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Simplicity and superrigidity of twin building lattices

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Dedicated to Jacques Tits with our admiration

Abstract. Kac–Moody groups over finite fields are finitely generated groups. Most of them can naturally be viewed as irreducible lattices in products of two closed automorphism groups of non-positively curved twinned buildings: those are the most important (but not the only) examples of *twin building lattices*. We prove that these lattices are simple if the corresponding buildings are irreducible and not of affine type (i.e. they are not Bruhat–Tits buildings). Many of them are finitely presented and enjoy property (T). Our arguments explain geometrically why simplicity fails to hold only for affine Kac–Moody groups. Moreover we prove that a nontrivial continuous homomorphism from a completed Kac–Moody group is always proper. We also show that Kac–Moody lattices fulfill conditions implying strong superrigidity properties for isometric actions on non-positively curved metric spaces. Most results apply to the general class of twin building lattices.

Introduction

Since the origin, Kac–Moody groups (both in their so-called minimal and maximal versions) have been mostly considered as natural analogues of semisimple algebraic groups arising in an infinite-dimensional Lie theoretic context (see e.g. [40] and [43]). A good illustration of this analogy is the construction of minimal Kac–Moody groups over arbitrary fields, due to

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J. Tits [71], by means of presentations generalizing in infinite dimension the so-called Steinberg presentations of Chevalley groups over fields [68]. This presentation provides not only a Kac–Moody group G, but also a family of root subgroups $\{U_\alpha\}_{\alpha\in\Phi}$ indexed by an abstract root system Φ and satisfying a list of properties shared by the system of root groups of any isotropic semisimple algebraic group. These properties constitute the group theoretic counterpart of the geometric notion of a twin building: any group endowed with such a family of root groups, which is called a *twin root datum*, has a natural diagonal action on a product of two buildings, and this action preserves a twinning. We refer to [74] and [62] for this combinatorial point of view.

In this paper, we are mainly interested in finitely generated Kac–Moody groups, i.e. minimal Kac–Moody groups over finite ground fields. In this special situation, it has been noticed more recently that another viewpoint, different from the aforementioned algebraic group theoretic one, is especially relevant: the arithmetic group viewpoint. The striking feature which justifies in the first place this more recent analogy is the fact that finitely generated Kac–Moody groups embed as irreducible lattices in the product of closed automorphism groups of the associated buildings, provided the ground field is sufficiently large, see [60]. A sufficient condition for this is that the order of the finite ground field is at least the number of canonical generators of the Weyl group. In fact, Kac–Moody theory is one of the few known sources of examples of irreducible lattices in products of locally compact groups outside the classical world of lattices in higher-rank Lie groups. On the other hand, the intersection between Kac–Moody groups and arithmetic groups is nonempty since Kac–Moody groups of affine type, namely those obtained by evaluating Chevalley group schemes over rings of Laurent polynomials, are indeed arithmetic groups. A standard example, which is good to keep in mind, is the arithmetic group $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$, which is an irreducible lattice of $\mathrm{SL}_n(\mathbf{F}_q((t))) \times \mathrm{SL}_n(\mathbf{F}_q((t^{-1})))$. This arithmetic group analogy, suggesting the existence of strong similarities between Kac–Moody groups of arbitrary type and the previous examples of affine type, is supported by several other results, see e.g. [1] for finiteness properties, [30] for continuous cohomology, [65, §1] for some structural properties, etc.

The main result of the present paper shows that for infinite Kac–Moody groups over finite fields, there is a sharp structural contrast between affine and non-affine groups. Indeed, affine Kac–Moody groups over finite fields, as finitely generated linear groups, are residually finite. On the other hand, non-affine Kac–Moody groups are subjected to the following:

Simplicity theorem (Kac–Moody version). *Let Λ be a split or almost split Kac–Moody group over a finite field $\mathbf{F}_q$. Assume that the Weyl group W of Λ is an irreducible, infinite and non-affine Coxeter group. Then every finite index subgroup of Λ contains the derived subgroup $[\Lambda, \Lambda]$, which is of finite index. Assume moreover that $q \geqslant |S|$. Then the group $[\Lambda, \Lambda]$, divided by its finite center, is simple.*

A more general result (Theorem 19 of Subsect. 4.4) holds in the abstract framework of twin root data; it was announced in [24].

It follows from the above that for any neither spherical nor affine, indecomposable generalized Cartan matrix A of size n, there exists a Kac–Moody group functor $\mathcal{G}_A$ such that for any finite field $\mathbf{F}_q$ with $q \geqslant n > 2$, the group $\Lambda = \mathcal{G}_A(\mathbf{F}_q)$, divided by its finite center, is an infinite finitely generated simple group. We also note that this simplicity result for Kac–Moody groups over finite fields implies strong non-linearity properties for Kac–Moody groups over *arbitrary* fields of positive characteristic (Theorem 25). To be more constructive, we add the following corollary (see Corollary 21). As pointed out to us by Y. Shalom, we obtain the first infinite *finitely presented* discrete groups to be both simple and Kazhdan. Note that finitely generated infinite simple Kazhdan groups were constructed by M. Gromov [33, Corollary 5.5.E] as quotients of hyperbolic groups with property (T).

Simple Kazhdan group corollary. *If the generalized Cartan matrix A is 2-spherical (i.e. every 2×2-submatrix is of spherical type) and if $q > \frac{1764^n}{25}$, then the group $\Lambda/Z(\Lambda)$ is finitely presented, simple and Kazhdan. Moreover there exist infinitely many isomorphism classes of infinite groups with these three properties.*

Another consequence is the possibility to exhibit a large family of inclusions of lattices in topological groups for which the density of the commensurator does not hold (see Corollary 17).

Non-arithmeticity corollary. *Let Λ be a split or almost split Kac–Moody group over a finite field $\mathbf{F}_q$. Assume that the Weyl group of Λ is irreducible, infinite and non-affine. Let $\mathcal{B}_+$, $\mathcal{B}_-$ be the buildings associated with Λ and let $\overline{\Lambda}_+$ and $\overline{\Lambda}_-$ be the respective closures of the images of the natural actions $\Lambda \to \mathrm{Aut}(\mathcal{B}_+)$ and $\Lambda \to \mathrm{Aut}(\mathcal{B}_-)$. We view $\Lambda/Z(\Lambda)$ as a diagonally embedded subgroup of $\overline{\Lambda}_- \times \overline{\Lambda}_+$. Then the commensurator $\mathrm{Comm}_{\overline{\Lambda}_- \times \overline{\Lambda}_+}(\Lambda/Z(\Lambda))$ contains Λ as a finite index subgroup; in particular it is discrete.*

Let us finally mention a consequence of the simplicity theorem concerning the word problem. It is a well known observation that a finitely presented simple group has solvable word problem. In fact, a theorem of W. Boone and G. Higman [8] asserts that a finitely generated group has solvable word problem if and only if it embeds in a simple group which itself embeds in a finitely presented group. Most finitely generated split or almost split Kac–Moody groups embed in adjoint Kac–Moody groups of irreducible 2-spherical non-affine type over large finite fields, which are simple by the theorem above and finitely presented by [3]. In particular, one obtains:

Solvable word problem corollary. *Let Λ be a split or almost split Kac–Moody group over an arbitrary finite field $\mathbf{F}_q$. Assume that the Weyl group of Λ is 2-spherical. Then Λ has solvable word problem.*

The proof of the above theorem follows a two-step strategy which owes much to a general approach initiated by M. Burger and Sh. Mozes [18] to construct simple groups as cocompact lattices in products of two trees; however, the details of arguments are often substantially different. The idea of [18] is to prove the normal subgroup property (i.e. normal subgroups either are finite and central or have finite index) following G.A. Margulis' proof for lattices in higher rank Lie groups [45, VIII.2] but to disprove residual finiteness by geometric arguments in suitable cases [18, Proposition 2.1]. This relies on a preliminary study of sufficiently transitive groups of tree automorphisms [17]. Let us also recall that a finitely generated just infinite group (i.e. all of whose proper quotients are finite) is either residually finite or is, up to finite index, a direct product of finitely many isomorphic simple groups [76].

In our situation, the first step of the proof, i.e. the normal subgroup property, had been established in previous papers, in collaboration with U. Bader and Y. Shalom: [5,64] and [67]. This fact, which is recalled here as Theorem 18, is one of the main results supporting the analogy with arithmetic groups mentioned above. One difference with [18, Theorem 4.1] is that the proof does not rely on the Howe–Moore property (i.e. decay of matrix coefficients). Instead, Y. Shalom and U. Bader use cohomology with unitary coefficients and Poisson boundaries. In fact, it can be seen that closed strongly transitive automorphism groups of buildings do not enjoy Howe–Moore property in general: whenever the ambient closed automorphism group of the buildings in consideration contains a proper parabolic subgroup which is not of spherical type (i.e. whose Weyl subgroup is infinite), then any such parabolic subgroup is an open subgroup which is neither compact nor of finite index.

The second half of the simplicity proof does not actually use the notion of residual finiteness. Instead, it establishes severe restrictions on the existence of finite quotients of a group endowed with a twin root datum of non-affine type (Theorem 15). Here, one confronts the properties of the system of root subgroups to a geometric criterion which distinguishes between the Tits cones of affine (i.e. virtually abelian) and of infinite non-affine Coxeter groups (see [46] and [58] for another illustration of this fact). This part of the arguments holds without any restriction on the ground field, and holds in particular for those groups over tiny fields for which simplicity is still an open question.

We note that the simplicity theorem above is thus obtained as the combination of two results which pertain respectively to each of the two analogies mentioned above. In this respect, it seems that the structure of Kac–Moody groups is enriched by the ambiguous nature of these groups, which are simultaneously arithmetic-group-like and algebraic-group-like.

In order to conclude the presentation of our simplicity results, let us compare quickly Burger–Mozes' groups with simple Kac–Moody lattices. The groups constructed thanks to [18, Theorem 5.5] are cocompact lattices in a product of two trees; they are always finitely presented, simple, torsion free and amalgams of free groups (hence cannot have property (T)). Simple Kac–Moody lattices are non-uniform lattices of products of buildings, possibly (and usually) of dimension $\geqslant 2$; they are often (not always, though) finitely presented and Kazhdan and contain infinite subgroups of finite exponent. It is still an open and challenging question to construct simple cocompact lattices in higher-dimensional buildings.

We now turn to the second series of results of this paper. It deals with restrictions on actions of Kac–Moody lattices on various non-positively curved spaces. This is a natural question in view of the analogy between Kac–Moody lattices and arithmetic groups, since the latter are known to yield some superrigidity phenomena. In fact one should expect even stronger rigidity results for simple lattices in view of the following known fact: a non residually finite group cannot be embedded injectively into a compact group (see Proposition 26). In particular a simple Kazhdan group acting non-trivially on a Gromov-hyperbolic proper metric space Y of bounded geometry cannot fix any point in the visual compactification $\overline{Y} = Y \sqcup \partial_\infty Y$.

Therefore, it makes sense to try to use the recent superrigidity results due to N. Monod (with CAT(0) target spaces [50]) and generalized further by T. Gelander, A. Karlsson and G. Margulis (with suitable Busemann non-positively curved, uniformly convex target spaces [31]). Note however that, as it is the case for Y. Shalom's result on property (T) [67], the non-cocompactness of Kac–Moody lattices is an obstruction to a plain application of these results (stated for uniform lattices). Nevertheless, all the previous references propose measure-theoretic or representation-theoretic substitutes for cocompactness. *Weak cocompactness* of a lattice Γ in a topological group G is the fact that the orthogonal of the constant functions in the regular representation $L^2(G/\Gamma)$ doesn't almost have invariant vectors [45, Sect. III.1.8]. It is still an interesting open question to determine whether all Kac–Moody lattices are weakly cocompact; it is of course the case for lattices enjoying Kazhdan's property (T). We prove here that another partial substitute for cocompactness holds (see Proposition 31).

Uniform integrability proposition. *Let Λ be a split or almost split Kac–Moody group over $\mathbf{F}_q$. Assume that Λ is a lattice of the product of its twinned buildings $\mathcal{B}_\pm$. Then the group Λ admits a natural fundamental domain with respect to which it is uniformly p-integrable for any $p \in [1; +\infty)$.*

For an arbitrary inclusion of a finitely generated lattice Γ in a locally compact group G, uniform integrability is a technical condition requiring the existence of a fundamental domain D with respect to which some associated cocycle is uniformly integrable (see Subsect. 7.2). This circle of ideas enables us to prove the following superrigidity statement [31, Theorem 1.1].

Superrigidity proposition. *Let Λ be as above and assume in addition that it is a weakly cocompact lattice of the product of its two completions*

$\overline{\Lambda}_- \times \overline{\Lambda}_+$ *(this is automatic if* Λ *is Kazhdan). Let* X *be a complete Busemann non-positively curved, uniformly convex metric space without nontrivial Clifford isometries. We assume that there exists a* Λ*-action by isometries on* X *with reduced unbounded image. Then the* Λ*-action extends uniquely to a* $\overline{\Lambda}_- \times \overline{\Lambda}_+$*-action which factors through* $\overline{\Lambda}_-$ *or* $\overline{\Lambda}_+$.

As already mentioned, this is a corollary of a result of N. Monod's when the target space is complete and CAT(0) [50, Theorem 6]. In fact the relevancy of reduced actions was pointed out in [loc. cit.]: a subgroup $L < \mathrm{Isom}(X)$ is called *reduced* if there is no unbounded closed convex proper subset Y of X such that gY is at finite Hausdorff distance from Y for any $g \in L$. We also recall that a *Clifford isometry* of X is a surjective isometry $T : X \to X$ such that $x \mapsto d(T(x), x)$ is constant on X.

These results about continuous extensions of group homomorphisms call for structure results for the geometric completions $\overline{\Lambda}_\pm$ of Kac–Moody groups over finite fields, i.e. the closures of the non-discrete Λ-actions on each building $\mathscr{B}_\pm$. Indeed, once a continuous extension has been obtained by superrigidity, it is highly desirable to determine whether this map is proper, e.g. to know whether infinite discrete subgroups can have a global fixed point in the target metric space. When the ambient topological groups are semisimple Lie groups, the properness comes as a consequence of the Cartan decomposition of such groups [16, Lemma 5.3]. The difficulty in our situation is that, with respect to structure properties, topological groups of Kac–Moody type are not as nice as semisimple algebraic groups over local fields. Unless the Kac–Moody group is of affine type, there is no Cartan decomposition in which double cosets modulo a maximal compact subgroup are indexed by an abelian semi-group: the Weyl group is not virtually abelian and roots cannot be put into finitely many subsets according to parallelism classes of walls in the Coxeter complex. This is another avatar of the strong Tits alternative for infinite Coxeter groups [46, 58]. Here is a slightly simplified version of our main properness result (Theorem 28).

Properness theorem. *Let* Λ *be a split or almost split simply connected Kac–Moody group over* $\mathbf{F}_q$ *and let* $\overline{\Lambda}_+$ *be its positive topological completion. Then any nontrivial continuous homomorphism* $\varphi : \overline{\Lambda}_+ \to G$ *to a locally compact second countable group* G *is proper.*

As an example of application of superrigidity results, we study actions of Kac–Moody lattices on CAT(-1)-spaces. In this specific case, the most appropriate results available are the superrigidity theorems of N. Monod and Y. Shalom [51]. Putting these together with the abstract simplicity of non-affine Kac–Moody lattices and the properness theorem above enables us to exhibit strong incompatibilities between higher-rank Kac–Moody groups and some negatively curved metric spaces (see Theorem 34 for more details).

"Higher-rank versus CAT(−1)" theorem. *Let Λ be a simple Kac–Moody lattice and Y be a proper* CAT(-1)*-space with cocompact isometry group. If the buildings $\mathcal{B}_\pm$ of Λ contain flat subspaces of dimension $\geqslant 2$ and if Λ is Kazhdan, then the group Λ admits no nontrivial action by isometries on Y.*

We show below, by means of a specific example, that the assumption that Isom(Y) is cocompact is necessary (see the remark following Theorem 34). This theorem was motivated by [16, Corollary 0.5]. Note that we made two assumptions (one on flat subspaces, one on property (T)) which, in the classical case, are implied by the same algebraic condition. Namely, if Λ were an irreducible lattice in a product of semisimple algebraic groups, and if each algebraic group were of split rank $\geqslant 2$, then both "higher-rank" assumptions would be fulfilled. In the Kac–Moody case, there is no connection between existence of flats in the buildings and property (T). The relevant rank here is the geometric one (the one involving flats in the buildings). According to [6] and [21], it has a more abstract interpretation relevant to the general theory of totally disconnected locally compact groups.

The proofs of most results of the present paper use in a very soft way that the construction of the lattices considered in this introduction pertains to Kac–Moody theory. The actual tool which is the most natural framework for our arguments is the notion of a *twin root datum* introduced in [74]. It turns out that the class of groups endowed with a twin root datum includes split and almost split Kac–Moody groups only as a (presumably small) subfamily (see Sect. 1 below). Several exotic constructions of such groups outside the strict Kac–Moody framework are known, see e.g. [73, §9] for groups acting on twin trees, [65] for groups acting on right-angled twin buildings and [55] for groups obtained by integration of Moufang foundations. All these examples are discrete subgroups of the product of the automorphism groups of the two halves of a twin building, which are actually mostly of finite covolume. These lattices, called *twin building lattices*, constitute the main object of study for the rest of this paper.

Structure of the paper. In the preliminary Sect. 0, we fix the conventions and notation. Section 1 is devoted to collect some basic results for later reference. Although these results are often stated in the strict Kac–Moody framework in the literature, we have been careful to state and prove them in the context of twin building lattices. Section 2 introduces a fixed-point property of root subgroups and it is shown that most examples of twin building lattices enjoy this property. It is then used to establish several useful structural properties of these completions. In Sect. 3, we prove the main fact needed for the simplicity theorem; it is the existence of a weakly hyperbolic geometric configuration of walls for non-affine infinite Coxeter complexes. In Sect. 4, the simplicity theorem is proved together with very strong restrictions on quotients of the groups for which the simplicity is still unknown. In Sect. 5, we prove a non-linearity property for Kac–Moody groups over arbitrary fields of positive characteristic. In Sect. 6, we study homomorphisms from Kac–Moody groups to locally compact groups. The main part deals with the geometric completions of Kac–Moody groups; it establishes that any nontrivial continuous homomorphism from such a group to a second countable group must be proper. In Sect. 7, we check some integrability conditions for Kac–Moody lattices and we derive superrigidity statements from work by Monod-Shalom; restrictions on actions in hyperbolic metric spaces in terms of "rank" are derived from this.

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0. Notation and general references

Let us fix some notation, conventions and make explicit our standard references.

0.1. About Coxeter groups. Throughout this paper, (W, S) denotes a Coxeter system [11, IV.1] of finite rank (i.e. with S finite) and ℓ or ℓ_S denotes the word length $W \to \mathbf{N}$ with respect to the generating set S. We denote by $W(t)$ the canonical growth series, i.e. the series $\sum_{n\geqslant 0} c_n t^n$ where c_n is the number of elements $w \in W$ such that $\ell_S(w) = n$. The combinatorial root system Φ of W is abstractly defined in [71, Sect. 5]. We adopt this point of view because, since it is purely set-theoretic, it is useful to connect several geometric realizations of the Coxeter complex of (W, S) [2,66]. A pair of opposite roots here is a pair of complementary subsets W which are permuted by a suitable conjugate of some canonical generator $s \in S$. The set of simple roots is denoted by Π.

Recall that a set of roots Ψ is called *prenilpotent* if both intersections $\bigcap_{\alpha\in\Psi} \alpha$ and $\bigcap_{\alpha\in\Psi} -\alpha$ are nonempty. Given a prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, we introduce the following finite sets of roots:

$$[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset -\gamma\} \quad \text{and}$$
$$]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

0.2. About geometric realizations. We denote by $\mathcal{A}$ the Davis complex associated to (W, S) and by d the corresponding CAT(0) distance on $\mathcal{A}$ [28]. The metric space $\mathcal{A}$ is obtained as a gluing of compact subsets, all isometric to one another and called chambers. The group W acts properly discontinuously on $\mathcal{A}$ and simply transitively on the chambers. The fixed point set of each reflection, i.e. of each element of the form wsw^{-1} for some $s \in S$ and $w \in W$, separates $\mathcal{A}$ into two disjoint halves, the closures of which are called root half-spaces. The fixed point set of a reflection is called a wall. The set of root half-spaces of $\mathcal{A}$ is denoted by $\Phi(\mathcal{A})$; it is naturally in W-equivariant bijection with Φ. We distinguish a base chamber, say c_+, which we call the standard chamber: it corresponds to 1_W in the above free W-action. We denote by Φ_+ the set of root half-spaces containing c_+ and by Π the set of simple roots, i.e. of positive roots bounded by a wall associated to some $s \in S$.

0.3. About group combinatorics. The natural abstract framework in which the main results of this paper hold is provided by the notion of a *twin root datum*, which was introduced in [74] and is further discussed for instance in [1, §1], [62, Sect. 1.5] or [2, §§7–8]. A twin root datum consists of a couple $(G, \{U_\alpha\}_{\alpha\in\Phi})$ where G is a group and $\{U_\alpha\}_{\alpha\in\Phi}$ is a collection of subgroups indexed by the combinatorial root system of some Coxeter system; the subgroups $\{U_\alpha\}_{\alpha\in\Phi}$, called *root groups*, are subjected to the following axioms, where $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and U_+ (resp. U_-) denotes the subgroup generated by the root groups indexed by the positive roots (resp. their opposites):

(TRD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.
(TRD1) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{]\alpha,\beta[} := \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.
(**TRD2**) For each $\alpha \in \Pi$ and each $u \in U_\alpha \backslash \{1\}$, there exist elements $u', u'' \in U_{-\alpha}$ such that the product $m(u) := u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$.
(TRD3) For each $\alpha \in \Pi$, the group $U_{-\alpha}$ is not contained in U_+ and the group U_α is not contained in U_-.
(TRD4) $G = T\langle U_\alpha \mid \alpha \in \Phi\rangle$.

