{Var}_k(\mathbf{P}V)$ the set of algebraic subvarieties of $\mathbf{P}V$ defined over k and by $\mathrm{supp}_Z : M^1(\mathbf{P}V_k) \to \mathrm{Var}_k(\mathbf{P}V)$ the map which to a probability measure μ associates $\overline{\mathrm{supp}(\mu)}^Z$, the Zariski closure of its support. For any n-dimensional k-vector space W_k we set $\mathrm{Gr}(W_k) := \bigsqcup_{l=0}^n \mathrm{Gr}_l(W_k)$, where $\mathrm{Gr}_l(W_k)$ is the compact Grassmannian of l-planes in W_k. By attaching to each projective variety $X \subset \mathbf{P}V$ its graded defining ideal I_X, we see $\mathrm{Var}_k(\mathbf{P}V)$ as a subspace of the compact space $\prod_{d=0}^{\infty} \mathrm{Gr}(k[V]_d)$, where $k[V]_d$ is the space of d-homogeneous polynomials on V_k. This induces a topology on $\mathrm{Var}_k(\mathbf{P}V)$, and it is proved in[A'CB, Sect. 5], by characteristic-free arguments, that the map supp_Z is measurable and $\mathrm{PGL}(V_k)$-equivariant. Therefore we obtain by composition a Γ-equivariant measurable map $\Phi : S \to \mathrm{Var}_k(\mathbf{P}V)$, sending each $s \in S$ to the Zariski closure of the support of $\phi(s)$. We denote it by Φ, and call it *boundary map.*

Theorem 5 [A'CB, Theorem 5.1]**.** *The boundary map Φ is not essentially constant.*

This result follows from the arguments in [A'CB, Sect. 5]. To see this, we first note that since H is k-simple, $\pi(\Gamma)$ is unbounded and $\pi(\Lambda)$ is Zariski dense, the inclusion $\Lambda < \mathrm{Comm}_G(\Gamma)$ and the fact that the identity component of an algebraic group is always a finite index subgroup imply that $\pi(\Gamma)$ is Zariski dense in H. The other facts needed in [A'CB, Sect. 5] are the ergodicity of Γ on S and on $S \times S$, and the Furstenberg lemma, all available in our context. It follows from Theorem 5 that there is a d for which $\Phi : S \to \mathrm{Gr}(k[V]_d)$ is not essentially constant. The ergodic argument of the previous section and the ergodicity of the Γ-action on S imply that Φ essentially sends S to an H_k-orbit in $\mathrm{Gr}(k[V]_d)$, which is homeomorphic to a space H_k/L_k for some proper algebraic subgroup L of H: we have thus obtained a Γ-equivariant measurable map $\phi : S \to H_k/L_k$.

The very last step consists in passing from Γ- to Λ-equivariance. Once maps as above are known to exist, the descending chain condition for algebraic subgroups and Zorn's lemma, as used in [A'CB, Sect. 7], prove the existence of a couple $(\phi, H_k/L_k)$ satisfying a universal property. The normalizer of L_k in H may only be k-closed, but if we denote by L' the Zariski closure of the normalizer of L_k in H_k, we get a k-subgroup, which is proper by k-simplicity of H and such that

Theorem 6 [A'CB, Corollary 7.2]**.** *The composed map $\theta : S \to H_k/L_k \to H_k/L'_k$ is Λ-equivariant and measurable.*

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