Recall that prenilpotent pairs of roots, as well as intervals of roots, where defined in Sect. 0.1.

We also set $N := T.\langle m(u) \mid u \in U_\alpha - \{1\},\ \alpha \in \Pi\rangle$. A basic fact is that the subquotient N/T is isomorphic to W; we call it the *Weyl group* of G.

0.4. About twin buildings. The geometric counterpart to twin root data is the notion of twin buildings. Some references are [74], [1, §2], [62, §2.5] or [2, §§7–8]. Roughly speaking, a group with a twin root datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type (W, S) admits two structures of BN-pairs which are not conjugate to one another in general. Let $(\mathcal{B}_+, \mathcal{B}_-)$ be the associated twinned buildings; their apartments are modelled on the Coxeter complex of (W, S). We will not need the combinatorial notion of a twinning between $\mathcal{B}_-$ and $\mathcal{B}_+$. The standard twin apartment (resp. standard positive chamber) is denoted by $(\mathcal{A}_+, \mathcal{A}_-)$ (resp. c_+). We identify the Davis complex $\mathcal{A}$ with the positive apartment $\mathcal{A}_+$. With this identification and when the root groups are all finite, the buildings $\mathcal{B}_\pm$ are locally finite CAT(0) cell complexes.

1. Twin building lattices and their topological completions

As mentioned in the introduction, the main results of this paper apply not only to split or almost split Kac–Moody groups over finite fields, but also to the larger class of groups endowed with a twin root datum with finite root groups. Some existing results in the literature are stated for split or almost split Kac–Moody groups, but remain actually valid in this more general context of twin root data. The purpose of this section is to collect some of these results and to restate them in this context for subsequent references.

1.1. Kac–Moody groups versus groups with a twin root datum. Although the notion of a twin root datum was initially designed as an appropriate tool to study Kac–Moody groups, it became rapidly clear that many examples of twin root data arise beyond the strict scope of Kac–Moody theory. This stands in sharp contrast to the finite-dimensional situation: as follows from the classification achieved in [70] (see also [75]), any group endowed with a twin root datum with *finite* Weyl group of rank at least 3 and of irreducible type is associated (in a way which we will not make precise) with some isotropic simple algebraic group over a field or with a classical group over a (possibly skew) field. Here is a list of known constructions which yield examples of twin root data with *infinite* Weyl group but not associated with split or almost split Kac–Moody groups:

(I) [73, §9] constructs a twin root datum with infinite dihedral Weyl group and arbitrary prescribed rank one Levi factors. The possibility of mixing ground fields prevent these groups from being of "Kac–Moody origin". The associated buildings are one-dimensional, i.e. trees.

(II) In [65], the previous construction is generalized to the case of Weyl groups which are arbitrary right-angled Coxeter groups. In particular, the associated buildings are of arbitrarily large dimension.

(III) Opposite to right-angled Coxeter groups are 2-spherical Coxeter groups, i.e. those Coxeter groups in which every pair of canonical generators generates a finite group. Twin root data with 2-spherical Weyl group are subjected to strong structural restrictions (see [57]) showing in particular that wild constructions as in the right-angled case are impossible. For instance, the following fact is a consequence of the main result of [55]: *a group* Λ *endowed with a twin root datum* $\{U_\alpha\}_{\alpha\in\Phi}$ *of irreducible type, such that the root groups are all finite of order* > 3 *and generate* Λ*, and that every rank* 2 *parabolic subgroup is of type* $A_1 \times A_1$*,* A_2*,* B_2*,* C_2 *or* G_2 *must be a split or almost split Kac–Moody group in the sense of* [62]. Furthermore, it was mentioned to us by B. Mühlherr, as a non-obvious strengthening of [55], that the preceding statement remains true if one allows the rank 2 subgroups to be twisted Chevalley groups of rank 2, with the exception of the Ree groups ${}^2F_4$. On the other hand, the theory developed in [55] allows to obtain twin root data by integrating arbitrary Moufang foundations of 2-spherical type. The groups obtained in this way are not Kac–Moody groups whenever the foundation contains a Moufang octagon (which corresponds to a rank 2 parabolic subgroup of type ${}^2F_4$).

The conventions adopted throughout the rest of this section are the following: we let (W, S) be a Coxeter system with root system Φ and $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ be a twin root datum of type (W, S). We assume that each root group is finite and that W is infinite. The associated twin buildings are denoted $(\mathscr{B}_+, \mathscr{B}_-)$. The groups $\mathrm{Aut}(\mathscr{B}_\pm)$ are endowed with the compact-open topology, which makes them locally compact totally disconnected second countable topological groups. We also consider the subgroups T, N, U_+, U_- of Λ defined in Sect. 0.3. The normal subgroup generated by all root groups is denoted $\Lambda^\dagger$. If $g \in \Lambda$ fixes the building $\mathscr{B}_+$ it fixes in particular the standard positive apartment and its unique opposite in $\mathscr{B}_-$ [1, Lemma 2] since g preserves the twinning; therefore we have: $g \in T$ by [62, Corollaire 3.5.4]. Moreover by the Moufang parametrization (by means of root group actions) of chambers having a panel in the standard apartment, such a g must centralize each root group. This argument shows that the kernel of the action of Λ on $\mathscr{B}_+$ (resp. $\mathscr{B}_-$) is the centralizer $Z_\Lambda(\Lambda^\dagger)$ and is contained in T.

1.2. Topological completions: the building topology. For $\epsilon \in \{+, -\}$, let $\Lambda_\epsilon^{\mathrm{eff}}$ be the image of the natural homomorphism $\pi_\epsilon : \Lambda \to \mathrm{Aut}(\mathscr{B}_\epsilon)$. Thus $\Lambda_+^{\mathrm{eff}} \simeq \Lambda/Z_\Lambda(\Lambda^\dagger) \simeq \Lambda_-^{\mathrm{eff}}$. The closure $\overline{\Lambda_+^{\mathrm{eff}}} \leqslant \mathrm{Aut}(\mathscr{B}_+)$ is the topological completion considered in [65]. In the Kac–Moody case, another approach was proposed in [27], using the so-called weight topology; this allows to obtain completions of Λ without taking the effective quotient. However, the weight topology is defined using Kac–Moody algebras and, hence, does not have an obvious substitute in the abstract framework considered here. Therefore, we propose the following.

For each non-negative integer n, let K_ϵ^n be the pointwise stabilizer in U_ϵ of the ball of $\mathscr{B}_\epsilon$ centered at c_ϵ and of combinatorial radius n. Clearly $\bigcap_n K_\epsilon^n \subset Z_\Lambda(\Lambda^\dagger) \subset T$ and, hence, $\bigcap_n K_\epsilon^n = \{1\}$ because $T \cap U_\epsilon = \{1\}$ by [62, Theorem 3.5.4]. Define a map $f_\epsilon : \Lambda \times \Lambda \to \mathbf{R}_+$ as follows:

$$f_\epsilon(g, h) = \left\{\begin{matrix} 2 & \text{if } g^{-1}h \notin U_\epsilon, \\ \exp\left(-\max\left\{n \,\middle|\, g^{-1}h \in K_\epsilon^n\right\}\right) & \text{if } g^{-1}h \in U_\epsilon \end{matrix}\right\}.$$

Since K_ϵ^n is a group for each n, it follows that f_ϵ is a left-invariant ultrametric distance on Λ. We let $\overline{\Lambda}_\epsilon$ be the completion of Λ with respect to this metric [9, II.3.7 Théorème 3 and III.3.4 Théorème 3.4]; this is the topological completion that we consider in this paper.

Definition. *The so-obtained topology is called the* building topology *on* $\overline{\Lambda}_\epsilon$.

By left-invariance of the metric, replacing U_ϵ and c_ϵ by Λ-conjugates leads to the same topology. Here is a summary of some of its basic properties; similar results hold with the signs $+$ and $-$ interchanged.

Proposition 1. *We have the following:*

(i) *The group $\overline{\Lambda}_+$ is locally compact and totally disconnected for the above topology. It is second countable whenever Λ is countable, i.e. whenever so is T.*
(ii) *The canonical map $\pi_+ : \Lambda \to \Lambda_+^{\mathrm{eff}}$ has a unique extension to a continuous surjective open homomorphism $\overline{\pi}_+ : \overline{\Lambda}_+ \to \overline{\Lambda}_+^{\mathrm{eff}}$.*
(iii) *The kernel of $\overline{\pi}_+$ is the discrete subgroup $Z_\Lambda(\Lambda^\dagger) < \overline{\Lambda}_+$.*
(iv) *We have* $\mathrm{Stab}_{\overline{\Lambda}_+}(c_+) \simeq T \ltimes \overline{U}_+$, *where $\overline{U}_+$ denotes the closure of U_+ in $\overline{\Lambda}_+$.*
(v) *Every element $g \in \overline{\Lambda}_+$ may be written in a unique way as a product $g = u_+ n u_-$, with $u_+ \in \overline{U}_+$, $n \in N$ and $u_- \in U_-$.*
(vi) *The sextuple $(\overline{\Lambda}_+, N, \overline{U}_+, U_-, T, S)$ is a refined Tits system, as defined in* [41].

Remark. It follows from (ii) and (iii) that the canonical map $\overline{\Lambda}_+/Z_\Lambda(\Lambda^\dagger) \to \overline{\Lambda}_+^{\mathrm{eff}}$ is an isomorphism of topological groups.

Proof. We start by noting that the restriction $\pi_+|_{U_+}$ is injective since $Z_\Lambda(\Lambda^\dagger) \cap U_\epsilon = \{1\}$ by [62, Théorème 3.5.4]. Therefore, it follows from the definitions that $\pi_+ : U_+ \to \pi_+(U_+)$ is an isomorphism of topological groups.

We now prove (ii). Let (λ_n) be a Cauchy sequence of elements of Λ. Let $n_0 \geqslant 0$ be such that $f_+(\lambda_{n_0}, \lambda_n) \leqslant 1$ for all $n > n_0$. It follows that $\pi_+(\lambda_{n_0}^{-1}\lambda_n)$ lies in the stabilizer of c_+ in $\mathrm{Aut}(\mathscr{B}_+)$, which is compact. This implies that $\pi_+(\lambda_n)$ is a converging sequence in $\mathrm{Aut}(\mathscr{B}_+)$. Hence π_+ has a unique continuous extension $\overline{\pi}_+ : \overline{\Lambda}_+ \to \overline{\Lambda}_+^{\mathrm{eff}}$ and it remains to prove that $\overline{\pi}_+$ is surjective. By the preliminary remark above, it follows that $\overline{\pi}_+ : \overline{U}_+ \to \overline{\pi_+(U_+)}$ is an isomorphism of topological groups. The surjectivity of $\overline{\pi}_+$ follows easily since $\overline{U}_+$ is an open neighborhood of the identity. Finally, since $\overline{U}_+$ contains a basis $\{\overline{K}_\epsilon^n\}$ of open neighborhoods of the identity, it follows that $\overline{\pi}_+$ maps an open subset to an open subset.

(iv). The inclusion $T.\overline{U}_+ < \mathrm{Stab}_{\overline{\Lambda}_+}(c_+)$ is clear. Let $g \in \mathrm{Stab}_{\overline{\Lambda}_+}(c_+)$ and let (λ_n) be a sequence in Λ such that $\lim_{n\to+\infty} \lambda_n = g$. Up to passing to a subsequence, we may – and shall – assume that $\lambda_n \in \mathrm{Stab}_\Lambda(c_+)$ for all n. We know by [62, §3.5.4] that $\mathrm{Stab}_\Lambda(c_+) \simeq T \ltimes U_+$. Hence each λ_n has a unique writing $\lambda_n = t_n u_n$ with $t_n \in T$ and $u_n \in U_+$. Again, up to extracting a subsequence, we have $f_+(\lambda_1, \lambda_n) < 1$ for all n. In view of the semidirect decomposition $\mathrm{Stab}_\Lambda(c_+) \simeq T \ltimes U_+$, this implies that $t_n = t_1$ for all n. In particular, the sequence (u_n) of elements of U_+ converges to $t_1^{-1}g$. This shows that $g \in T.\overline{U}_+$ as desired. For every nontrivial element $t \in T$, we have $f_+(1, t) = 2$ because $T \cap U_+ = \{1\}$. On the other hand, for all $u \in \overline{U}_+$, we have $f_+(1, u) \leqslant 1$. Therefore, we have $T \cap \overline{U}_+ = \{1\}$.

(iii). The fact that $Z_\Lambda(\Lambda^\dagger)$ is discrete follows from $Z_\Lambda(\Lambda^\dagger) \cap U_+ = \{1\}$. Clearly we have $Z_\Lambda(\Lambda^\dagger) < \mathrm{Ker}(\overline{\pi}_+)$. We must prove the reverse inclusion.

Let $k \in \operatorname{Ker}(\overline{\pi}_+)$. By (iv), we have $k = tu$ for unique elements $t \in T$ and $u \in \overline{U}_+$. Applying (iv) to the effective group $\overline{\Lambda}_+^{\mathrm{eff}}$, we obtain $\pi_+(t) = 1$ and $\overline{\pi}_+(u) = 1$. Since the restriction of $\overline{\pi}_+$ to $\overline{U}_+$ is injective by the proof of (ii), we deduce $u = 1$ and hence $k \in T < \Lambda$. Therefore $k \in Z_\Lambda(\Lambda^\dagger)$ as desired.

(i). The building topology comes from a metric, so the Hausdorff group Λ injects densely in its completion $\overline{\Lambda}_+$ [9, II.3.7 Corollaire] and the latter group is second countable whenever Λ is countable. It is locally compact because $\overline{U}_+$ is a compact open subgroup by the proof of (ii). Furthermore, $\overline{\pi}_+$ annihilates the connected component of $\overline{\Lambda}_+$ because $\overline{\Lambda}_+^{\mathrm{eff}}$ is totally disconnected. On the other hand, the kernel of $\overline{\pi}_+$ is discrete by (iii). Hence $\overline{\Lambda}_+$ itself is totally disconnected.

(v). The group U_- acts on $\mathscr{B}_+$ with the apartment $\mathscr{A}_+$ as a fundamental domain. The group N stabilizes $\mathscr{A}_+$ and acts transitively on its chambers. In view of (iv), it follows that $\overline{\Lambda}_+ = \overline{U}_+.N.U_-$. On the other hand, it follows easily from the definition of f_ϵ that

$$\overline{U}_\epsilon = \{g \in \overline{\Lambda}_\epsilon \mid f_\epsilon(1, g) \leqslant 1\}.$$

Therefore, the uniqueness assertion follows immediately from [62, §1.5.4] and the fact that $\Lambda \cap \overline{U}_+ = U_+$.

(vi). The main axiom of a refined Tits system is the property of assertion (v), which has just been proven. For the other axioms to be checked, the arguments are the same as [65, Proof of Theorem 1.C.(i)]. □

1.3. Twin building lattices. Let $q_{\min} = \min\{|U_\alpha| : \alpha \in \Pi\}$, where $\Pi \subset \Phi$ is the set of simple roots. The following is an adaptation of the main result of [60]:

Proposition 2. *The image of the diagonal injection*

$$\Lambda \to \overline{\Lambda}_+ \times \overline{\Lambda}_- : \lambda \mapsto (\lambda, \lambda)$$

is a discrete subgroup of $\overline{\Lambda}_+ \times \overline{\Lambda}_-$. It is an irreducible lattice if and only if $W(1/q_{\min}) < +\infty$ and $Z_\Lambda(\Lambda^\dagger)$ is finite.

Proof. Let $\Delta(\Lambda) = \{(\lambda, \lambda) \mid \lambda \in \Lambda\} < \overline{\Lambda}_+ \times \overline{\Lambda}_-$. The subgroup $\overline{U}_+ \times \overline{U}_- < \overline{\Lambda}_+ \times \overline{\Lambda}_-$ is an open neighborhood of the identity. We have $\Delta(\Lambda) \cap (\overline{U}_+ \times \overline{U}_-) = \Delta(U_+ \cap U_-)$. By [62, §3.5.4], $U_+ \cap U_- = \{1\}$. Thus $\Delta(\Lambda)$ is discrete. The second assertion follows from the proofs of [64, Proposition 5 and Corollary 6], which apply here without any modification: the only requirement is that $(\overline{\Lambda}_+, N, \overline{U}_+, U_-, T, S)$ and $(\overline{\Lambda}_-, N, \overline{U}_-, U_+, T, S)$ be refined Tits systems. This follows from Proposition 1(vi). □

Note that the group $Z_\Lambda(\Lambda^\dagger)$ may be arbitrarily large, since one may replace Λ by the direct product of Λ with an arbitrary group; the root groups of Λ also provide a twin root datum for this direct product. However it is always possible to make the group $Z_\Lambda(\Lambda^\dagger)$ finite by taking appropriate quotients; note that if $\Lambda = \Lambda^\dagger$, then $Z_\Lambda(\Lambda^\dagger) = Z(\Lambda)$ is abelian. Finally, since $Z_\Lambda(\Lambda^\dagger) < T$, it follows that $Z_\Lambda(\Lambda^\dagger)$ is always finite when Λ is a split or almost split Kac–Moody group.

Definition. *If the twin root datum $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ is such that $W(1/q_{\min}) < +\infty$ and $Z_\Lambda(\Lambda^\dagger)$ is finite, then Λ is called a* twin building lattice.

1.4. Structure of $\overline{U}_+$. The first assertion of the following proposition was proved in [65, Theorem 1.C(ii)]. Recall that a set Ψ of roots is called *prenilpotent* if the intersections $\bigcap_{\alpha\in\Psi}\alpha$ and $\bigcap_{\alpha\in-\Psi}\alpha$ are both non-empty sets of chambers; in this case Ψ is finite.

Proposition 3. *We have the following:*

(i) *Assume that each root group U_α is a finite p-group. Then $\overline{U}_+$ is pro-p.*
(ii) *Assume that each root group U_α is solvable. Then $\overline{U}_+$ is pro-solvable.*
(iii) *Assume that each root group U_α is nilpotent. Then, for every prenilpotent set of roots $\Psi \subset \Phi$, the group $U_\Psi = \langle U_\alpha \mid \alpha \in \Psi\rangle$ is nilpotent.*

Remark. One might expect that, under the assumption that all root groups are nilpotent, the group $\overline{U}_+$ is pro-nilpotent. This is however not true in general. Counterexamples are provided by twin root data over ground fields of mixed characteristics, constructed in [65]. More precisely, consider a twin root datum $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ of rank 2 with infinite Weyl group, such that the rank one subgroups are $\mathrm{SL}_2(\mathbf{F}_3)$ and $\mathrm{SL}_2(\mathbf{F}_4)$. The associated twin building is a biregular twin tree. Then the U_+-action induced on the ball of combinatorial radius 2 centered at c_+ is not nilpotent: indeed, the corresponding finite quotient of U_+ contains a subgroup isomorphic to the wreath product $(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \wr \mathbf{Z}/3\mathbf{Z}$, which is not nilpotent.

Proof. For (i), see [65, 1.C Lemma 1 p.198]. The arguments given in [loc. cit.] can be immediately adapted to provide a proof of (ii): the essential fact is that an extension of a solvable group (resp. a p-group) by a solvable group (resp. a p-group) is again solvable (resp. a p-group).

(iii). A set of roots Ψ is called *nilpotent* if it is prenilpotent and if, moreover, for each pair $\{\alpha, \beta\} \subset \Psi$ one has $[\alpha, \beta] \subset \Psi$. Since every prenilpotent set of roots is contained in a nilpotent set (see [62, §1.4.1 and §2.2.6]), it suffices to prove the assertion for nilpotent sets. The proof is by induction on the cardinality of Ψ, the result being obvious when Ψ is a singleton. The elements of Ψ can be ordered in a nibbling sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$; hence the sets $\Psi_1 = \Psi\backslash\{\alpha_1\}$ and $\Psi_n = \Psi\backslash\{\alpha_n\}$ are nilpotent [loc. cit., §1.4.1]. Furthermore, one has $[U_{\alpha_1}, U_{\Psi_1}] \leqslant U_{\Psi_1}$ and $[U_{\alpha_n}, U_{\Psi_n}] \leqslant U_{\Psi_n}$ as a consequence of (TRD1). Therefore, the subgroups U_{Ψ_1} and U_{Ψ_n} are normal in U_Ψ, and are nilpotent by the induction hypothesis. It follows that U_Ψ is nilpotent [34, Theorem 10.3.2]. This part of the proof does not require that the root groups be finite. $\square$

2. Further properties of topological completions

In this section, we introduce a property of fixed points of root subgroups of a group Λ endowed with a twin root datum; this property is called (FPRS). We first provide sufficient conditions which ensure that this property holds for any split or almost split Kac–Moody group, as well as for all exotic twin building lattices mentioned in Sect. 1.1. We then show that (FPRS) implies that the topological completion $\overline{\Lambda}_+$ is topologically simple (modulo the kernel of the action on the building, see Proposition 11). Property (FPRS) will be used again below, as a sufficient condition implying that any nontrivial continuous homomorphism whose domain is $\overline{\Lambda}_+$ is proper (Theorem 28).

Throughout this section, we let Λ be a group endowed with a twin root datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type (W, S) and let $(\mathscr{B}_+, \mathscr{B}_-)$ be the associated twin buildings.

2.1. Fixed points of root subgroups. For any subgroup $\Gamma \leqslant \Lambda$, we define $r(\Gamma)$ to be the supremum of the set of all non-negative real numbers r such that Γ fixes pointwise the (combinatorial) closed ball $B(c_+, r)$ of (combinatorial) radius r centered at c_+. In the present subsection, we consider the following condition:

(FPRS) *Given any sequence of roots $(\alpha_n)_{n\geqslant 0}$ of $\Phi(\mathscr{A})$ such that $\lim_{n\to+\infty} d(c_+, \alpha_n) = +\infty$, we have: $\lim_{n\to+\infty} r(U_{-\alpha_n}) = +\infty$.*

Remark. This property can be seen as a non-quantitative generalization of [15, Prop. 7.4.33].

In other words, this means that if the sequence of roots $(\alpha_n)_{n\geqslant 0}$ is such that $\lim_{n\to+\infty} d(c_+, \alpha_n) = +\infty$, then the sequence of root subgroups $(U_{-\alpha_n})_{n\geqslant 0}$ tends uniformly to the identity in the building topology. The purpose of this subsection is to establish sufficient conditions on the twin root datum $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ which ensure that property (FPRS) holds. To this end, we will need the following conditions:

(PP) *For any prenilpotent pair of roots $\{\alpha, \beta\}$ such that $\langle r_\alpha, r_\beta\rangle$ is infinite, either $[U_\alpha, U_\beta] = \{1\}$ or there exists a root ϕ such that $r_\phi(\alpha) = -\beta$, $[U_\alpha, U_\beta] \leqslant U_\phi$ and $[U_\alpha, U_\phi] = \{1\} = [U_\beta, U_\phi]$.*

(2-sph) *The Coxeter system (W, S) is 2-spherical and Λ possesses no critical rank 2 subgroup.*
This means that any pair of elements of S generates a finite group and moreover that for any pair $\{\alpha, \beta\} \subset \Pi$, the group $X_{\alpha,\beta}$ generated by the four root groups $U_{\pm\alpha}$, $U_{\pm\beta}$, divided by its center, is not isomorphic to any of the groups $B_2(2)$, $G_2(2)$, $G_2(3)$ or ${}^2F_4(2)$.

The main result of this section is the following:

Proposition 4. *Assume that the twin root datum* $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ *satisfies* (PP) *or* (2-sph) *or that Λ is a split or almost split Kac–Moody group. Then property* (FPRS) *holds.*

Remark. The exotic examples of twin root data mentioned in Sect. 1.1(I) and (II) satisfy condition (PP). In fact, they satisfy even the stronger condition that all commutation relations are trivial: for any prenilpotent pair of distinct roots $\{\alpha, \beta\}$, one has $[U_\alpha, U_\beta] = \{1\}$. Moreover, the examples of type (III) satisfy (2-sph). However, it was communicated to us by B. Mühlherr that there exists an example of a group endowed with a twin root datum, which does not satisfy condition (FPRS). In this example, whose construction is nontrivial, the Weyl group is the free Coxeter group of rank 3 (i.e., a free product of 3 copies of the group of order 2) and the ground field is $\mathbf{F}_2$.

The proof of Proposition 4 splits into a sequence of lemmas which we prove separately.

Lemma 5. *Assume that* (PP) *holds. For each integer* $n \geqslant 0$*, each root* $\alpha \in \Phi(\mathcal{A})$ *and each chamber* $c \in \mathcal{A}_+$*, if* $d(c, \alpha) \geqslant \frac{4^{n+1}-1}{3}$*, then* $U_{-\alpha}$ *fixes* $B(c, n)$ *pointwise. In particular* (FPRS) *holds.*

Proof. We work by induction on n. If $d(c, \alpha) \geqslant 1$, then $c \notin \alpha$ whence $c \in -\alpha$. In particular c is fixed by $U_{-\alpha}$. Thus the desired property holds for $n = 0$.

Assume now that $n > 0$ and let α be a root such that $d(c, \alpha) \geqslant \frac{4^{n+1}-1}{3}$. By induction, the group $U_{-\alpha}$ fixes the ball $B(c, n-1)$ pointwise. Furthermore, if c' is a chamber contained in $\mathcal{A}_+$ and adjacent to c, then $d(c', \alpha) \geqslant d(c, \alpha) - 1$; therefore, the induction hypothesis also implies that $U_{-\alpha}$ fixes the ball $B(c', n-1)$ pointwise.

Let now x be a chamber at distance n from c. Let $c_0 = c, c_1, \ldots, c_n = x$ be a minimal gallery joining c to x. We must prove that $U_{-\alpha}$ fixes x. If c_1 is contained in $\mathcal{A}_+$ then we are done by the above. Thus we may assume that c_1 is not in $\mathcal{A}_+$. Let c' be the unique chamber of $\mathcal{A}_+$ such that c, c_1 and c' share a panel. Let $\beta \in \Phi(\mathcal{A})$ be one of the two roots such that the wall $\partial\beta$ separates c from c'. Upon replacing β by its opposite if necessary, we may – and shall – assume by [73, Proposition 9] that the pair $\{-\alpha, \beta\}$ is prenilpotent. Let $u \in U_\beta$ be the (unique) element such that $u(c_1)$ belongs to $\mathcal{A}_+$; thus we have $u(c_1) = c$ or c'. Since $u(c_1), u(c_2), \ldots, u(c_n)$ is a minimal gallery, it follows that $u(x)$ is contained in $B(c, n-1) \cup B(c', n-1)$.

There are three cases.

Suppose first that $[U_{-\alpha}, U_\beta] = \{1\}$. For any $g \in U_{-\alpha}$, we have $g = u^{-1}gu$ whence $g(x) = u^{-1}gu(x) = x$ because $g \in U_{-\alpha}$ fixes $B(c, n-1) \cup B(c', n-1)$ pointwise.

Suppose now that $[U_{-\alpha}, U_\beta] \neq \{1\}$ and that $\langle r_\alpha, r_\beta \rangle$ is infinite. By property (PP) there exists a root $\phi \in \Phi(\mathcal{A})$ such that $[U_{-\alpha}, U_\beta] \leqslant U_\phi$ and $r_\phi(\alpha) = \beta$. Let $y_0 = c, y_1, \ldots, y_k$ be a gallery of minimal possible length joining c to a chamber of $-\phi$. Thus we have $y_k \in -\phi$, $y_{k-1} \in \phi$ and $k = d(c, -\phi)$. Since $r_\phi(\beta) = \alpha$ and since either c or c' belongs to β, it follows by considering the (possibly non-minimal) gallery

$$c = y_0, \ldots, y_{k-1}, y_k = r_\phi(y_{k-1}), r_\phi(y_{k-2}), \ldots, r_\phi(c), r_\phi(c')$$

of length $2k$, that $d(c,\alpha)\leqslant 2k$, whence $d(c,-\phi)\geqslant\frac{1}{2}d(c,\alpha)\geqslant\frac{4^{n+1}-1}{6}>\frac{4^n-1}{3}$. A similar argument shows that $d(c',-\phi)\geqslant\frac{1}{2}d(c',\alpha)\geqslant\frac{4^{n+1}-4}{6}>\frac{4^n-1}{3}$. Therefore, the induction hypothesis shows that U_ϕ fixes $B(c,n-1)\cup B(c',n-1)$ pointwise. Now, for any $g\in U_{-\alpha}$, we have

$$g(x)=[g,u^{-1}]u^{-1}gu(x)=[g,u^{-1}](x)$$

because $g\in U_{-\alpha}$ fixes $B(c,n-1)\cup B(c',n-1)$ pointwise. By (PP), the commutator $[g,u^{-1}]$ commutes with u and, hence, we have $[g,u^{-1}](x)=u^{-1}[g,u^{-1}]u(x)=x$ because $[g,u^{-1}]\in U_\phi$ fixes $B(c,n-1)\cup B(c',n-1)$ pointwise.

Suppose finally that $[U_{-\alpha},U_\beta]\neq\{1\}$ and that $\langle r_\alpha,r_\beta\rangle$ is finite. This implies that the pairs $\{-\alpha,\beta\}$ and $\{-\alpha,-\beta\}$ are both prenilpotent. Therefore, up to replacing β by its opposite if necessary, we may – and shall – assume that $c\notin\beta$, whence $u(c_1)=c$. Note that $\langle r_\alpha,r_\beta\rangle$ is contained in a rank 2 parabolic subgroup P of W. Since any such subgroup is the Weyl group of a Levi factor of rank 2 of Λ, which is itself endowed with a twin root datum of rank 2, it follows from [75, Theorem 17.1] that P is of order at most 16. Let $[-\alpha,\beta]=\{\gamma\in\Phi(\mathcal{A})\mid(-\alpha)\cap\beta\subseteq\gamma,\ \alpha\cap(-\beta)\subseteq-\gamma\}$; thus $[-\alpha,\beta]$ has at most 8 elements because for every $\gamma\in[-\alpha,\beta]$, the reflection r_γ belongs to P. Order the elements of $[\beta,-\alpha]$ in a natural cyclic order: $[\beta,-\alpha]=\{\beta=\beta_0,\beta_1,\dots,\beta_m=-\alpha\}$; this means that $r_{\beta_i}(\beta_{i-1})=\beta_{i+1}$ for $i=1,\dots,k-1$. Such an ordering does exist because the group $\langle r_\gamma\mid\gamma\in[\beta,-\alpha]\rangle$ is (finite) dihedral. Let $c=y_0,y_1,\dots,y_k$ be a gallery of minimal possible length joining c to a chamber of $-\beta_1$. Thus we have $y_k\in-\beta_1$, $y_{k-1}\in\beta_1$ and $k=d(c,-\beta_1)$. Since $r_{\beta_1}(\beta)=-\beta_2$ and since c' belongs to β, it follows from considering the gallery

$$c=y_0,\dots,y_{k-1},y_k=r_{\beta_1}(y_{k-1}),r_{\beta_1}(y_{k-2}),\dots,r_{\beta_1}(c),r_{\beta_1}(c')$$

of length $2k$, that $d(c,-\beta_2)\leqslant 2k$. A straightforward induction yields $d(c,-\beta_i)\leqslant ik$ for $i=1,\dots,m$. In particular, we have $d(c,\alpha)\leqslant mk=m.d(c,-\beta_1)$. Recall that $m+1$ is the cardinality of $[-\alpha,\beta]$.

We may now choose a natural cyclic order $[-\beta,-\alpha]=\{-\beta=\beta'_0,\beta'_1,\dots,\beta'_{m'}=-\alpha\}$ and repeat the same arguments with c replaced by c', β replaced by $-\beta$ and each β_i replaced by $r_\beta(\beta_i)$. This yields $d(c',-\beta'_i)\leqslant id(c',-\beta'_1)$ for each i. Note that $d(c,-\beta_1)=d(r_\beta(c),-r_\beta(\beta_1))=d(c',-\beta'_1)$. We obtain that $d(c,\alpha)-1\leqslant d(c',\alpha)\leqslant m'.d(c,-\beta_1)$, where $m'+1$ is the cardinality of $[-\alpha,-\beta]$. Observe now that $m+m'=\frac{|P|}{2}\leqslant 8$. In particular, we have $\min\{m,m'\}\leqslant 4$. Therefore, we deduce from the inequalities above that for each $i=1,\dots,m$, we have

$$d(c,-\beta_i)\geqslant d(c,-\beta_1)\geqslant\frac{d(c,\alpha)-1}{4}\geqslant\frac{4^n-1}{3}.$$

By the induction hypothesis, it follows that for each $\gamma\in\,]-\alpha,\beta[=[-\alpha,\beta]\setminus\{-\alpha,\beta\}$, the root subgroup U_γ fixes the ball $B(c,n-1)$ pointwise.

Now, for any $g \in U_{-\alpha}$, we have $g(x) = [g, u^{-1}]u^{-1}gu(x) = [g, u^{-1}](x)$ because $g \in U_{-\alpha}$ fixes $B(c, n-1)$ pointwise. Moreover, we have $[g, u^{-1}] \in U_{]-\alpha,\beta[} = \langle U_\gamma \mid \gamma \in]-\alpha, \beta[\rangle$ by (TRD1). Therefore, the commutator $[g, u^{-1}]$ fixes $u(x)$ and we have

$$g(x) = [g, u^{-1}](x) = ([[g, u^{-1}], u^{-1}])u^{-1}[g, u^{-1}]u(x)$$
$$= [[g, u^{-1}], u^{-1}](x).$$

Repeating the argument m times successively, we finally obtain $g(x) = [\dots[[g, u^{-1}], u^{-1}], \dots, u^{-1}](x)$ where the commutator is iterated m times. By (TRD1), we have $[\dots[[g, u^{-1}], u^{-1}], \dots, u^{-1}] \in U_{]\beta_1,\beta[}$, which is trivial since $]\beta_1, \beta[$ is empty. Therefore, we deduce finally that g fixes x, as desired. □

Lemma 6. *Suppose that* Λ *is a split Kac–Moody group and that* $\{U_\alpha\}_{\alpha\in\Phi}$ *is its natural system of root subgroups. Then the twin root datum* $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ *satisfies* (PP).

Proof. This follows by combining [54, Theorem 2] with some results from [7] (see also [72, Sect. 3.2]). In order to be more precise, we freely use the notation and terminology of these references in the present proof. In particular, we use the 'linear' root system of the Lie algebra associated with the Kac–Moody group Λ, instead of the 'abstract' root system introduced above and which is appropriate to the case of general twin root data. A comprehensive introduction to linear root systems can be found in [53, Chapt. 5].

Commutation relations in split Kac–Moody groups are described precisely by [54, Theorem 2]. Combining the latter result together with [7, Proposition 1], one sees easily that if $\{\alpha, \beta\}$ is a prenilpotent pair such that $\langle r_\alpha, r_\beta\rangle$ is infinite and $[U_\alpha, U_\beta] \neq \{1\}$, then $[U_\alpha, U_\beta] \leqslant U_{\alpha+\beta}$ and the α-string through β is of length $\geqslant 5$ and contains exactly 4 real roots, which are $\beta - \langle\beta, \alpha^\vee\rangle\alpha$, $\beta - (\langle\beta, \alpha^\vee\rangle - 1)\alpha$, β and $\beta + \alpha$. In particular, $\beta + 2\alpha$ is not a root, whence $\{-\alpha, \alpha + \beta\}$ is W-conjugate to a Morita pair by [7, Proposition 3(i)]. In particular, we have $\langle -\alpha, (\alpha + \beta)^\vee\rangle = -1$ by [7, Proposition 2]. We deduce that

$$r_{\alpha+\beta}(-\alpha) = -\alpha + \langle -\alpha, (\alpha + \beta)^\vee\rangle(\alpha + \beta) = \beta.$$

Finally, since $2\alpha + \beta$ is not a root, it follows from [54, Theorem 2] that $[U_\alpha, U_{\alpha+\beta}] = \{1\}$. Hence property (PP) holds, as desired. □

Lemma 7. *Suppose that* Λ *is an almost split Kac–Moody group and that* $\{U_\alpha\}_{\alpha\in\Phi}$ *is its natural system of root subgroups. Then the twin root datum* $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ *satisfies* (FPRS).

Proof. A reference for almost split Kac–Moody groups is [62, Chap. 11–13]. Let $\mathbf{K}$ be the ground field of Λ, let $\mathbf{K}_s$ be a separable closure of $\mathbf{K}$, let $\Gamma = \mathrm{Gal}(\mathbf{K}_s/\mathbf{K})$ and let $\widetilde{\Lambda}$ be a split Kac–Moody group over $\mathbf{K}_s$, such that Λ is the fixed point set of a Γ-action on $\widetilde{\Lambda}$. We henceforth view Λ as a subgroup of $\widetilde{\Lambda}$. We denote by $(\widetilde{U}_\alpha)_{\alpha\in\widetilde{\Phi}}$ the natural system of root subgroups of

$\widetilde{\Lambda}$ and by $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$ the twin building associated with the twin root datum $(\widetilde{\Lambda}, (\widetilde{U}_\alpha)_{\alpha\in\widetilde{\Phi}})$. By [62, Théorème 12.4.4], the twin building $(\mathscr{B}_+, \mathscr{B}_-)$ is embedded in a Λ-equivariant way in $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$, as the fixed point set of Γ-action on $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$. This embedding maps chambers of $(\mathscr{B}_+, \mathscr{B}_-)$ to **K**-chambers of $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$, which are minimal Γ-invariant spherical residues. Let r be the rank of such a spherical residue. Two **K**-chambers are adjacent (as chambers of $(\mathscr{B}_+, \mathscr{B}_-)$) if they are contained in a common spherical residue of rank $r+1$ and either coincide or are opposite in that residue. This shows that bounded subsets of $(\mathscr{B}_+, \mathscr{B}_-)$ are also bounded in $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$ and, moreover, that every ball of large radius in $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$ which is centered at a point of a **K**-chamber contains a ball of large radius of $(\mathscr{B}_+, \mathscr{B}_-)$.

Let $(\alpha_n)_{n\geqslant 0}$ be a sequence of roots of $\Phi = \Phi(\mathscr{A})$ such that $d(c_+, \alpha_n)$ tends to infinity with n. We must prove that $r(U_{-\alpha_n})$ also tends to infinity with n. We choose the base chamber $\widetilde{c}_+$ of $(\widetilde{\mathscr{B}}_+, \widetilde{\mathscr{B}}_-)$ such that it is contained in the **K**-chamber c_+, and denote by $\widetilde{r}(H)$ the supremum of the radius of a ball centered at $\widetilde{c}_+$ which is pointwise fixed by H. In view of the preceding paragraph, it suffices to show that $\widetilde{r}(U_{-\alpha_n})$ tends to infinity with n. To this end, we will use the fact that $(\widetilde{\Lambda}, (\widetilde{U}_\alpha)_{\alpha\in\widetilde{\Phi}})$ satisfies property (FPRS) by Lemmas 5 and 6. Let $\beta \in \Phi$ be a **K**-root and consider the root subgroup U_β. Let x, y be two adjacent **K**-chambers such that β contains x but not y. Let $\widetilde{\Phi}(\beta)$ be the set of ($\mathbf{K}_s$-)roots containing x but not y; it is independent of the choice of x and y. Furthermore $\widetilde{\Phi}(\beta)$ is a prenilpotent subset of $\widetilde{\Phi}$ and $U_\beta \subset \widetilde{U}_{\widetilde{\Phi}(\beta)} = \langle \widetilde{U}_\gamma \mid \gamma \in \widetilde{\Phi}(\beta)\rangle$ by [62, §12.4.3]. Therefore, in order to finish the proof, it suffices to show that $\min\{d(\widetilde{c}_+, \gamma) \mid \gamma \in \widetilde{\Phi}(\alpha_n)\}$ tends to infinity with n.

Assume for a contradiction that this is not the case. Then there exists a subsequence $(\alpha_{n_j})_{j\geqslant 0}$ and an element $\gamma_j \in \widetilde{\Phi}(\alpha_{n_j})$ such that $d(\widetilde{c}_+, \gamma_j)$ is a bounded function of j. Since the apartments are locally finite, it follows that, up to extracting a subsequence, we may – and shall – assume that γ_j is constant. On the other hand, we claim (and prove below) that if α, α' are two distinct **K**-roots, then the sets $\widetilde{\Phi}(\alpha)$ and $\widetilde{\Phi}(\alpha')$ are disjoint; this implies that the sequence $(\alpha_{n_j})_{j\geqslant 0}$ is constant, contradicting the fact that $d(c_+, \alpha_{n_j})$ tends to infinity with j. It remains to prove the claim. This is most easily done using the notion of (combinatorial) projections in buildings. Let π (resp. π') be a **K**-panel stabilized by the **K**-reflection r_α (resp. $r_{\alpha'}$). Thus π and π' are spherical residues of $\widetilde{\mathscr{B}}_+$ of rank $r+1$. Note that $\mathrm{proj}_\pi(\pi')$ (resp. $\mathrm{proj}_{\pi'}(\pi)$) contains a **K**-chamber. Furthermore, given any $\gamma \in \widetilde{\Phi}(\alpha) \cap \widetilde{\Phi}(\alpha')$, the ($\mathbf{K}_s$-)reflection r_γ stabilizes both π and π', but it does not stabilize any **K**-chamber. Therefore $\mathrm{proj}_\pi(\pi')$ (resp. $\mathrm{proj}_{\pi'}(\pi)$) cannot be reduced to a single **K**-chamber; since $\mathrm{proj}_\pi(\pi')$ (resp. $\mathrm{proj}_{\pi'}(\pi)$) is a sub-residue of π (resp. π'), it must be of rank $r+1$, which yields $\mathrm{proj}_\pi(\pi') = \pi$ (resp. $\mathrm{proj}_{\pi'}(\pi) = \pi'$). In other words the **K**-panels π and π' are *parallel*. Since r_α stabilizes any **K**-panel which is parallel to π by [23, Proposition 2.7], this implies that the **K**-reflections r_α and $r_{\alpha'}$ stabilize a common **K**-panel. Since $\alpha \neq \alpha'$, we deduce that $\alpha = -\alpha'$, which implies that $\widetilde{\Phi}(\alpha) = -\widetilde{\Phi}(\alpha') = \{-\gamma \mid \gamma \in \widetilde{\Phi}(\alpha')\}$. In particular, the sets $\widetilde{\Phi}(\alpha)$ and $\widetilde{\Phi}(\alpha')$ are disjoint, a contradiction. □

Lemma 8. *Suppose that* Λ *satisfies condition* (2-sph) *and that all root groups are finite. Then property* (FPRS) *holds.*

Remark. It appears that the proof given below uses the finiteness of the root subgroups in exactly one place, in order to show that $\overline{U}_+$ is pro-nilpotent in the building topology. It turns out that this holds in the general case of groups satisfying (2-sph), without any assumption on the cardinality of the root subgroups (recall from the Remark following Proposition 3 that condition (2-sph) is necessary for this to hold). However, although in the case of finite root subgroups, this will be established in an elementary way using the fact that finite p-groups are nilpotent, the infinite case is more delicate and requires B. Mühlherr's embedding theorem (see [56] and references therein), showing that any group satisfying (2-sph) admits a geometric embedding in some split Kac–Moody group. The desired assumption then follows from the fact that in the case of Kac–Moody groups, the group $\overline{U}_+$ is always pro-nilpotent in the building topology.

In fact, since split Kac–Moody groups satisfy (FPRS) by Lemmas 6 and 5, Mühlherr's embedding theorem may be used to deduce directly that groups satisfying (2-sph) also satisfy (FPRS), as was done in the proof of Lemma 7.

In order to keep the paper reasonably self-contained, we shall content ourselves with a detailed proof in the case of finite root subgroups, without appealing to the aforementioned embedding theorem. This is in fact the only relevant case for all the applications discussed in the rest of the paper.

Proof. As before, we denote by (W, S) the Coxeter system consisting of the Weyl group W together with its canonical generating set S. If (W, S) is not of irreducible type, then the buildings $\mathscr{B}_+$ and $\mathscr{B}_-$ split into direct products of irreducible components, and it is easy to see that checking (FPRS) for the Λ-action on $\mathscr{B}_+$ is equivalent to checking (FPRS) for the induced action on each irreducible component. We henceforth assume that (W, S) is of irreducible type. If W is finite, then there is no sequence of roots $(\alpha_n)_{n\geqslant 0}$ such that $d(c_+, \alpha_n)$ tends to infinity with n. Assume now that W is infinite; in particular (W, S) is of rank $\geqslant 3$. Consider a sequence of roots $(\alpha_n)_{n\geqslant 0}$ such that $d(c_+, \alpha_n)$ tends to infinity with n. We must prove that $r(U_{-\alpha_n})$ tends to infinity with n.

Given any two basis roots α, α', there exists a sequence of basis roots $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \alpha'$ such that $r_{\alpha_{i-1}}$ does not commute with r_{α_i} for $i = 1, \ldots, k$ because (W, S) is irreducible. This implies that each rank two subgroup $X_{\alpha_{i-1},\alpha_i} = \langle U_{\pm\alpha_{i-1}}, U_{\pm\alpha_i}\rangle$ is endowed with a twin root datum of irreducible spherical type and rank 2, since (W, S) is 2-spherical. By the classification of such groups [75], it follows that $U_{\alpha_{i-1}}$ and U_{α_i} are both p_i-groups (of nilpotency degree ≤ 3) for some prime p_i. Since this is true for each i, we have $p_i = p_{i-1}$, whence the sequence (p_i) is constant. This shows that there exists a prime p such that each root group is a finite p-group.

By Proposition 3(i), it follows that $\overline{U_+}$ is pro-p, whence pro-nilpotent. In particular, the descending central series $(U_+^{(n)})_{n\geqslant 0}$ tends to the identity in the building topology when $n \to +\infty$. In other words, this means that for each k there is some n such that the group $U_+^{(n)}$ acts trivially on the ball of radius k centered at the base chamber c_+. Therefore, in order to finish the proof, it suffices to show that $\lim_{n\to+\infty} q(-\alpha_n) = +\infty$, where $q(\alpha) = \max\{n \geqslant 0 : U_\alpha \leqslant U_+^{(n)}\}$ for each positive root $\alpha \in \Phi_+$.

For each integer $n \geqslant 0$, we set $\Phi_+^n = \{\alpha \in \Phi_+ : q(\alpha) = n\}$. We claim that each Φ_+^n is finite. By assumption, for every n there exists n' such that the set $\{\alpha_j : j \geqslant n'\}$ does not contain any element of Φ_+^n, the desired assertion follows from the claim. In order to prove the claim, we proceed by induction on n. We first need to recall a consequence of condition (2-sph).

A pair $\{\alpha, \beta\} \subset \Phi_+$ is called *fundamental* if the following conditions hold:

(FP1) The group $\langle r_\alpha, r_\beta\rangle$ is finite.
(FP2) For each $\gamma \in \Phi_+$ such that the group $\langle r_\alpha, r_\beta, r_\gamma\rangle$ is dihedral, we have $\gamma \in [\alpha, \beta]$. In other words, this means that the pair $\{\alpha, \beta\}$ is a basis of the root subsystem it generates.

We have the following:

(a) Let $\{\alpha, \beta\} \subset \Phi_+$ be a fundamental pair. Then, for all $\gamma \in]\alpha, \beta[$, we have $U_\gamma \leqslant [U_\alpha, U_\beta]$ by [1, Proposition 7].
(b) Let $\gamma \in \Phi_+$ be a root such that $d(c_+, -\gamma) > 1$. Then there exists a fundamental pair $\{\alpha, \beta\} \subset \Phi_+$ such that $\gamma \in]\alpha, \beta[$. This follows from [14, Lemma 1.7] together with the fact that (W, S) is 2-spherical.

We now prove by induction on n that Φ_+^n is finite. The set Φ_+^0 coincides with Π. Indeed, for each simple root $\alpha \in \Pi$, the group U_α fixes c_+ but acts non-trivially on the chambers adjacent to c_+. Since on the other hand, the derived group $U_+^{(1)}$ fixes the ball $B(c_+, 1)$ pointwise, we deduce that U_α is not contained in $U_+^{(1)}$, whence $q(\alpha) = 0$. Thus $\Pi \subset \Phi_+^0$. Conversely, if $\alpha \in \Phi_+$ does not belong to Π, then $d(c_+, -\alpha) > 1$ and property (a) implies that $U_\alpha \leqslant U_+^{(1)}$. Thus $q(\alpha) \geqslant 1$ and $\alpha \notin \Phi_+^0$. This shows that $\Phi_+^0 = \Pi$. In particular Φ_+^0 is finite.

Let now $n \geqslant 1$ and assume that Φ_+^k is finite for all $k < n$. We must prove that Φ_+^n is finite. Let us enumerate its elements: $\Phi_+^n = \{\gamma_1, \gamma_2, \dots\}$. Since $n \geqslant 1$ and since $\Phi_+^0 = \Pi$, we have $d(c_+, -\gamma_i) > 1$ for all $i \geqslant 1$. Hence, by property (b), for each i there is a fundamental pair $\{\alpha_i, \beta_i\}$ such that $\gamma_i \in]\alpha_i, \beta_i[$. By property (b), this implies $U_{\gamma_i} < [U_{\alpha_i}, U_{\beta_i}]$. Therefore, we have $n = q(\gamma) > \max\{q(\alpha_i), q(\beta_i)\}$. In particular:

$$\bigcup_{i>0}\{\alpha_i, \beta_i\} \subset \bigcup_{k=0}^{n-1} \Phi_+^k.$$

The set $\bigcup_{i>0} \{\alpha_i, \beta_i\}$ is thus finite. By the definition of the γ_i's, we have

$$\Phi_+^n \subset \bigcup_{i>0}]\alpha_i, \beta_i[.$$

Since each interval $]\alpha_i, \beta_i[$ is finite [62, 2.2.6], this shows that Φ_+^n is finite. □

This concludes the proof of Proposition 4.

2.2. Density of the commutator subgroup. As before, $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ is a twin root datum of type (W, S) and $(\mathscr{B}_+, \mathscr{B}_-)$ is the associated twin buildings. We assume moreover, for the rest of this section, that all root groups are finite.

Lemma 9. *Assume that property* (FPRS) *holds, that the Weyl group W is infinite and that the associated Coxeter system (W, S) is irreducible. If Λ is generated by its root subgroups, then the commutator subgroup $[\overline{\Lambda}_+, \overline{\Lambda}_+]$ is dense in $\overline{\Lambda}_+$.*

Remark. If each rank one subgroup $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$ of Λ is perfect and if Λ is generated by its root subgroups, then $[\overline{\Lambda}_+, \overline{\Lambda}_+] \supset \Lambda$ and, hence, $[\overline{\Lambda}_+, \overline{\Lambda}_+]$ is dense in $\overline{\Lambda}_+$. However, there are many examples of groups endowed with a twin root datum satisfying (FPRS) but whose rank one subgroups are not perfect, e.g. Kac–Moody groups over $\mathbf{F}_2$ or $\mathbf{F}_3$, or twin building lattices as in Sect. 1.1(II) where the rank one subgroups may be solvable.

Proof. Let $\varphi : \overline{\Lambda}_+ \to G$ be a continuous homomorphism to an abelian topological group G. Let Π be the standard root basis of Φ, where Φ is the root system of (W, S) indexing the twin root datum of Λ. For each $\alpha \in \Pi$, let $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$.

Assume by contradiction that φ is nontrivial. Since $\Lambda = \langle U_\alpha \mid \alpha \in \Phi\rangle = \langle U_\alpha \mid \pm\alpha \in \Pi\rangle$, it follows that there is some $\alpha \in \Pi$ such that $\varphi(U_\alpha)$ is nontrivial. Let $u \in U_\alpha$ be such that $\varphi(u) \neq 1$. Since W is infinite and (W, S) is irreducible, there exists $\beta \in \Phi$ such that $\alpha \cap \beta = \varnothing$ [37, Proposition 8.1 p. 309]. Let $t = r_\beta r_\alpha \in W$ and $\alpha_n = t^n(\alpha)$ for all $n \geqslant 0$. By definition, we have $\lim_{n\to+\infty} d(c_+, -\alpha_n) = +\infty$. Let $\tau \in N$ be such that $\nu(\tau) = t \in W$, where $\nu : N \to N/T = W$ is the canonical projection. For each $n \geqslant 0$, let $u_n = \tau^n.u.\tau^{-n}$. Since G is abelian, we have $\varphi(u_n) = \varphi(u) \neq 1$ for all n. On the other hand, by definition $u_n \in U_{\alpha_n}$ and, hence, $\lim_{n\to+\infty} u_n = 1$ by (FPRS). This contradicts the continuity of φ. □

The following lemma will be used again below, in order to establish restrictions on finite quotients of a group endowed with a twin root datum.

Lemma 10. *Let $(X, \{U_\alpha, U_{-\alpha}\})$ be a twin root datum of rank one. We have the following:*

(i) *The group X is not nilpotent.*
(ii) *Given a homomorphism $\varphi : X \to G$ whose kernel does not centralize $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$, we have $\varphi(U_\alpha) = \varphi(U_{-\alpha})$.*

Proof. We identify the building $\mathcal{B}$ associated with the twin root datum $(X, \{U_\alpha, U_{-\alpha}\})$ with the conjugacy class $\{gU_\alpha g^{-1}\}_{g\in X}$ of U_α in X, on which X acts by conjugation. The axioms of a root datum imply that there exists $n \in X_\alpha$ such that $nU_\alpha n^{-1} = U_{-\alpha}$ and that U_α acts simply transitively on the X-conjugates of U_α different from U_α itself; the latter conjugates can be described as the subgroups $uU_{-\alpha}u^{-1}$ for u varying in U_α. Let $g \in X$ act trivially on $\mathcal{B}$. We deduce first that g normalizes U_α and $U_{-\alpha}$ and then from the previous simple transitivity property, we deduce that g actually centralizes U_α. Therefore, up to switching α and $-\alpha$, we deduce that the kernel of the X-action on the building is the centralizer $Z_X(X_\alpha)$.

(i). This implies that the group X_α is not nilpotent: up to dividing X_α by its center, we obtain a group endowed with a twin root datum of rank one which acts faithfully on the associated building, and is therefore center-free.

(ii). Since $(X, \{U_\alpha, U_{-\alpha}\})$ is a twin root datum of rank one, the group X acts 2-transitively, hence primitively, on the conjugacy class $\mathcal{B}$ of U_α in X. The preliminary remark shows that the kernel of the X-action on $\mathcal{B}$ is $Z_X(X_\alpha)$, so from the assumption on $\mathrm{Ker}(\varphi)$ and the previous primitivity property, we deduce that $\mathrm{Ker}(\varphi)$ acts transitively on $\mathcal{B}$. In particular, this proves that $\varphi(U_\alpha) = \varphi(U_{-\alpha})$. □

2.3. Topological simplicity. The following proposition is an improvement of the topological simplicity theorem of [63] (see also [26, Theorem 3.2]). We also note that, under some additional assumptions, topological completions of Kac–Moody lattices have recently been shown to be *abstractly* simple by L. Carbone, M. Ershov and G. Ritter [26].

Proposition 11. *Let (W, S) be an irreducible Coxeter system of non-spherical type with associated root system Φ. Let $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ be a twin root datum of type (W, S) with finite root groups and let $\overline{\Lambda}_+$ be its positive topological completion. We assume that the root groups are all solvable and that $[\overline{\Lambda}_+, \overline{\Lambda}_+]$ is dense in $\overline{\Lambda}_+$. Then:*

(i) *Every closed subgroup of $\overline{\Lambda}_+$ normalized by $\Lambda^\dagger$ either contains $\Lambda^\dagger$ or centralizes $\Lambda^\dagger$. In particular, the group $\overline{\Lambda}_+^\dagger/Z(\Lambda^\dagger)$ is topologically simple.*

(ii) *Let J be an irreducible non-spherical type in S and let $\overline{G}_J$ be the closure in $\overline{\Lambda}_+$ of the group generated by the root groups indexed by the simple roots in J and their opposites. Assume that $[\overline{G}_J, \overline{G}_J]$ is dense in $\overline{G}_J$. Then any proper closed normal subgroup of $\overline{G}_J$ is contained in the center $Z(\overline{G}_J)$.*

Remark. This is the opportunity to correct a mistake in [63, Proposition 2.B.1(iv)]. The factor groups there are not topologically simple but simply have property (ii) above: their proper closed normal subgroups fix inessential buildings, but this does not seem to imply easily that the whole ambient building is fixed. This does not affect the rest of the paper. The second author thanks M. Ershov for pointing out this mistake to him.

Proof. For both (i) and (ii), the proof is an easy "topological" adaptation of the "abstract" arguments from Bourbaki. The argument is provided in detail in [25, §7.2]; here, we sketch the proof of (ii). The essential point is that the group $\overline{U}_+$, and hence also $\overline{G}_J \cap \overline{U}_+$, is pro-solvable by Proposition 3(ii). Let H be a normal subgroup of $\overline{G}_J$ not contained in the center. As the closure of a Levi subgroup, $\overline{G}_J$ has an irreducible Tits system, with Borel subgroup $\widehat{B}_+ \cap \overline{G}_J$. Therefore, by Tits' transitivity lemma [11, Lemma 2 of Sect. IV.2.7], we have: $\overline{G}_J = H.(\widehat{B}_+ \cap \overline{G}_J)$. Since $\overline{G}_J$ is topologically generated by the root groups indexed by the simple roots in J, we can even obtain $\overline{G}_J = H.(\widehat{U}_+ \cap \overline{G}_J)$. It follows that $\overline{G}_J/H \simeq (\widehat{U}_+ \cap \overline{G}_J)/(\widehat{U}_+ \cap H)$. Since $\overline{G}_J$ is assumed to be topologically perfect, so is $\overline{G}_J/H$. But $\widehat{U}_+$ is pro-solvable, hence the derived series of $\widehat{U}_+ \cap \overline{G}_J$ meets any open neighborhood of the identity in $\widehat{U}_+ \cap \overline{G}_J$. This implies that the only topologically perfect continuous quotient of $\widehat{U}_+$ is the trivial one, hence $H = \overline{G}_J$. □

3. Non-affine Coxeter groups

This section is mainly Coxeter theoretic. We prove that in any non-affine infinite Coxeter complex, given any root there exist two other roots such that any two roots in the so-obtained triple have empty intersection. Such a triple is called a *fundamental hyperbolic configuration* and used in the next section to prove strong restrictions on finite index normal subgroups for twin root data.

3.1. Parabolic closure. Let (W, S) be a Coxeter system. Given a subset R of W, we denote by $\mathrm{Pc}(R)$ the *parabolic closure* of R, namely the intersection of all parabolic subgroups of W containing R. This notion is defined in D. Krammer's Ph.D. thesis [42]. It is itself a parabolic subgroup which can be characterized geometrically as follows. Let $\mathcal{C}$ be the Coxeter complex associated with (W, S). Given $R \subset W$ and any simplex ρ of maximal dimension stabilized by $\langle R \rangle$, we have: $\mathrm{Pc}(R) = \mathrm{Stab}_W(\rho)$.

By a *Euclidean triangle group*, we mean a reflection subgroup of $\mathrm{Isom}(\mathbb{E}^2)$ which is the automorphism group of a regular tessellation of the Euclidean plane $\mathbb{E}^2$ by triangles. Recall that there are three isomorphism classes of such groups, corresponding respectively to tessellations by triangles with angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$.

Lemma 12. *Let (W, S) be a Coxeter system and let r, s be reflections in W. Assume that the product $\tau = rs$ is of infinite order. Then the following holds.*

(i) *The Coxeter diagram of* $\mathrm{Pc}(\tau)$ *is irreducible.*
(ii) *The reflections r and s belong to* $\mathrm{Pc}(\tau)$.
(iii) *Let t be a reflection which does not centralize τ and such that $\langle r, s, t \rangle$ is isomorphic to a Euclidean triangle group. Then t belongs to* $\mathrm{Pc}(\tau)$.

Proof. We first prove that (ii) implies (i). Let us assume that (ii) holds. By a suitable conjugation in W, we may – and shall – assume that $\mathrm{Pc}(\tau) = W_J$ for some subset $J \subseteq S$. Let J_r be the connected component of J such that the irreducible factor W_{J_r} contains r. If s did not belong to W_{J_r} then r and s would generate a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, contradicting that τ is of infinite order. Therefore $s \in J_r$ and by definition of the parabolic closure we have $J = J_r$.

Suppose (ii) fails. Without loss of generality, this means that $r \notin \mathrm{Pc}(\tau)$. Let $\mathcal{C}$ be the Coxeter complex associated with (W, S) and ρ be a simplex of maximal dimension which is stabilized by $\langle \tau \rangle$. Since $r \notin \mathrm{Pc}(\tau)$, it follows that ρ is contained in the interior of one of the two half-spaces determined by r. Let α be this half-space. We have $\rho \subset \alpha$, and hence $\rho \subset \bigcap_{n \in \mathbf{Z}} \tau^n.\alpha$ because ρ is τ-invariant. This is absurd since $\bigcap_{n \in \mathbf{Z}} \tau^n.\alpha$ is empty.

The proof of (iii) is similar. Let t be a reflection which does not centralize τ and such that $\langle r, s, t \rangle$ is isomorphic to a Euclidean triangle group. Let β be any of the two half-spaces associated with t. Using the fact that τ does not centralize t, it is immediate to check in the Euclidean plane that the intersection $\bigcap_{n \in \mathbf{Z}} \tau^n.\beta$ is empty. Hence the same argument as in the proof of (ii) can be applied and yields $t \in \mathrm{Pc}(\tau)$. □

We also need the following result due to D. Krammer. It is a first evidence that non-affine infinite Coxeter groups have some weak hyperbolic properties.

Proposition 13. *Let (W, S) be an irreducible, non-affine Coxeter system. Let $w \in W$ be such that $\mathrm{Pc}(w) = W$. Then the cyclic group generated by w is of finite index in its centralizer.*

Reference. This is [42, Corollary 6.3.10]. □

Remark. This result is of course false for affine Coxeter groups whose subgroup of translations is isomorphic to $\mathbf{Z}^n$ with $n \geqslant 2$, since the centralizer of any translation in such a group contains the translation subgroup.

3.2. Fundamental hyperbolic configuration. The non-linearity proof in [63, §4] makes crucial use (for a very specific case of Weyl groups) of the fundamental hyperbolic configuration defined in the introduction of this section. We prove here that the Coxeter complex of any infinite non-affine irreducible Coxeter group contains many such configurations. Note that an affine Coxeter complex does not contain any fundamental hyperbolic configuration. We do not assume the generating set S to be finite.

Theorem 14. *Let (W, S) be an irreducible non-affine and non-spherical Coxeter system and let $\mathcal{C}$ be the associated Coxeter complex. Let α, β be two disjoint non-opposite root half-spaces of $\mathcal{C}$. Then there exists a root half-space γ such that $\gamma \cap \alpha = \gamma \cap \beta = \varnothing$.*

Proof. Let us first deal with the case when S is infinite. The pair $\{r_\alpha, r_\beta\}$ is contained in a finitely generated standard parabolic subgroup of W: take explicit (minimal) writings of r_α and r_β in the generating system S; the union of all elements used in these writings defines a finite non-spherical subdiagram. Up to adding a finite number of vertices to this subdiagram, we may assume that it is irreducible and non-affine. The corresponding standard parabolic subgroup of W is finitely generated and contains r_α and r_β.

We henceforth assume that the generating system S is finite and denote by $|S|$ its cardinality. We prove the assertion by induction on $|S|$. The roots α and β being non-opposite, the corresponding reflections r_α and r_β generate an infinite dihedral subgroup in W. This excludes $|S| = 1$ and $|S| = 2$, except possibly when the two vertices are connected by an edge labelled by ∞. But since the latter diagram is affine, the induction starts at $|S| = 3$.

Assume first that $|S| = 3$, i.e. that the Coxeter diagram of (W, S) is a triangle. Denoting by a, b and c the labels of its edges, we have $a, b, c \geqslant 3$, and also $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ because (W, S) is non-affine. Let $\mathbb{H}^2$ denote the hyperbolic plane and let $\mathcal{T}$ be a geodesic triangle in $\mathbb{H}^2$ of angles $\frac{\pi}{a}$, $\frac{\pi}{b}$ and $\frac{\pi}{c}$ (an angle equal to 0 correspond to a vertex in the boundary of $\mathbb{H}^2$). It follows from Poincaré's polyhedron theorem that the reflection group generated by $\mathcal{T}$ is isomorphic to W and that the so-obtained hyperbolic tiling is a geometric realization of the Coxeter complex of (W, S) [47, Sect. IV.H]. Thanks to this geometric realization, the result is then clear when $|S| = 3$.

Assume now that $|S| > 3$ and that the result is proved for any Coxeter system as in the theorem and whose canonical set of generators has less than $|S|$ elements. Denote by τ the infinite order element $r_\alpha r_\beta$. Using a suitable conjugation, we may – and shall – assume that $\mathrm{Pc}(\tau)$ is standard parabolic, i.e. $\mathrm{Pc}(\tau) = W_J$ for some $J \subseteq S$. According to Lemma 12, the Coxeter system (W_J, J) is irreducible by (i) and we have $r_\alpha, r_\beta \in W_J$ by (ii). Then two cases occur.

The first case is when (W_J, J) is non-affine. By the induction hypothesis, we only have to deal with the case $J = S$ and $W_J = W$. If all canonical generators in S centralized τ, then we would have $W = Z_W(\tau)$; but τ cannot be central in W since $\tau = r_\alpha r_\beta$ does not centralize r_α and r_β. Therefore there exists a reflection $t \in S$ such that t does not centralize τ. Let T be the subgroup generated by t, r_α and r_β. If T were isomorphic to a Euclidean triangle group, then $Z_T(\tau)$ would contain a free abelian group of rank 2. This is impossible by Proposition 13. Therefore, T is isomorphic to a hyperbolic triangle group and we can conclude as in the case $|S| = 3$.

The remaining case is when (W_J, J) is affine. Then J is properly contained in S because W is non-affine and there exists an element $s \in S \setminus J$ which does not normalize W_J. In particular s does not centralize τ because $\mathrm{Pc}(\tau) = W_J$. Let T' be the subgroup generated by s, r_α and r_β. If T' is isomorphic to a Euclidean triangle group, then Lemma 12 (iii) implies that $s \in W_J$, which is excluded. Thus T' is isomorphic to a hyperbolic triangle group and we are again reduced to the case $|S| = 3$. $\square$

4. Simplicity of twin building lattices

As mentioned in the introduction, the proof of the main simplicity theorem applies to the general setting of twin building lattices: the only required assumption is that root groups are nilpotent (see Theorem 19). The proof splits into two parts, each of which is presented in a separate subsection below. These two parts have each their own specific hypotheses and are each of independent interest.

4.1. Finite quotients of groups with a twin root datum. Here, we prove strong restrictions on finite index normal subgroups of a group endowed with a twin root datum, under the assumption that root groups are nilpotent. These conditions are fulfilled by Kac–Moody groups over arbitrary fields since their Levi factors are abstractly isomorphic to reductive algebraic groups.

Theorem 15. *Let (W, S) be a Coxeter system with associated root system Φ and let Π be the root basis associated to S. Let G be a group endowed with a twin root datum $\{U_\alpha\}_{\alpha\in\Phi}$ indexed by Φ. Suppose that:*

(1) *The Coxeter system (W, S) is irreducible, non-spherical and non-affine;*
(2) *For any $\alpha \in \Pi$, the root group U_α is nilpotent.*

Let H be a normal subgroup of G such that $N/T.(N \cap H)$ is finite. Let $G^\dagger$ be the subgroup of G generated by the root groups, let $H^\dagger = H \cap G^\dagger$, let $\pi : G^\dagger \to G^\dagger/H^\dagger$ be the canonical projection and for each $\alpha \in \Pi$, denote by f_α the inclusion $U_\alpha \to G^\dagger$. Then the composed map:

$$\prod_{\alpha\in\Pi} U_\alpha \xrightarrow[\prod f_\alpha]{} G^\dagger \xrightarrow[\pi]{} G^\dagger/H^\dagger$$

is a surjective homomorphism. In particular, the group $G^\dagger/H^\dagger$ is nilpotent.

Remark. The finiteness of $N/T.(N \cap H)$ is automatically satisfied when H has finite index in G.

Proof. We identify the elements of Φ with the half-spaces of the Davis complex $\mathcal{A}$ associated with (W, S). We set $h = [N : T.(N \cap H)]$.

Let α be an arbitrary root. By [37, Proposition 8.1, p. 309] there is a root η such that $\alpha \cap \eta = \varnothing$. The product $\tau = r_\eta r_\alpha$ has infinite order. We set $\beta = \tau^h.(-\alpha) \in \Phi$. We have $\beta \subset \eta$ and, hence, the roots α and β are disjoint (see Fig. 1). By Theorem 14, there exists a root $\xi \in \Phi$ such that $\alpha \cap \xi = \eta \cap \xi = \varnothing$. In particular $\beta \cap \xi = \varnothing$. Again the product $\tau' = r_\xi r_\beta$ has infinite order. We set $\gamma = (\tau')^h.(-\beta)$.

By construction, we have $\gamma \subset \xi$ (see Fig. 1). Hence the roots α, β and γ are pairwise disjoint. Therefore it follows from Assumption (2) and Proposition 3(iii) that the group $U' = \langle U_\alpha \cup U_{-\gamma}\rangle$ is nilpotent, and so is its image $\pi(U')$. But by (TRD2) we have

$$U_\beta = \tau^h U_{-\alpha} \tau^{-h} \quad \text{and} \quad U_{-\gamma} = (\tau')^h U_\beta (\tau')^{-h}.$$

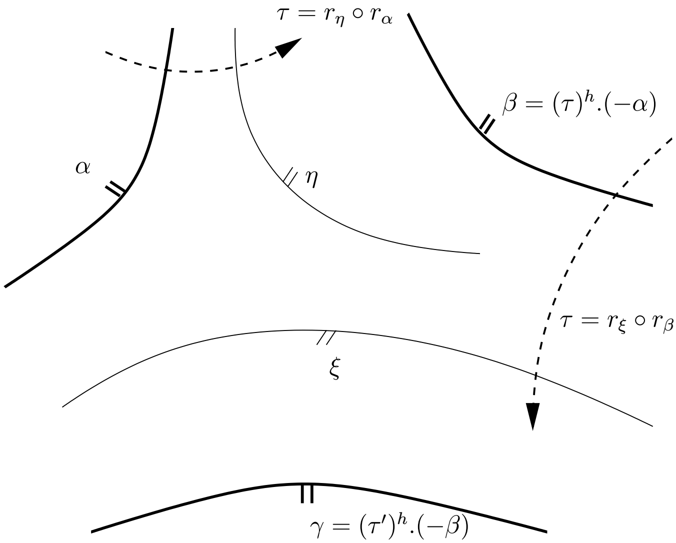

Fig. 1 Proof of Theorem 15

Note that $N/T.(N \cap H)$ is the quotient of the Weyl group $W = N/T$ induced by $H \lhd G$. Since h is the order of the quotient $N/T.(N \cap H)$, applying π provides $\pi(U_{-\alpha}) = \pi(U_\beta) = \pi(U_{-\gamma})$, which implies

$$\pi(U') = \langle \pi(U_\alpha) \cup \pi(U_{-\alpha}) \rangle = \pi(X_\alpha).$$

This shows that $\pi(X_\alpha)$ is a nilpotent group.

Note that $(X_\alpha, \{U_\alpha, U_{-\alpha}\})$ is a twin root datum of rank one. By Lemma 10(i), the group X_α is not nilpotent and, hence, $X_\alpha \cap H = \mathrm{Ker}(\pi|_{X_\alpha})$ is not central in X_α. Therefore, we have $\pi(U_\alpha) = \pi(U_{-\alpha})$ by Lemma 10(ii).

Finally, for any two distinct roots $\alpha, \beta \in \Pi$, we have $[U_\alpha, U_{-\beta}] = 1$ by axiom (TRD1). In view of the preceding paragraph, this implies that $[\pi(U_\alpha), \pi(U_\beta)] = 1$ for all distinct $\alpha, \beta \in \Pi$. The desired result follows by noticing that $G^\dagger$ is generated by $\bigcup_{\pm\alpha \in \Pi} U_\alpha$. This is easily seen using axiom (TRD2) of twin root data to produce elements in N and then to conjugate the simple root groups by these elements to produce any desired root group. □

The following corollary applies to all split and almost split Kac–Moody groups over finite fields.

Corollary 16. *Let G be a group as in Theorem 15, maintain the assumptions* (1) *and* (2) *and assume moreover that root groups are finite. Here we let $H^\dagger$ denote the intersection of all finite index normal subgroups of $G^\dagger$. Then*

$$[G^\dagger : H^\dagger] \leqslant \prod_{\alpha \in \Pi} |U_\alpha|.$$

Furthermore, we have $H^\dagger = G^\dagger$ *whenever one of the following holds:*

(3) *Each group* X_α, $\alpha \in \Pi$, *is a finite group of Lie type and the minimal order* $q_{\min} = \min\{|U_\alpha| : \alpha \in \Pi\} > 3$*;*
(4) *The Coxeter system* (W, S) *is* 2*-spherical, i.e. every* 2*-subset of* S *generates a finite group, and* $q_{\min} > 2$*;*
(5) *The Coxeter system* (W, S) *is simply laced, i.e. every* 2*-subset of* S *generates a group of order* 4 *or* 6*.*

Remark. The above group $H^\dagger$ is contained in any finite index subgroup of G.

Proof. Let H be a finite index normal subgroup in $G^\dagger$. Applying Theorem 15 to $G^\dagger$ we see that the index $[G^\dagger : H]$ is uniformly bounded, so that finite index subgroups of $G^\dagger$ are finite in number. This implies that the intersection defining $H^\dagger$ is finite, so that $H^\dagger$ is itself a finite index subgroup. It remains to apply again Theorem 15 to obtain the desired bound on $[G^\dagger : H^\dagger]$.

If condition (3) holds, then each X_α is perfect (in fact: simple modulo center), so admits no non-trivial nilpotent quotient. We combine this remark with Theorem 15 applied to $G^\dagger$: this shows that the image of each subgroup X_α is trivial in $G^\dagger/H^\dagger$. Since $G^\dagger$ is generated by the subgroups X_α, this implies the equality $H^\dagger = G^\dagger$. Similarly, if (4) or (5) holds then for each $\alpha \in \Pi$ there exists $\beta \in \Pi - \{\alpha\}$ such that $X_{\alpha,\beta} = \langle X_\alpha, X_\beta \rangle$ is a rank 2 finite group of Lie type. All such groups are perfect except $B_2(2)$ and $G_2(2)$ (which contain both a simple subgroup of index 2). Since (4) implies $q_{\min} > 2$ and (5) implies that $X_{\alpha,\beta}$ is of type A_2, the group $X_{\alpha,\beta}$ is isomorphic to neither of the latter groups and we have again $H^\dagger = G^\dagger$. □

Theorem 15 and its corollary imply that any split Kac–Moody group over a finite field of irreducible non-spherical and non-affine type, admits at most a finite number of finite quotients, which are necessarily abelian since so are root groups in the split case. Then $\Lambda/\Lambda^\dagger$ is a quotient of a finite split torus, that is a quotient of finitely many copies of the multiplicative group of the finite ground field.

Furthermore, if the ground field is of cardinality at least 4 and if the group is generated by its root groups, e.g. because it is simply connected, then all finite quotients are trivial.

We close this subsection with an example of a Kac–Moody group which admits nontrivial finite quotients when the ground field is $\mathbf{F}_2$ or $\mathbf{F}_3$. We set $I = \{1, 2, 3\}$ and consider the generalized Cartan matrix

$$A = (A_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

Let $\mathcal{G}_A$ be the simply connected Tits functor of type A [71, 3.7.c]. We set $\Lambda = \mathcal{G}_A(\mathbf{F}_2)$. For each $i \in I$, we let $\varphi_i : \mathrm{SL}_2(\mathbf{F}_2) \to \Lambda$ be the standard homomorphism [71, §2 and Sect. 3.9] and let $f_i : \mathrm{SL}_2(\mathbf{F}_2) \to \mathbf{F}_2$ be the surjective homomorphism defined by $f_i\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f_i\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1$. Using the defining relations of Λ [62, §8.3], we see that there is a unique homomorphism $f : \Lambda \to \prod_{i\in I} \mathbf{F}_2$ such that $f \circ (\prod_{i\in I} \varphi_i) = \prod_{i\in I} f_i$. By the definition of f_i, the homomorphism f is surjective.

4.2. Non-arithmeticity. For the next statement we recall that for a group inclusion $A < B$, the *commensurator* of A in B, denoted $\mathrm{Comm}_B(A)$, consists of the elements $b \in B$ such that A and bAb^{-1} share a finite index subgroup. According to a well-known theorem of G. Margulis, a lattice in a semisimple Lie group is arithmetic if and only if its commensurator is dense in the ambient Lie group [77, Th. 6.2.5].

Corollary 17. *Let Λ be a group as in Theorem 15. Suppose moreover that assumption (3) of Corollary 16 holds and let $\overline{\Lambda}_+$ (resp. $\overline{\Lambda}_-$) be the positive (resp. negative) topological completion of Λ. Then the commensurator* $\mathrm{Comm}_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger)$ *is a discrete subgroup of $\overline{\Lambda}_+ \times \overline{\Lambda}_-$.*

Proof. Recall that $\Lambda^\dagger$ is viewed here as a diagonal subgroup of $\overline{\Lambda}_+ \times \overline{\Lambda}_-$. By Corollary 16, the commensurator $\mathrm{Comm}_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger)$ is equal to the normalizer $N_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger)$ because any finite index subgroup of a given group contains a finite index normal subgroup. Furthermore, the centralizer $Z_{\overline{\Lambda}_+}(\Lambda^\dagger)$ (resp. $Z_{\overline{\Lambda}_-}(\Lambda^\dagger)$) is nothing but the kernel of the $\overline{\Lambda}_+$-action (resp. $\overline{\Lambda}_-$-action) on the positive (resp. negative) building associated with Λ. By Proposition 1(iii), we have $Z_{\overline{\Lambda}_+}(\Lambda^\dagger) = Z_\Lambda(\Lambda^\dagger)$ (resp. $Z_{\overline{\Lambda}_-}(\Lambda^\dagger) = Z_\Lambda(\Lambda^\dagger)$). Therefore, we have an exact sequence:

$$1 \longrightarrow Z_\Lambda(\Lambda^\dagger) \times Z_\Lambda(\Lambda^\dagger) \longrightarrow N_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger) \longrightarrow \mathrm{Aut}(\Lambda^\dagger).$$

This yields an exact sequence

$$1 \longrightarrow \big(Z_\Lambda(\Lambda^\dagger) \times Z_\Lambda(\Lambda^\dagger)\big).\Lambda^\dagger \longrightarrow N_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger) \longrightarrow \mathrm{Out}(\Lambda^\dagger),$$

where $\mathrm{Out}(\Lambda^\dagger) = \mathrm{Aut}(\Lambda^\dagger)/\mathrm{Inn}(\Lambda^\dagger)$ is the outer automorphism group. By [23, Corollary B], the group $\mathrm{Out}(\Lambda^\dagger)$ is finite. Since $(Z_\Lambda(\Lambda^\dagger) \times Z_\Lambda(\Lambda^\dagger)).\Lambda^\dagger$ is a discrete subgroup of $\overline{\Lambda}_+ \times \overline{\Lambda}_-$ by Propositions 1(iii) and 2, it finally follows that $\mathrm{Comm}_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger) = N_{\overline{\Lambda}_+\times\overline{\Lambda}_-}(\Lambda^\dagger)$ is discrete as well. $\square$

Remark. In fact, the non-existence of any proper finite index subgroup for Λ implies that its group of *abstract* commensurators coincides with $\mathrm{Aut}(\Lambda)$, which by [23, Corollary B] is a finite extension of Λ.

4.3. Normal subgroup property. In view of Corollary 16, the complementary property necessary in order to obtain simplicity of the group $H^\dagger$

(modulo center) is that any non-central normal subgroup has finite index. This is called the *normal subgroup property* and is well-known for irreducible higher rank lattices in Lie groups [45, Th. IV.4.9]. The generalization to irreducible cocompact lattices in products of topological groups follows from work by U. Bader and Y. Shalom, following Margulis' general strategy (see [67] and mostly [5, Introduction] for an explanation of the substantial differences with the classical case). In the attempt of adapting these results to Kac–Moody groups over finite fields, one has to overcome the fact that Kac–Moody lattices are never cocompact. This was done in [64] by proving that one can find a fundamental domain D for Λ in $\overline{\Lambda}_+ \times \overline{\Lambda}_-$ with respect to which the associated cocycle is square integrable.

The following result is a restatement of the normal subgroup theorem proved in [5] and [64], in the general framework of twin building lattices.

Theorem 18. *Let* $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ *be a twin root datum of type* (W, S). *We assume that:*

(NSP1) *Each root group* U_α *is finite.*
(NSP2) *With the notation of Sect. 1.3, the series* $W(1/q_{\min})$ *converges.*

Then any subgroup of Λ *which is normalized by* $\Lambda^\dagger$, *either centralizes* $\Lambda^\dagger$ *or contains a finite index subgroup of* $\Lambda^\dagger$.

Remarks. 1. There is no condition excluding affine diagrams. Indeed, Kac–Moody groups of affine type are $\{0, \infty\}$-arithmetic groups and as such are irreducible lattices in higher-rank algebraic groups: this case was already covered by Margulis' theorem.

2. As pointed out to us by M. Burger, an infinite group with the normal subgroup property cannot be hyperbolic since it is incompatible with SQ-universality, the property that any countable group embeds in a suitable quotient of the group under consideration. Any non-elementary hyperbolic group is SQ-universal [29,59]. The fact that no Kac–Moody group can be hyperbolic can also be derived from the specific property that Kac–Moody groups over finite fields contain infinitely many conjugacy classes of finite subgroups.

Proof. Note that a subgroup of Λ (resp. $\overline{\Lambda}_\pm$) centralizes $\Lambda^\dagger$ if and only if it acts trivially on the building $\mathscr{B}_\pm$. Without loss of generality, we may – and shall – assume that $Z(\Lambda^\dagger)$ is trivial. Hence Λ and $\overline{\Lambda}_+$ act faithfully on the building $\mathscr{B}_+$. Let H be a nontrivial normal subgroup of Λ and set $H^\dagger = H \cap \Lambda^\dagger$. We must show that the index of $H^\dagger$ in $\Lambda^\dagger$ is finite. To this end, we apply the main results of [5]. This requires to ensure that two conditions are fulfilled. The first condition is that the closure of $H^\dagger$ in $\overline{\Lambda}^\dagger_\pm$ is cocompact. Since $H^\dagger$ is normal in $\Lambda^\dagger$, it follows from Tits' transitivity lemma (see [11, Ch. IV, §2, Lemma 2]) that $H^\dagger$ is transitive on the chambers of both $\mathscr{B}_+$ and $\mathscr{B}_-$. Since the chamber-stabilizers are compact-open subgroups of $\overline{\Lambda}_\pm$, the desired cocompactness condition clearly holds.

The second condition is the existence of a fundamental domain D for Λ with respect to which the associated cocycle is square integrable; this is provided by the same arguments as in [64]. We do not go into details here because this question is more carefully examined Subsect. 7.2, where we prove a refinement of the square integrability. We merely remark that the group combinatorics needed to prove the existence of D, namely the structure of refined Tits system defined in [41], is available for arbitrary twin root data, and not only for those arising from Kac–Moody groups, see Proposition 1(vi). □

4.4. Simplicity of lattices. We can now put together the two ingredients needed to prove the simplicity theorem for twin building lattices.

Theorem 19. *Let $(\Lambda, \{U_\alpha\}_{\alpha\in\Phi})$ be a twin root datum of type (W, S). Let $\Lambda^\dagger$ be the subgroup generated by all root groups and assume that:*

- **(S0)** *The Coxeter system (W, S) is irreducible, non-spherical and non-affine.*
- **(S1)** *Each root group U_α is finite and nilpotent.*
- **(S2)** *With the notation of Sect. 1.3, the series $W(1/q_{\min})$ converges.*

Then the quotient $\Lambda^\dagger/Z(\Lambda^\dagger)$ is infinite virtually simple and all of its finite quotients are nilpotent.

Assume moreover that:

- **(S3)** *Each rank one group $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$ is perfect.*

Then any subgroup of Λ, normalized by $\Lambda^\dagger$, either centralizes $\Lambda^\dagger$ or contains $\Lambda^\dagger$.

Remark. This theorem applies to the groups of mixed characteristics defined in [65] provided the minimal size of the finite ground fields is large enough with respect to the growth of the (right-angled) Weyl group. In this case the lattices are by definition generated by their root groups and condition (S3) is fulfilled. One can push a little further this construction by replacing the rank 1 Levi factors, isomorphic to some suitable $SL_2(q)$'s, by affine groups. In this case, the root groups are isomorphic to multiplicative groups of finite fields, so the thicknesses are prime powers, and rank 1 subgroups are solvable.

Proof. Let $H^\dagger$ be the intersection of all finite index normal subgroups of $\Lambda^\dagger$. The center of $H^\dagger$ is a normal subgroup of $\Lambda^\dagger$, which must be central in $\Lambda^\dagger$ in view of Theorem 18. In particular the canonical projection of $H^\dagger$ in $\Lambda^\dagger/Z(\Lambda^\dagger)$ is isomorphic to $H^\dagger/Z(H^\dagger)$ and coincides with the intersection of all finite index subgroups of $\Lambda^\dagger/Z(\Lambda^\dagger)$.

By Corollary 16, the index of $H^\dagger$ in $\Lambda^\dagger$ is finite. On the other hand, it follows from Theorem 18 that $\Lambda^\dagger/Z(\Lambda^\dagger)$ is just infinite (i.e. every non-trivial quotient is finite). Therefore, it follows from [76, Proposition 1] that

$H^\dagger/Z(H^\dagger)$ is a direct product of finitely many isomorphic simple groups. Write $H^\dagger/Z(H^\dagger) = H_1 \times \cdots \times H_k$. We must prove that $k = 1$.

Notice that $H^\dagger$, viewed as a diagonally embedded subgroup, is a lattice in $\overline{\Lambda}_+^\dagger \times \overline{\Lambda}_-^\dagger$, because it is a finite index subgroup of the lattice $\Lambda^\dagger$. Furthermore, $H^\dagger$ is irreducible. Indeed, since $H^\dagger$ is a finite index normal subgroup of $\Lambda^\dagger$, its closure $\overline{H}^\dagger$ in $\overline{\Lambda}_+^\dagger$ is a non-central closed normal subgroup, which must coincide with $\overline{\Lambda}_+^\dagger$ by Proposition 11(i).

Assume now that $k > 1$. It follows that the simple group H_1 is a quotient of $H^\dagger$ which is not co-central, since we have a composed map

$$H^\dagger \to H^\dagger/Z(H^\dagger) = H_1 \times \cdots \times H_k \to H_1.$$

The closure of the projection of the corresponding normal subgroup of $H^\dagger$ in $\overline{\Lambda}_+^\dagger$ is thus a non-central closed normal subgroup of $\overline{\Lambda}_+^\dagger$. Hence it coincides with $\overline{\Lambda}_+^\dagger$ by Proposition 11(i). By [5, Theorem 1.3], this implies that H_1 is amenable. Since the H_i's are all isomorphic, it follows that $H^\dagger/Z(H^\dagger)$ is amenable, and so is $\Lambda^\dagger$ since $Z(H^\dagger)$ and $[\Lambda^\dagger : H^\dagger]$ are finite. Recall that $\Lambda^\dagger$ acts on the associated positive building $\mathcal{B}_+$, which may be viewed as a proper CAT(0)-space. Amenability of $\Lambda^\dagger$ implies that its action on $\mathcal{B}_+$ stabilizes a Euclidean flat or fixes a point in the visual boundary at infinity [4]. Both eventualities are absurd. This shows that $k = 1$ as desired.

Assume now that (S3) also holds. Note that a subgroup of Λ (resp. $\overline{\Lambda}_+$) centralizes $\Lambda^\dagger$ if and only if it acts trivially on the building $\mathcal{B}_+$. Hence, in view of what has already been proven, it suffices to show that $H^\dagger = \Lambda^\dagger$. This follows from Corollary 16. □

Here is now the Kac–Moody specialization of this theorem:

Theorem 20. *Let Λ be a split or almost split Kac–Moody group over a finite field $\mathbf{F}_q$ of order q. Let us denote by (W, S) the natural Coxeter system of the Weyl group W and by $W(t)$ the growth series of W with respect to S. Assume that (W, S) is irreducible, neither of spherical nor of affine type and that $W(\frac{1}{q}) < +\infty$. Then the derived group of Λ, divided by its center, is simple.*

Proof. All root groups of Λ are nilpotent (of class at most 2). Thus conditions (S0), (S1) and (S2) are clearly satisfied. In order to deduce the desired statement from Theorem 19 and its proof, it remains to show that the derived group $[\Lambda, \Lambda]$ coincides with the intersection $H^\dagger$ of all finite index subgroups of $\Lambda^\dagger$.

Each rank one subgroup $X_\alpha = \langle U_\alpha \cup U_{-\alpha} \rangle$ is isomorphic to the $\mathbf{F}_q$-points of a simple algebraic group of relative rank one. Therefore, the group X_α is perfect except if U_α is of order 2 or 3 in which case it is abelian. In view of Theorem 15, this implies in particular that the quotient $\Lambda/H^\dagger$ is abelian. Thus $[\Lambda, \Lambda] \subset H^\dagger$. It follows from the proof of Theorem 19 that the latter inclusion cannot be proper, as desired. □

Note that if $q > 3$ then every rank one subgroup of the Kac–Moody group Λ is perfect and, hence, condition (S3) holds. In that case, we have $[\Lambda, \Lambda] = \Lambda^\dagger$.

Kac–Moody groups are the values over fields of group functors defined by J. Tits thanks to combinatorial data called *Kac–Moody root data* [71]. The main information in a Kac–Moody root datum is given by a generalized Cartan matrix, say A. Once A is fixed we can still make some choices in order for Λ to be generated by its root groups. In this case, e.g. when we choose the *simply connected Kac–Moody root datum* [71, Sect. 3.7.c], we have $\Lambda = \Lambda^\dagger = [\Lambda, \Lambda]$ (if $q > 3$) and we recover the situation described in the comment to the simplicity theorem (Introduction).

Let us now state a corollary on property (T). Its proof is a straightforward combination of work by J. Dymara and T. Januszkiewicz and by P. Abramenko and B. Mühlherr, but the corollary has interesting rigidity consequences (Theorem 34).

Corollary 21. *Let Λ be a group endowed with a twin root datum satisfying* (S0)*,* (S1)*,* (S2) *and* (S3) *of Theorem 19. We assume furthermore that any two canonical reflections in S generate a finite subgroup of W. If $q_{\min} > \frac{1764^n}{25}$, then Λ has Kazhdan's property* (T)*. In particular there exist infinitely many isomorphism classes of finitely presented infinite simple groups with Kazhdan's property* (T).

Proof. This is a straightforward application of [30, Theorem E], which provides the vanishing of the first cohomology useful to a well-known criterion for property (T) [36, Chapitre 4]. Finite presentation follows from [3] under the hypothesis that $q_{\min} > 3$. Finally, it follows from [23] that Kac–Moody groups over non isomorphic finite fields (or of different types) are not isomorphic. □

Concretely, in order to produce infinite simple Kazhdan groups, it is enough to pick a generalized Cartan matrix $A = [A_{s,t}]_{s,t \in S}$ such that $A_{s,t}A_{t,s} \leqslant 3$ for each $s \neq t$ and a finite ground field, whose order is at least the size of A. The above simple groups seem to be the first examples of infinite finitely presented simple groups enjoying property (T). The simple lattices in products of trees constructed by M. Burger and Sh. Mozes [18] are finitely presented but they cannot have property (T) since they act fixed-point-freely on trees. However these lattices are torsion free, while a Kac–Moody group over a finite field of characteristic p contains infinite abelian subgroups of exponent p [61, proof of Theorem 4.6]. Note that finitely generated infinite simple Kazhdan groups were constructed by M. Gromov [33, Corollary 5.5.E] as quotients of hyperbolic groups with property (T).

4.5. Application to the word problem. As mentioned in the introduction, combining Theorem 20 with the theorem of W. Boone and G. Higman [8], one deduces that large classes of finitely generated Kac–Moody groups have solvable word problem. It is however a delicate problem to determine exactly which Kac–Moody groups can be embedded in a finitely presented simple Kac–Moody group. Here, we limit ourselves to recording the following statement:

Corollary 22. *Let Λ be a split or almost split Kac–Moody group over an arbitrary finite field $\mathbf{F}_q$ of order q. Let us denote by (W, S) the natural Coxeter system of the Weyl group W. If (W, S) is 2-spherical, then Λ has solvable word problem.*

Proof. In view of Theorem 20, [8] and [3], if a group Γ can be embedded in a split adjoint Kac–Moody group of irreducible 2-spherical non-affine type over a sufficiently large finite field, then Γ has solvable word problem. Therefore, the desired result follows from the following three observations:

Observation 1: *Any finitely generated almost split Kac–Moody group of 2-spherical type embeds in a finitely generated split Kac–Moody group of 2-spherical type.*

By definition, an almost split Kac–Moody group Λ embeds in a split group $\overline{\Lambda}$. If Λ is finitely generated, then we may assume that $\overline{\Lambda}$ is defined over a finitely generated field and, hence, is itself finitely generated. Let $\bar{W}$ be the Weyl group of $\overline{\Lambda}$; thus the Weyl group W of Λ embeds in $\bar{W}$. If now W is 2-spherical, then it has Serre's property (FA) and it is easy to deduce that its parabolic closure in $\bar{W}$ is itself 2-spherical. This implies that Λ is conjugate to a Levi subgroup of $\overline{\Lambda}$ which is of 2-spherical type.

Observation 2: *Any finitely generated split Kac–Moody group of 2-spherical type embeds (possibly modulo a finite normal subgroup) in a finitely generated split adjoint Kac–Moody group of irreducible 2-spherical non-affine type.*

Clearly, any generalized Cartan matrix of 2-spherical type can be embedded as a top-left submatrix of a generalized Cartan matrix of irreducible 2-spherical non-affine type. The claim follows since an embedding of Cartan matrices induces an embedding of Kac–Moody groups (possibly modulo a finite normal subgroup), the smaller one as a Levi subgroup of the bigger one.

Observation 3: *A split Kac–Moody group over a given finite field embeds in the group of rational points over any extension of that field.*

Immediate by functoriality. □

5. Non-linearity of Kac–Moody groups

A result of Mal'cev's asserts that any finitely generated linear group is residually finite. In particular, the groups covered by Corollary 16 are not linear over any field. Note that with the notation and assumptions of this corollary, the group $G^\dagger$ is finitely generated. In this section, we show that the latter corollary actually implies a strong non-linearity statement for Kac–Moody groups over arbitrary fields of positive characteristic.

5.1. Normal subgroups (arbitrary ground field). The proof of the non-linearity theorem below (Theorem 25) requires the following statement, which is a complement to Theorem 15. The reader familiar with infinite-dimensional Lie algebras will recognize some similarity with [39, Proposition 1.7].

Proposition 23. *Let A be a generalized Cartan matrix which is indecomposable and non-affine, let $\mathcal{G}_A$ be a Tits functor of type A and let $\mathbf{K}$ be an infinite field. We set $G = \mathcal{G}_A(\mathbf{K})$ and $G^\dagger = \langle U_\alpha : \alpha \in \Phi \rangle$, where $\{U_\alpha\}_{\alpha\in\Phi}$ is the twin root datum given by the root groups. Then given any normal subgroup H of G, either H contains $G^\dagger$ or $H \cap U_\Psi = \{1\}$ for each nilpotent set of roots Ψ.*

Proof. Let $H \lhd G$ be such that $H \cap U_\Psi \neq \{1\}$ for some nilpotent set of roots Ψ. We must prove that $H \supset G^\dagger$. The set Ψ is finite [62, Cor. 2.2.6]; we assume that it is of minimal cardinality with respect to the property that $H \cap U_\Psi \neq \{1\}$ and we set $n = |\Psi|$.

Suppose that $n > 1$. The elements of Ψ can be ordered in a nibbling sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ [loc. cit., 1.4.1]. Now let $g \in H \cap U_\Psi - \{1\}$. The group U_Ψ decomposes as a product $U_\Psi = U_{\alpha_1} U_{\alpha_2} \dots U_{\alpha_n}$ [loc. cit., 1.5.2], so we have $g = u_1 u_2 \dots u_n$ with $u_i \in U_{\alpha_i}$ for each $i = 1, \dots, n$. By the minimality assumption on Ψ, the elements u_1 and u_n must be nontrivial. We set $j = \min\{i > 1 : u_i \neq 1\}$. By Lemma 24 below, we can pick some $h \in Z_T(U_{\alpha_1})$ not centralizing U_{α_j}. By the defining relations of $\mathcal{G}_A$, we see in a suitable parametrization of U_{α_j} by the additive group $(\mathbf{K}, +)$ that the action of h on U_{α_j} by conjugation is merely a multiplication by an element of $\mathbf{K}^\times$. Therefore h centralizes no nontrivial element of U_{α_j} and we obtain successively:

$$\begin{aligned} g^{-1}h^{-1}gh &= u_n^{-1} \dots u_j^{-1} u_1^{-1} u_1^h u_j^h \dots u_n^h \\ &= u_n^{-1} \dots u_j^{-1} u_1^{-1} u_1 u_j^h \dots u_n^h \\ &= u_n^{-1} \dots u_j^{-1} u_j^h \dots u_n^h \\ &= u_j^{-1} u_j^h u'_{j+1} \dots u'_n \end{aligned}$$

for some $u'_i \in U_{\alpha_i}$ $(i = j+1, \dots, n)$ and where the last equality follows from the commutation relations satisfied by the U_α's in view of (TRD 1). Since T normalizes H, we have $g^{-1}h^{-1}gh \in H$. Moreover the definition of j and the choice of h imply that $u_j^{-1}u_j^h \neq 1$ so in particular $g^{-1}h^{-1}gh \neq 1$. This shows that $H \cap U_{\Psi - \{\alpha_1\}} \neq \{1\}$, which contradicts the minimality of $|\Psi|$. Thus $n = 1$.

The group $X_\alpha = \langle U_\alpha \cup U_{-\alpha} \rangle$ is quasi-simple. More precisely, any proper normal subgroup is contained in the center, which intersects U_α trivially. Therefore, since $H \cap U_\alpha$ is non-trivial, we deduce that $H \cap X_\alpha$ coincides with X_α. Let Π be a basis of Φ containing α and let $\beta \in \Pi - \{\alpha\}$ be such that the associated reflections r_α and r_β do not commute. Since $\mathbf{K}$ is infinite, it follows that $T \cap X_\alpha \not\subset Z_T(U_\beta)$. In particular, there exists $h' \in T \cap H$ and $u \in U_\beta$ such that $h'u(h')^{-1}u^{-1} \neq 1$. Thus $H \cap U_\beta$ is nontrivial and, as above, this implies that H contains $U_\beta \cup U_{-\beta}$. Finally, since A is indecomposable we obtain $U_\gamma < H$ for any $\gamma \in \Phi$, that is to say $H \supset G^\dagger$. □

Let us now immediately prove the lemma we used in the previous proof.

Lemma 24. *Maintain the notation and assumptions of Proposition 23 and set $T = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$. Then, for any positive roots $\alpha, \beta \in \Phi^+$, the inclusion $Z_T(U_\alpha) \subset Z_T(U_\beta)$ implies $\alpha = \beta$.*

Proof. Let X^* (resp. X_*) be the lattice of algebraic characters (resp. cocharacters) of the maximal split torus T [62, §8.4.3]. We may – and shall – identify the abstract root system Φ with a subset of X^* and use the identification $T \simeq \mathrm{Hom}_{\mathrm{groups}}(X^*, \mathbf{K}^\times)$. For $\alpha \in \Phi$, $x \in \mathbf{K}$ and $t \in T$, we have:

$$(**) \qquad t.\mathfrak{u}_\alpha(x).t^{-1} = \mathfrak{u}_\alpha(t(\alpha).x),$$

where $\mathfrak{u}_\alpha : (\mathbf{K}, +) \to U_\alpha$ is a standard isomorphism (see [71, §3]).

Assume now that $\alpha \neq \beta$. We claim that there exists $\gamma \in X_*$ such that $\langle \alpha \mid \gamma \rangle = 0$ and $\langle \beta \mid \gamma \rangle \neq 0$. Let us set $k(\alpha, \beta) = \langle \alpha \mid \beta^\vee \rangle \langle \beta \mid \alpha^\vee \rangle$. If $k \neq 4$, then it is easy to see that there exists such a γ in the group $\mathbf{Z}\alpha^\vee + \mathbf{Z}\beta^\vee$. If $k = 4$, then the order of $s_\alpha s_\beta$ is infinite and we are in the position to apply Theorem 14. This yields a non-degenerate infinite rank 3 root subsystem of Φ containing α and β. Then it is again easy to check the existence of γ inside this subsystem. In both cases, the claim above holds. Given $t \in \mathbf{K}^\times$, let t^γ denote the element of T defined by $t^\gamma : \lambda \mapsto t^{\langle \lambda \mid \gamma \rangle}$. Since $\mathbf{K}$ is infinite, there exists some $z \in \mathbf{K}^\times$ such that $z^{\langle \beta \mid \gamma \rangle} \neq 1$. In view of $(**)$, it is now straightforward to check that z^γ is an element of T which centralizes U_α but not U_β. □

5.2. Non-linearity. We can now state the main non-linearity theorem of this section. Note that it is known that Kac–Moody groups of indefinite type over infinite fields of arbitrary characteristic do not admit any *faithful* finite-dimensional linear representation over any field [20, Theorem 7.1], but the simplicity for Kac–Moody groups over infinite fields is still an open question.

Theorem 25. *Let A be a generalized Cartan matrix, let $\mathcal{G}_A$ be a Tits functor of type A and let $\mathbf{K}$ be a field of characteristic $p > 0$. Assume that A is indecomposable, of indefinite type, i.e. neither spherical nor affine, that each rank one subgroup of $\mathcal{G}_A(\mathbf{K})$ is perfect and that $\mathcal{G}_A(\mathbf{K})$ is generated by its root subgroups. Then any finite-dimensional linear representation of $\mathcal{G}_A(\mathbf{K})$ is trivial.*

Proof. We set $G = \mathcal{G}_A(\mathbf{K})$ and we let $\varphi : G \to \mathrm{GL}_n(\mathbf{F})$ be any representation. If $\mathbf{K}$ is finite, then $\varphi(G)$ is residually finite by Mal'cev's theorem.

On the other hand G does not have any finite quotient by Corollary 16. Hence $\varphi(G)$ is trivial in this case.

We henceforth assume that $\mathbf{K}$ is infinite. Let us denote by G_p the group $\mathcal{G}_A(\mathbf{F}_p)$ (where $\mathbf{F}_p$ is the prime field of $\mathbf{K}$). We view G_p as a subgroup of G. By Mal'cev's theorem, $\varphi(G_p)$ is residually finite, so by Corollary 16 it is finite. Since the root system Φ contains nilpotent subsets of arbitrary large cardinality, the kernel H of φ meets non-trivially the $\mathbf{F}_p$-points of U_Ψ for some nilpotent set of roots Ψ. In particular, we have $H \cap U_\Psi \neq \{1\}$, which implies by Proposition 23 that H contains $G^\dagger$. Since the field $\mathbf{K}$ is infinite, we have $G^\dagger = [G, G]$ because the rank one subgroups $X_\alpha = \langle U_\alpha \cup U_{-\alpha} \rangle$ are perfect. The conclusion follows since $[G, G] = G$ by hypothesis. $\square$

6. Homomorphisms to topological groups

In this section we study homomorphisms from Kac–Moody groups to locally compact groups. In the first result, we collect some basic facts which show that the only interesting group homomorphisms from finitely generated Kac–Moody groups are those with totally disconnected targets. However, the main part of this section is devoted to proving that any nontrivial continuous homomorphism whose domain is the topological completion of a twin building lattice is a proper map. This is a useful result to be combined with superrigidity.

6.1. Homomorphisms from simple discrete groups. We collect here some basic (and probably well-known) facts about abstract group homomorphisms from simple discrete to locally compact groups.

Proposition 26. *Let Λ be an infinite finitely generated group endowed with the discrete topology.*

(i) *The group Λ is residually finite if, and only if, there exists an injective homomorphism from Λ to a compact group.*
(ii) *If Λ is simple (resp. simple and Kazhdan), any group homomorphism from Λ to a compact (resp. amenable) group is trivial.*

We henceforth assume that Λ is simple.

(iii) *There exists no nontrivial group homomorphism from Λ to a Lie group with finitely many connected components.*
(iv) *Let $\varphi : \Lambda \to G$ be a nontrivial group homomorphism to a locally compact group G and let $\pi : G \to G/G^\circ$ be the projection onto the group of connected components. Then $\pi \circ \varphi$ is a continuous, injective, unbounded homomorphism.*
(v) *Let X be a* CAT(0) *or hyperbolic proper metric space. Then if Λ fixes a point, say ξ, in the visual boundary $\partial_\infty X$, it stabilizes each horosphere centered at ξ.*

Point (i) was pointed out to us as a folklore result by N. Monod. A proof appears in [35, Proposition 4(i) and (ii)]; since it is short and elegant, we reproduce it here. The key is to use Peter–Weyl's theorem.

Proof. (i). By definition, a residually finite group injects in its profinite completion, so one direction is clear. Now let Λ admit an injective homomorphism $\varphi : \Lambda \to K$ into a compact group K and let $\lambda \in \Lambda - \{1\}$. The regular representation ρ_K of K in $L^2(K)$ is injective; we will use its Peter–Weyl decomposition [38, Theorem 27.40]. The image $\bar{\lambda}$ of λ in some suitable finite-dimensional irreducible submodule, say V, is nontrivial. The projection Λ_V of $(\rho_K \circ \varphi)(\Lambda)$ to $\mathrm{GL}(V)$ is a finitely generated linear group containing $\bar{\lambda}$. By Mal'cev's theorem [44, Window 7 §4 Proposition 8], the group Λ_V is residually finite, so it admits a finite quotient in which $\bar{\lambda}$ is nontrivial: this is a finite quotient of Λ in which the image of the arbitrary nontrivial element λ is nontrivial.

(ii). The case when Λ is simple follows immediately from (i). We assume that Λ is both simple and Kazhdan. Let $\varphi : \Lambda \to P$ be a homomorphism to an amenable group P. The closure $\overline{\varphi(\Lambda)}$ is a Kazhdan group because so is Λ [77, Proposition 7.1.6] and it is amenable as a closed subgroup of P [32, Th. 2.3.2]. Therefore it is compact [45, Th. III.2.5] and it remains to apply the first case of this point.

(iii). Let $\varphi : \Lambda \to G$ be a homomorphism to a Lie group with finitely many connected components. By simplicity, the group Λ has no finite index subgroup, so we are reduced to the case when G is connected. We compose this map with the adjoint representation of G, whose kernel is the center $Z(G)$ [10, Cor. 4 of Sect. II.6.4], in order to obtain a continuous homomorphism $\mathrm{Ad} \circ \varphi$ to the general linear group of the Lie algebra of G. This map is not injective since the group Λ is simple and finitely generated, hence non-linear. Therefore $(\mathrm{Ad} \circ \varphi)(\Lambda)$ is trivial. Finally, again by simplicity, we successively obtain $\varphi(\Lambda) < Z(G)$ and $\varphi(\Lambda) = \{1\}$.

(iv). By simplicity of Λ, the map φ is injective since it is not trivial. Moreover the kernel of $\pi \circ \varphi$ is equal to $\{1\}$ or Λ. We have to exclude the case when $\mathrm{Ker}(\pi \circ \varphi) = \Lambda$. Let us assume the contrary, i.e. $\varphi(\Lambda) < G^\circ$, in order to obtain a contradiction. It follows from [52, Th. 4.6] that there exists a compact normal subgroup $K \lhd G^\circ$ such that G°/K is a connected Lie group. Let us consider the composed map $\Lambda \xrightarrow{\varphi} G^\circ \xrightarrow{p} G^\circ/K$ where $p : G^\circ \to G^\circ/K$ denotes the canonical projection. By (iii) we have $(p \circ \varphi)(\Lambda) = \{1\}$ so $\varphi(\Lambda) < K$. It remains to apply (ii) to obtain the desired contradiction. The unboundedness of $\pi \circ \varphi$ follows from (ii) as well.

(v). For each $y \in X$, we denote by $\beta_{\xi,y}$ the Busemann function $\beta_{\xi,y} : X \to \mathbf{R}$ centered at ξ and such that $\beta_{\xi,y}(y) = 0$ [13, Def. II.8.17]. We pick $x \in X$ and define the function $\varphi_{\xi,x} : \Lambda \to \mathbf{R}$ by setting $\varphi_{\xi,x}(g) = \beta_{\xi,x}(g.x)$. Then for $g, h \in \Lambda$, we compute $\varphi_{\xi,x}(gh) - \varphi_{\xi,x}(h)$, i.e. $\beta_{\xi,x}(gh.x) - \beta_{\xi,x}(h.x)$ by definition. By equivariance, this is $\beta_{\xi,x}(gh.x) - \beta_{g.\xi,g.x}(gh.x)$, that is also $\beta_{\xi,x}(gh.x) - \beta_{\xi,g.x}(gh.x)$ because ξ is fixed under the Λ-action. But the latter quantity is also $\beta_{\xi,x}(g.x)$, i.e. $\varphi_{\xi,x}(g)$, by the cocycle property of Busemann functions. In other words, the function $\varphi_{\xi,x}$ is a group homomorphism from Λ to $(\mathbf{R}, +)$. By simplicity of Λ, it is trivial, from which we deduce that x and $g.x$ are on the same horosphere centered at ξ for any $g \in \Lambda$ and any $x \in X$. □

Note that if in (v) we replace Λ by a topologically simple group acting continuously on X, the same conclusion holds (the argument is the same: the above map $\varphi_{\xi,x}$ is a continuous group homomorphism).

6.2. Diverging sequences in Coxeter groups. In the present subsection, we consider a Coxeter system (W, S) and the associated Davis complex $\mathcal{A}$.

We say that a sequence $(w_n)_{n\geqslant 0}$ of elements of W *diverges* if $\lim_{n\to+\infty} \ell(w_n) = +\infty$.

Lemma 27. *Let $(w_n)_{n\geqslant 0}$ be a diverging sequence in W. Given any $x \in \mathcal{A}$, there exists a root half-space $\alpha \in \Phi(\mathcal{A})$ and a subsequence $(w_{n_k})_{k\geqslant 0}$ such that* $\lim_{k\to+\infty} d(x, w_{n_k}.\alpha) = +\infty$.

Proof. The sequence $(w_n)_{n\geqslant 0}$ diverges if and only if so does $(w_n^{-1})_{n\geqslant 0}$. Therefore, it suffices to find a root $\alpha \in \Phi$ and a subsequence $(w_{n_k})_{k\geqslant 0}$ such that $\lim_{k\to+\infty} d(w_{n_k}.x, \alpha) = +\infty$. We set $x_n = w_n.x$. Since $(w_n)_{n\geqslant 0}$ diverges, we have $\lim_{n\to+\infty} d(x, x_n) = +\infty$. Therefore $(x_n)_{n\geqslant 0}$ possesses a subsequence $(x_{n_k})_{k\geqslant 0}$ which converges to a point ξ of the visual boundary $\partial_\infty \mathcal{A}$.

Let $\rho : [0, +\infty) \to \mathcal{A}$ be the geodesic ray such that $\rho(0) = x$ and $\rho(+\infty) = \xi$. Since ρ is unbounded and since chambers are compact, it follows that ρ meets infinitely many walls of the Davis complex $\mathcal{A}$. On the other hand, the ray ρ is contained in finitely many walls, otherwise its pointwise stabilizer would be infinite, contradicting the fact that W acts properly discontinuously on $\mathcal{A}$. Therefore, there exists a wall $\partial\alpha$ which meets ρ and such that ρ is not contained in it. This wall determines two roots, one of which containing no subray of ρ. We let α be that root: $\alpha \cap \rho$ is a bounded (nonempty) segment.

Since α is a closed convex subset of $\mathcal{A}$, the map $d(\cdot, \alpha) : \mathcal{A} \to \mathbf{R}_+$ is convex [13, Cor. II.2.5] and so is $f : \mathbf{R}_+ \to \mathbf{R}_+ : t \mapsto d(\rho(t), \alpha)$. Therefore, if f is bounded, it is constant and since ρ meets α we have $f(t) = 0$ for all t. This is excluded because by construction the ray ρ is not contained in α. Thus f is an unbounded convex function and we deduce that $\lim_{t\to+\infty} d(\rho(t), \alpha) = +\infty$. Finally, since $\lim_{t\to\infty} \rho(t) = \lim_{k\to\infty} x_{n_k} = \xi$, it follows that $\{x_{n_k} : k \geqslant 0\}$ is at finite Hausdorff distance from $\rho([0, +\infty))$. Therefore, we obtain $\lim_{k\to+\infty} d(x_{n_k}, \alpha) = +\infty$ as desired. □

6.3. Properness of continuous homomorphisms. We now settle our main properness result for continuous group homomorphisms from topological completions of twin building lattices.

Theorem 28. *Let Λ be a group endowed with a twin root datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type (W, S) with finite root groups, and such that $\Lambda = \langle U_\alpha \mid \alpha \in \Phi \rangle$*

and $Z(\Lambda)$ is finite. Assume that W is infinite, that (W, S) is irreducible, that all root groups are solvable and that property (FPRS) *of Sect. 2.1 holds. Let $\overline{\Lambda}_+$ be the positive topological completion of Λ and let $\varphi : \overline{\Lambda}_+ \to G$ be a continuous nontrivial homomorphism to a locally compact second countable group G. Then φ is proper.*

Remark. In view of the proof below, we can consider Proposition 4 and Lemma 27 as substitutes for results such as the contracting or expanding properties of torus actions on root groups in the classical algebraic group case [16, Lemma 5.3].

Proof. Let us assume that φ is not proper in order to obtain a contradiction. There exists a sequence $(\gamma_j)_{j\geqslant 0}$ eventually leaving every compact subset of $\overline{\Lambda}_+$ and such that $\lim_{j\to+\infty}\varphi(\gamma_j)$ exists in G. Let us recall that we can view the Weyl group W as the quotient $\widehat{N_{\mathcal{A}}}/\Omega_{\mathcal{A}}$ where $\widehat{N_{\mathcal{A}}} = \mathrm{Stab}_{\overline{\Lambda}_+}(\mathcal{A}_+)$ and $\Omega_{\mathcal{A}} = \mathrm{Fix}_{\overline{\Lambda}_+}(\mathcal{A}_+)$. We also have a Bruhat decomposition: $\overline{\Lambda} = \bigsqcup_{w\in W} B_+wB_+$, where B_+ is the Iwahori subgroup $B_+ = \mathrm{Fix}_{\overline{\Lambda}_+}(c_+)$. We use it to write $\gamma_j = k_j.n_j.k'_j$ with $k_j, k'_j \in B_+$ and $n_j \in \widehat{N_{\mathcal{A}}}$. Up to passing to a subsequence, we may assume that $(k_j)_{j\geqslant 0}$ and $(k'_j)_{j\geqslant 0}$ are both converging in the compact open subgroup B_+. We set $w_j = n_j\Omega_{\mathcal{A}}$. The hypothesis on $(\gamma_j)_{j\geqslant 0}$ implies that $(w_j)_{j\geqslant 0}$ is a diverging sequence in W and that $\lim_{j\to+\infty}\varphi(n_j)$ exists in G. We denote this limit by g. In view of Lemma 27, up to passing to a subsequence, there exists a root $\alpha \in \Phi(\mathcal{A}_+)$ such that $\lim_{j\to+\infty} d(c_+, w_j.\alpha) = +\infty$. Let $u \in U_{-\alpha} - \{1\}$. Recall that $n_j.u.n_j^{-1} \in U_{w_j.(-\alpha)}$ for all j. Therefore by (FPRS) we have: $\lim_{j\to+\infty} n_j.u.n_j^{-1} = 1$. Applying φ we obtain: $1 = \lim_{j\to+\infty}\varphi(n_j.u.n_j^{-1}) = g.\varphi(u).g^{-1}$. Thus we have $u \in \mathrm{Ker}(\varphi)$. By Proposition 11(i), this implies that $\overline{\Lambda}_+^\dagger < \mathrm{Ker}(\varphi)$. Since $\Lambda = \Lambda^\dagger$ by assumption, it follows that φ is trivial, providing the desired contradiction: φ is proper. $\square$

7. Superrigidity

In this section, we show that recent superrigidity theorems can be applied to twin building lattices. They concern actions on CAT(0)-spaces. We also derive some consequences: non-linearity of irreducible cocompact lattices in some Kac–Moody groups, homomorphisms of twin building lattices with Kac–Moody targets, restrictions for actions on negatively curved complete metric spaces.

7.1. Actions on CAT(0)**-spaces.** The possibility to apply [50] to irreducible cocompact lattices enables us to prove the following. Note that the existence of irreducible cocompact lattices in this context is an open problem.

Proposition 29. *Let Λ be a twin building lattice generated by its root groups $\{U_\alpha\}_{\alpha\in\Phi}$, with associated twinned buildings $\mathcal{B}_\pm$. We assume that the*

Weyl group W is infinite, irreducible and non-affine, that root groups are nilpotent and that $Z(\Lambda) = 1$. Let Γ be an irreducible cocompact lattice in $\overline{\Lambda}_+ \times \overline{\Lambda}_-$. Then any linear image of Γ is finite.

Remark. The important assumption here is irreducibility. Indeed, the twin buildings associated to some Kac–Moody groups are right-angled Fuchsian. For each of the two buildings $\mathscr{B}_\pm$, the completion $\overline{\Lambda}_\pm$ contains a cocompact lattice isomorphic to a convex cocompact subgroup of the isometry group of a well-chosen real hyperbolic space [65, Prop. 4B]. Taking the product of two such lattices, we obtain a cocompact lattice in $\overline{\Lambda}_- \times \overline{\Lambda}_+$ which is linear over the real numbers.

The arguments are classical, so we only sketch the proof. Note that one could appeal here to Monod's alternative [49, Theorem 2.4] which yields the above proposition when combined with the nonlinearity of twin building lattices. However, the alternative in [loc. cit.] does not apply to linear images over fields of characteristic 2 and 3 (although this restriction is probably not necessary, see [49, Remark 2.4(c)]). The arguments below, relying on [50], avoid any consideration of characteristics.

Proof. By Proposition 11(i), the groups $\overline{\Lambda}_\pm$ are topologically simple. By [5, Corollary 1.4], it follows that Γ is just infinite, i.e. all its proper quotients are finite. Hence any group homomorphism from Γ with infinite image is injective; the same holds for any finite index subgroup of Γ. Let $\mathbf{F}$ be a field with algebraic closure $\overline{\mathbf{F}}$ and let $n \geqslant 2$ be an integer such that there is an injective group homomorphism $\eta : \Gamma \to \mathrm{GL}_n(\mathbf{F})$. We must obtain a contradiction. Let H be the Zariski closure of $\eta(\Gamma)$ in $\mathrm{GL}_n(\overline{\mathbf{F}})$. We denote by Γ° the preimage by η of the identity component H°. It is a finite index normal subgroup of Γ, so as a lattice in $\overline{\Lambda}_- \times \overline{\Lambda}_+$ it is still irreducible because $\overline{\Lambda}_\pm$ is topologically simple. We denote by $R(H^\circ)$ the radical of H° and by $\pi : H^\circ \to H^\circ/R(H^\circ)$ the natural projection. Then $\pi \circ \eta$ is still injective since otherwise, by the normal subgroup property for Γ°, the group Γ would be virtually solvable, hence amenable, while $\overline{\Lambda}_- \times \overline{\Lambda}_+$ is not. We thus obtain a semisimple group G over $\overline{\mathbf{F}}$ and an injective group homomorphism $\varphi : \Gamma^\circ \to G$ with Zariski dense image. We choose an algebraic group embedding in some general linear group: $G < \mathrm{GL}_r$. Being cocompact, the lattice Γ° is finitely generated. Taking the matrix coefficients of the elements of some finite symmetric generating system implies that $\varphi(\Gamma^\circ)$ lies in $\mathrm{GL}_r(\mathbf{E})$ for some finitely generated field $\mathbf{E}$. The group Γ° is finitely generated, linear and non-amenable so by Tits' alternative it contains a non-abelian free group [69]. We can find an element with one eigenvalue, say λ, of infinite multiplicative order, so there is a local field $\hat{K}$ with absolute value $|\cdot|$ and a field extension $\sigma : \mathbf{E} \to \hat{K}$ such that $|\sigma(\lambda)| \neq 1$. In particular, the subgroup $\varphi(\Gamma^\circ)$ is unbounded in $G(\hat{K})$. The map $\varphi : \Gamma^\circ \to G(\hat{K})$ satisfies the two conditions required to apply Monod's superrigidity [50, Corollary 4 and Lemma 59]: the homomorphism φ extends to a continuous homomorphism $\tilde{\varphi} : \overline{\Lambda_-} \times \overline{\Lambda_+} \to G(\hat{K})$. By topological simplicity of $\overline{\Lambda_\pm}$, we obtain an injective homomorphism $\Lambda \to G(\hat{K})$, which is impossible since Λ is infinite, virtually simple and finitely generated, hence non-linear. $\square$

7.2. Uniform p-integrability. We now check an integrability condition which is a partial substitute for cocompactness of lattices. Let us recall the general context: G is a locally compact group, Γ is a lattice in G. We assume that Γ contains a finite generating subset Σ and denote by $|\cdot|_\Sigma$ the length function on Γ with respect to it. Following [31, Sect. 7], each time we have a right fundamental domain Ω for Γ, we define the function $\chi_\Omega : G \to \Gamma$ by $g \in \chi_\Omega(g)\Omega$. For each real number $p > 1$, we say that Γ is *p-integrable* if there is a right fundamental domain Ω such that for any $c \in G$, we have: $\int_\Omega (|\chi_\Omega(gc)|_\Sigma)^p \, \mathrm{d}g < +\infty$. The main result of [64] is that Kac–Moody lattices are p-integrable for any $p > 1$. This amounts to saying that the function $g \mapsto |\chi_\Omega(gc)|_\Sigma$ belongs to $L^p(\Omega)$ for any $c \in G$. We are interested in a stronger property. We denote by $\|\cdot\|_{\Omega,p}$ the L^p-norm of measurable functions on Ω.

Definition 30. *Given $p \in [1;\infty)$, the lattice Γ in G is called* uniformly p-integrable *if there is a right fundamental domain Ω as above such that for any compact subset C in G we have:*

$$\sup_{c\in C} \int_\Omega (|\chi_\Omega(gc)|_\Sigma)^p \mathrm{d}g < +\infty,$$

i.e. the real valued function $\varphi_{\Omega,p} : c \mapsto \| |\chi_\Omega(\cdot c)|_\Sigma \|_{\Omega,p}$ is bounded on compact subsets of G.

The relation with Y. Shalom's condition [67] is the following. Given a left fundamental domain D for the inclusion $\Gamma < G$, we can define a map $\alpha_D : G \times D \to \Gamma$ by setting $\alpha_D(g,d) = \gamma$ if, and only if, we have $gd\gamma \in D$. Up to translating it, we may – and shall – assume that 1_G belongs to D. The set $\Omega = D^{-1}$ is a right fundamental domain and the following equivalences hold:

$$\begin{aligned} \alpha_D(g, 1_G) = \gamma \iff g\gamma \in D \iff \gamma^{-1}g^{-1} \in \Omega \iff g^{-1} \in \gamma\Omega \\ \iff \gamma = \chi_\Omega(g^{-1}). \end{aligned}$$

In other words, we have: $\chi_\Omega(g) = \alpha_D(g^{-1}, 1_G)$ and 2-integrability amounts to Y. Shalom's condition (1.5) in [67, 1.II p. 14].

Let us now turn to the specific case of twin building lattices. For the rest of the section, we let $\Gamma = \Lambda$ be a twin building lattice with twin root datum $\{U_\alpha\}_{\alpha\in\Phi}$. We also let $G = \overline{\Lambda_-} \times \overline{\Lambda_+}$. In [64, Prop. 5] a left fundamental domain D is defined by means of refined Tits systems arguments. It is a union $D = \bigsqcup_{w\in W} D_w$, indexed by the Weyl group W, of compact open subsets D_w in G and we have $1_G \in D_{1_W}$. For the rest of the subsection, Σ denotes the finite symmetric generating subset used in [65, Definition 1], i.e. the union of the rank one subgroups $X_\alpha \cdot T = \langle U_\alpha, U_{-\alpha}\rangle \cdot T$ indexed by the simple roots $\alpha \in \Pi$.

Theorem 31. *There exists a right fundamental domain Ω such that for any $p \in [1; +\infty)$ the function $\varphi_{\Omega,p} : c \mapsto \| |\chi_\Omega(\cdot c)|_\Sigma \|_{\Omega,p}$ is bounded from above by a function which is constant on each product of double cosets modulo the standard Iwahori subgroups in $\overline{\Lambda}_-$ and $\overline{\Lambda}_+$. In particular, twin building lattices are uniformly p-integrable for any $p \in [1; +\infty)$.*

Remark. The second assertion implies the first one because the standard Iwahori subgroups $B_\pm$ are open and compact, so products of double cosets modulo Iwahori subgroups $B_+ w_+ B_+ \times B_- w_- B_-$ are open and compact (and disjoint when distinct).

Proof. We first note that as for the normal subgroup property (Theorem 18), though the results in [64] are stated for Kac–Moody groups, we can use them in the more general context of the above statement thanks to Proposition 1(vi). Moreover since the groups $\overline{\Lambda}_\pm$ have BN-pairs, they are unimodular [11, IV.2.7, Lemme 2]. Hence so is G. We denote by dg a Haar measure on G and compute $\varphi_{\Omega,p}(c)^p$ for $c \in G$. It is:

$$\int_\Omega (|\chi_\Omega(gc)|_\Sigma)^p \mathrm{d}g = \int_{D^{-1}} \left(\left|\alpha_D\left(c^{-1}g^{-1}, 1_G\right)\right|_\Sigma\right)^p \mathrm{d}g$$
$$= \int_D \left(\left|\alpha_D\left(c^{-1}g, 1_G\right)\right|_\Sigma\right)^p \mathrm{d}g.$$

The first equality follows from the remarks before the statement and the last equality from the unimodularity of G. Therefore it is enough to check that the map $h \mapsto \int_D (|\alpha_D(hg, 1_G)|_\Sigma)^p \, \mathrm{d}g$ is bounded from above by a function which is constant on products of double cosets modulo the standard Iwahori subgroups B_- and B_+, in $\overline{\Lambda}_-$ and $\overline{\Lambda}_+$ respectively. We are now back to objects studied in [64]. An element $h \in G$ is a couple (h_-, h_+) with $h_\pm \in \overline{\Lambda}_\pm$ and we denote by $L_\pm(h)$ the combinatorial distance (i.e. the length of a minimal gallery) in the building $\mathscr{B}_\pm$ from the standard chamber $c_\pm$ to the chamber $h_\pm^{-1}.c_\pm$. The function $h \mapsto L_\pm(h)$ is constant on each double coset of $\overline{\Lambda}_\pm$ modulo the standard Iwahori subgroup of sign $\pm$ since $L_\pm(h)$ is nothing but the length in W of the Weyl group element indexing the double class $B_\pm h_\pm B_\pm$. We set $L(h) = L_-(h) + L_+(h)$ and introduce the polynomial Q_h defined in [64, Lemma 17] by $Q_h(X) = 3X^2 + (6L(h)+3)X + (3L(h)^2 + 3L(h) + 1)$. Then by the proof of the main theorem of [loc. cit., p. 39] there exists a constant $|T|$ such that we have: $\varphi_{\Omega,p}(h)^p \leqslant |T| \cdot \sum_{n\in\mathbf{N}} \frac{Q_h(n)^p}{q^n}$. We are done since $h \mapsto Q_h$ is constant on the products of double cosets $B_+ w_+ B_+ \times B_- w_- B_-$. □

Recall that weak cocompactness of a lattice Γ in a topological group G is the fact that the space of functions of zero mean in $L^2(G/\Gamma)$ doesn't almost have invariant vectors [45, Sect. III.1.8]. It is possible that all Kac–Moody lattices enjoy this property, which is implied by Kazhdan's property (T). As already mentioned in the introduction (where a specific statement is provided), using the above p-integrability for weakly cocompact Kac–Moody groups enables us to derive several superrigidity results.

Proposition 32. *Let Λ be a twin building lattice, which we assume to be weakly cocompact in $\overline{\Lambda}_- \times \overline{\Lambda}_+$. Then the results in* [50]*, as well as Theorems 1.1, 1.3, 1.4 and 2.7 of* [31] *can be applied to $\Lambda < \overline{\Lambda}_- \times \overline{\Lambda}_+$.*

Proof. This is a straightforward consequence of [50, Theorem 7] and of [31, §7]. □

Remark. In this subsection, no assumption was made on the type of the Coxeter diagram of the Weyl group.

7.3. Homomorphisms of twin building lattices with Kac–Moody targets. The purpose of this subsection is to present a concrete application of superrigidity of twin building lattices, and more specifically of Proposition 32. Recall that the main application of superrigidity of lattices in Lie groups is arithmeticity, see e.g. [45] and [50]. In the context of twin building lattices, and in view of the simplicity Theorem 19 and its corollaries, it is rather natural to apply superrigidity to homomorphisms with non-linear targets. The main result of this section is an example of such an application.

Let $(G, \{U_\alpha\}_{\alpha\in\Phi})$ be a twin root datum of type (W, S), with finite root groups, such that the centralizer $Z_G(G^\dagger)$ is trivial, and let $\overline{G}_+$ be its positive topological completion. Since $Z_G(G^\dagger)$ is the kernel of $\overline{G}_+$-action on the associated building $\mathscr{B}_+$, we may – and shall – view $\overline{G}_+$ as a subgroup of $\mathrm{Aut}(\mathscr{B}_+)$ (see Proposition 1(iii)).

Theorem 33. *Let Λ be a weakly cocompact (e.g. Kazhdan) twin building lattice of irreducible type, whose root groups are solvable, and such that Λ is generated by the root groups and that property* (FPRS) *of Sect. 2.1 holds. Let $\varphi : \Lambda \to \overline{G}_+$ be a homomorphism with dense image. Assume that (W, S) is irreducible, non-spherical and 2-spherical (i.e. any 2-subset of S generates a finite group). Assume also that $q_{\min} = \min\{|U_\alpha| : \alpha \in \Phi\} > 3$. Then $\varphi(\Lambda)$ is conjugate to $G^\dagger$ in $\mathrm{Aut}(\mathscr{B}_+)$. In particular $G^\dagger$ is isomorphic to $\Lambda/Z(\Lambda)$.*

Proof. Details of the arguments involve some rather delicate considerations pertaining to the theory of twin buildings. Since a detailed proof would therefore require somewhat lengthy preparations which are too far away from the main topics of this paper, we only give a sketch.

Since the $\overline{G}_+$-action on the associated building $\mathscr{B}_+$ is strongly transitive and since $\varphi(\Lambda)$ is dense in $\overline{G}_+$, it follows that the Λ-action on $\mathscr{B}_+$ induced by φ is reduced (in the sense of [50]). By Proposition 32, we may therefore apply [31, Theorem 1.1], which ensures that the homomorphism φ extends uniquely to a continuous homomorphism $\overline{\Lambda}_+ \times \overline{\Lambda}_- \to \overline{G}_+$, factoring through $\overline{\Lambda}_+$ or $\overline{\Lambda}_-$. Up to exchanging $+$ and $-$, we assume that φ extends to a continuous homomorphism $\overline{\Lambda}_+ \to \overline{G}_+$, also denoted φ. Since φ is proper by Theorem 28, it follows that φ is surjective. Furthermore, by Proposition 11(i) the kernel of φ is contained in the discrete center $Z(\overline{\Lambda}_+) = Z(\Lambda)$.

For both $\overline{\Lambda}_+$ and $\overline{G}_+$, the maximal compact subgroups are precisely the maximal (spherical) parahoric subgroups. Moreover, since (W, S) is irreducible and 2-spherical, a maximal parahoric subgroup of $\overline{G}_+$ has a Levi decomposition as semi-direct product $L \ltimes \overline{U}$, where L is a finite group of Lie type and $\overline{U}$ is a pro-p group for some prime p depending only on G (see [63, Theorem 1.B.1(ii)] and Proposition 3(i)). Furthermore, in the decomposition $L \ltimes \overline{U}$, the "congruence subgroup" $\overline{U}$ is characterized as the maximal normal pro-p subgroup. Using the fact that maximal parahoric subgroups of $\overline{\Lambda}_+$ also admit Levi decompositions, it is not difficult to deduce that the Levi factors in $\overline{\Lambda}_+$ are also finite groups of Lie type and, then, that φ induces isomorphisms between the Levi factors in $\overline{\Lambda}_+$ and in $\overline{G}_+$. In view of the description of the respective buildings of the latter groups as coset geometries modulo parahoric subgroups, this in turn implies that φ induces an isomorphism between the building $\mathcal{X}_+$ of $\overline{\Lambda}_+$ and the building $\mathcal{B}_+$ of $\overline{G}_+$; moreover this isomorphism is φ-equivariant. In particular $\mathcal{X}_+$ is of 2-spherical irreducible type with infinite Weyl group. By assumption, the building $\mathcal{X}_+$ (resp. $\mathcal{B}_+$) admits a twin $\mathcal{X}_-$ (resp. $\mathcal{B}_-$) such that the diagonal action of Λ (resp. G) on the product $\mathcal{X}_+ \times \mathcal{X}_-$ (resp. $\mathcal{B}_+ \times \mathcal{B}_-$) preserves the twinning. By the main result of [57], these twinnings must be isomorphic since $\mathcal{X}_+$ and $\mathcal{B}_+$ are. More precisely, these twinnings are conjugate under some element of $\mathrm{Aut}(\mathcal{B}_+)$. Up to conjugating $\varphi(\Lambda)$ by this element of $\mathrm{Aut}(\mathcal{B}_+)$, we may – and shall – assume that $\varphi(\Lambda)$ preserves the twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

Since the isomorphism between $\mathcal{X}_+$ and $\mathcal{B}_+$ is φ-equivariant, it follows from standard description of root group actions in Moufang twin buildings (see e.g. [74]) that φ maps each root group of Λ to a root group of G, and that every root group of G is reached in this manner. Since Λ is generated by its root groups, we deduce that $\varphi(\Lambda) = G^\dagger$. Finally, since $G^\dagger$ is center-free, so is $\varphi(\Lambda)$, from which it follows that $\mathrm{Ker}(\varphi) \cap \Lambda = Z(\Lambda)$. □

7.4. Actions on $\mathrm{CAT}(-1)$**-spaces.** Another consequence is the existence of strong restrictions on actions of "higher-rank" Kac–Moody lattices on $\mathrm{CAT}(-1)$-spaces. Of course, in this case we must discuss the notion of rank which is relevant to the situation; this is done just after the statement. Since we are dealing with hyperbolic target spaces, it is most convenient to use a superrigidity theorem due to N. Monod and Y. Shalom [51].

Theorem 34. *Let Λ be a split or almost split adjoint Kac–Moody group over $\mathbf{F}_q$ which is a lattice in the product of the associated buildings $\mathcal{B}_\pm$. We assume that the Weyl group W is infinite, irreducible and non-affine and* *that* $q \geqslant 4$. *Let* Y *be a proper* CAT(-1)*-space with cocompact isometry group. We assume that* Λ *acts on* Y *by isometries and we denote by* $\varphi : \Lambda \to \mathrm{Isom}(Y)$ *the corresponding homomorphism.*

(i) *If the* Λ*-action is nontrivial but has a global fixed point in the compactification* $\overline{Y} = Y \sqcup \partial_\infty Y$, *then the fixed point is unique and lies in the visual boundary* $\partial_\infty Y$.
(ii) *We assume that the* Λ*-action has no global fixed point at all. Then there exists a nonempty, closed, convex,* Λ*-stable subset* $Z \subseteq Y$ *on which it extends to a continuous homomorphism* $\tilde{\varphi} : \overline{\Lambda_-} \times \overline{\Lambda_+} \to \mathrm{Isom}(Z)$ *which factors through* $\overline{\Lambda_-}$ *or* $\overline{\Lambda_+}$.
(iii) *We assume that the buildings* $\mathscr{B}_\pm$ *contain flat subspaces of dimension* $\geqslant 2$. *Then the* Λ*-action, if not trivial, has a unique global fixed point in* $\overline{Y}$, *which lies in the visual boundary* $\partial_\infty Y$.
(iv) *Assume that* $\mathrm{Isom}(Y)$ *is non-elementary, or else that* Λ *is Kazhdan. If the buildings* $\mathscr{B}_\pm$ *contain flat subspaces of dimension* $\geqslant 2$, *then the* Λ*-action is trivial.*

Remarks. 1. The assumption that Y has cocompact isometry group in the theorem is necessary. Consider indeed the minimal adjoint Kac–Moody group $\Lambda = \mathcal{G}_A(\mathbf{F}_q)$ over $\mathbf{F}_q$, where A is the generalized Cartan matrix of size 4 defined by $A_{ii} = 2$ for $i = 1, \ldots, 4$ and $A_{ij} = -1$ for $1 \leqslant i \neq j \leqslant 4$. Thus the group Λ satisfies all hypotheses of the theorem if $q \geqslant 4$; furthermore Λ has property (T) provided $q > \frac{1764^4}{25}$ [30]. The specificity is here that the Weyl group W is a Coxeter group which is a non-uniform lattice of SO(3, 1). In fact W acts on the real hyperbolic 3-space $\mathbb{H}^3$ with a non-compact simplex as a fundamental domain. It turns out that the whole building $\mathscr{B}_+$ has a geometric realization X in which apartments are isomorphic to the tiling of $\mathbb{H}^3$ by the above simplex as fundamental tile. This geometric realization is a locally finite simplicial complex which admits a global CAT(-1)-metric induced by the metric of the apartments [2, Proposition 11.31]. Hence, although the building $\mathscr{B}_+$ has 2-dimensional flats in its usual geometric realization, the CAT(-1)-space X is endowed with a natural Λ-action which has no global fixed point in the visual compactification $\overline{X}$. The point is of course that chambers are not compact in X (while they are of course compact in $\mathscr{B}_+$), which implies that $\mathrm{Isom}(X)$ is not cocompact.

Note that in the above example, the rank 2 Levi factors of Λ are (virtually) nothing but arithmetic groups $\mathrm{SL}_3(\mathbf{F}_q[t, t^{-1}])$. The action of these subgroups on X induced by the Λ-action has no global fixed points in X, which shows in particular that the assumption that $\mathrm{Isom}(Y)$ has finite critical exponent is necessary in [16, Corollary 0.5] as well.

2. The prototype of "higher-rank versus CAT(-1)" result we have in mind is [16, Corollary 0.5]. In the latter case the target space is also a CAT(-1)-space without any required connection with Lie groups, but the irreducible lattice lies in a product of algebraic groups. Then the fact that each factor in the associated product of symmetric spaces and Bruhat–

Tits buildings contains higher-dimensional flats implies property (T) for the lattice. In the Kac–Moody case, the existence of higher-dimensional flats no longer implies property (T); this explains the distinction between (iii) and (iv).

3. Another interesting rigidity result for actions of "higher-rank" groups on CAT(-1)-spaces is obtained in [12, Theorem 2]. More precisely, it is shown in [loc. cit.] that a finitely generated group whose first L^p-cohomology vanishes for all $p > 1$ cannot act properly on a CAT(-1)-space with cocompact isometry group. Note that these cohomological conditions are satisfied by 2-spherical Kac–Moody groups over sufficiently large finite fields by the results of [30].

4. The notion of flat rank considered in [6] for groups of building automorphisms is relevant here. According to [21] and [6, Theorem A], knowing whether the buildings $\mathcal{B}_\pm$ contain higher-dimensional flat subspaces is equivalent to the fact that the groups $\overline{\Lambda}_\pm$ are of flat rank $\geqslant 2$ or, still equivalently, that the Weyl group contains a free abelian subgroup of rank $\geqslant 2$. In particular, the flat rank can be explicitly computed from the Dynkin diagram of the twin building lattice group Λ.

Proof of Theorem 34. (i). We first note that the hypotheses (S0)–(S3) of Theorem 19 are fulfilled. Moreover $\Lambda = \Lambda^\dagger$ and $Z(\Lambda) = 1$ since Λ is adjoint. Hence the group Λ is simple. In particular, the non-triviality of φ implies that it is injective. Moreover for any $y \in Y$ the stabilizer $\mathrm{Stab}_{\mathrm{Iso}(Y)}(y)$ is a compact group, so by Proposition 26 a nontrivial Λ-action on Y cannot have any fixed point in Y. Now let ξ and η be two distinct Λ-fixed points in the visual boundary $\partial_\infty Y$. Then the unique geodesic $(\eta\xi)$ is stable and by simplicity of Λ the restriction of the Λ-action on $(\eta\xi)$ has to be trivial: this implies the existence of a global fixed point in Y, which again is excluded when φ is nontrivial.

(ii). Let us first show that the closure group $\overline{\varphi(\Lambda)}$ is non-amenable. Assume the contrary. Then there is a probability measure μ which is fixed by this group. Since the Λ-action has no global fixed point in Y, the group $\overline{\varphi(\Lambda)}$ is not compact so by the CAT(-1) Furstenberg's lemma [16, Lemma 2.3] the support of μ contains at most two points. This support is Λ-stable, so by simplicity it is pointwise fixed by Λ; this is excluded because the Λ-action has no global fixed point in $\partial_\infty Y$.

We henceforth know that $\overline{\varphi(\Lambda)}$ is non-amenable. We apply [51, Theorem 1.3]: there exists a compact normal subgroup $M \lhd \overline{\varphi(\Lambda)}$ such that the induced homomorphism $\Lambda \to \overline{\varphi(\Lambda)}/M$ extends to a continuous homomorphism $\overline{\Lambda}_- \times \overline{\Lambda}_+ \to \overline{\varphi(\Lambda)}/M$ factoring through $\overline{\Lambda}_-$ or $\overline{\Lambda}_+$. Therefore, choosing a suitable sign we obtain an injective continuous group homomorphism

$$\bar{\varphi} : \overline{\Lambda}_\pm \to \overline{\varphi(\Lambda)}/M.$$

Let us denote by Z the fixed-point set of M in Y. The subset Z is nonempty because M is compact, it is closed and convex by uniqueness of geodesic segments. Since M is normal in $\overline{\varphi(\Lambda)}$, the subset Z is stable under the $\overline{\varphi(\Lambda)}$-action. The restriction map $r_Z : \overline{\varphi(\Lambda)} \to \mathrm{Isom}(Z)$ factors through the canonical projection $\pi_M : \overline{\varphi(\Lambda)} \to \overline{\varphi(\Lambda)}/M$ and provides a natural homomorphism $\bar{r}_Z : \overline{\varphi(\Lambda)}/M \to \mathrm{Isom}(Z)$ which we can compose with $\bar{\varphi}$ to obtain the desired homomorphism $\tilde{\varphi} = \bar{r}_Z \circ \bar{\varphi}$.

(iii). Let us assume that the Λ-action has no fixed point in $\overline{Y}$ in order to obtain a contradiction. By applying (ii), we have an injective continuous homomorphism $\tilde{\varphi} : \overline{\Lambda}_\pm \to \mathrm{Isom}(Z)$ as above. Moreover Proposition 4 and Theorem 28 imply that $\tilde{\varphi}$ is proper.

Root system preliminaries. By the remarks before the proof, the existence of flats of dimension $\geqslant 2$ implies the existence of an abelian subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$ in the Weyl group W. Moreover it follows from [42, Theorem 6.8.3] that if W has a subgroup isomorphic to $\mathbf{Z}\times\mathbf{Z}$, then it has two reflection subgroups, both isomorphic to D_∞, whose translations commute, where D_∞ is the infinite dihedral group. Let $\alpha, \alpha', \beta, \beta' \in \Phi$ be roots such that $\tau = r_\alpha r_\beta$ and $\tau' = r_{\alpha'} r_{\beta'}$ are mutually commuting and both of infinite order. Let V_+ (resp. V_-) the group generated by the root groups indexed by roots in $\bigcup_{n\in\mathbf{Z}} \tau^n.\{\alpha; -\beta\}$ (resp. by the opposite roots). Note that each of these two groups is normalized by any element lifting τ in N.

Reduction to hyperbolic isometries. Recall that the torus T is finite, hence the fixed-point-set Y^T of T in Y is nonempty. Thus Y^T is a closed convex subset of Y on which the group N acts since N normalizes T. Recall that the quotient N/T is nothing but the Weyl group W. Obviously T acts trivially on Y^T and, hence, the action of N on Y^T factors through W. Therefore, we may – and shall – consider that W acts on Y^T.

Let us now pick n_τ such an element, i.e. such that $n_\tau T = \tau$ in $W = N/T$. By properness of $\tilde{\varphi}$, the group generated by $\tilde{\varphi}(n_\tau)$ is unbounded, therefore the isometries $\tilde{\varphi}(n_\tau)^{\pm 1}$ are either both hyperbolic or both parabolic because n_τ and n_τ^{-1} are mutually conjugate by r_α. We claim that we obtain the desired contradiction if we manage to prove that these isometries (as well as $n_{\tau'}$) are hyperbolic. Indeed, τ together with τ' generate a free abelian group of rank 2 which acts on Y^T. Since the group N is a discrete subgroup of $\overline{\Lambda}_+$ (because it acts properly discontinuously on $\mathscr{B}_+$) and since $\tilde{\varphi}$ is proper, it follows that $\langle \tau, \tau' \rangle$ acts freely on Y^T. By the flat torus theorem [13, Corollary II.7.2], we deduce that Y^T contains a 2-flat. This is absurd because Y is CAT(-1).

Fixed points of "unipotent" subgroups. On the one hand, we claim that $\overline{\tilde{\varphi}(V_\pm)}$ cannot stabilize any geodesic line in Z. Indeed, any element $g \in V_\pm$ is torsion so it fixes a point in Z. If L were a $\tilde{\varphi}(V_\pm)$-stable geodesic line then, using orthogonal projection, $\tilde{\varphi}(g)$ would fix a point of L. This would imply that the subgroup of index at most 2 in $\tilde{\varphi}(V_\pm)$ fixing the extremities of L would in fact fix the whole line L: this is excluded because, by properness of $\tilde{\varphi}$, the groups $\overline{\tilde{\varphi}(V_\pm)}$ are not compact. On the other hand, the groups $V_\pm$ are metabelian [61, 3.2 Example 2] so the closures $\overline{\tilde{\varphi}(V_\pm)}$ are amenable groups [77, Lemma 4.1.13], hence fix a probability measure on $\partial_\infty Z$. By [16, Lemma 2.3] the support of such a measure contains at most two points. By the previous point, the support must consist of one single point at infinity and the same argument shows that this point is the unique $\tilde{\varphi}(V_\pm)$-fixed point in $\partial_\infty Z$: we call it $\eta_\pm$.

Images of translations are not parabolic. Since n_τ normalizes $V_\pm$ and since $(\partial_\infty Z)^{\tilde{\varphi}(V_\pm)} = \{\eta_\pm\}$, the isometry $\tilde{\varphi}(n_\tau)$ fixes η_- and η_+. In order to see that $\tilde{\varphi}(n_\tau)$ is hyperbolic, it suffices to show that $\eta_- \neq \eta_+$. Let us assume that η_- and η_+ are the same boundary point, which we call η. We need to obtain a very last contradiction. Let us consider the group $H = \overline{\langle V_-, V_+\rangle}$, by definition topologically generated by V_- and V_+. It is non-amenable because, as a completion of a group with twin root datum of type $\tilde{A}_1$, it admits a proper strongly transitive action on a semi-homogeneous locally finite tree. Theorem 28 implies that the maps $\tilde{\varphi}$ and $\bar{\varphi}$ are proper. On the one hand, properness of $\bar{\varphi}$ implies that the group $\bar{\varphi}(H)$ is non-amenable and neither is $\pi_M{}^{-1}(\bar{\varphi}(H))$ since π_M is proper and surjective. On the other hand, properness of $\tilde{\varphi}$ implies that $\tilde{\varphi}(H)$ is a closed subgroup of $\mathrm{Isom}(Z)$, which implies that $r_Z^{-1}(\tilde{\varphi}(H))$ is a closed subgroup of $\mathrm{Stab}_{\overline{\varphi(\Lambda)}}(\eta)$. Moreover we have: $\pi_M{}^{-1}(\bar{\varphi}(H)) < r_Z^{-1}(\tilde{\varphi}(H))$, so the non-amenable group $\pi_M{}^{-1}(\bar{\varphi}(H))$ is a closed subgroup of $\mathrm{Stab}_{\overline{\varphi(\Lambda)}}(\eta)$. The contradiction comes from the fact that $\mathrm{Stab}_{\mathrm{Isom}(Y)}(\eta)$ is amenable, since $\mathrm{Isom}(Y)$ acts co-compactly on Y [16, Propositions 1.6 and 1.7].

(iv). The stabilizer $\mathrm{Stab}_{\mathrm{Isom}(Y)}(\xi)$ of every point at infinity of Y is amenable [16, Propositions 1.6 and 1.7]. Therefore, it cannot contain a Kazhdan subgroup. If one assumes moreover that $\mathrm{Isom}(Y)$ is non-elementary, then by [48, Theorem 21] $\mathrm{Isom}(Y)$ is either virtually connected or totally disconnected. In the first case, $\mathrm{Isom}(Y)$ does not contain any finitely generated infinite simple group by the solution to Hilbert 5th problem [52] (see also Proposition 26). In the second case, no amenable subgroup of $\mathrm{Isom}(Y)$ contains a finitely generated infinite simple group by [19, Corollary 1.2]. Thus the desired conclusion follows from (iii). □

In view of recent results from [22], it is known that if X is a proper CAT(0) space such that $X/\mathrm{Isom}(X)$ is compact, then the closure of any finitely generated infinite simple subgroup $\Gamma < \mathrm{Isom}(X)$ is non-amenable. This implies that the conclusion of (iv) holds even without assuming that Λ is Kazhdan or that $\mathrm{Isom}(Y)$ is non-elementary.

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