Classical and Non-Linearity Properties of Kac–Moody Lattices

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Abstract This paper presents a new class of geometries and groups satisfying algebraic and combinatorial rules. These were initially produced in the context of Kac–Moody theory to obtain infinite-dimensional analogues of semisimple algebraic groups. Adopting the point of view of discrete groups, we obtain in this way lattices for buildings which are both negatively curved and have dimension at least two, properties which are incompatible for Euclidean buildings. The problem then is to know to what extent the groups are new and whether classical properties of lattices of Lie groups are relevant. The first question leads to discussing linearity properties. The second one is partially answered by positive results concerning Kazhdan property (T) and cohomological finiteness properties, proved by several authors. Our guideline is the analogy with semisimple groups over local fields of positive characteristic.

1 Introduction

Kac–Moody groups were defined over arbitrary fields by J. Tits about fifteen years ago. The article [42] provides at the same time an abstract functorial approach, a new combinatorial structure and a concrete definition of the groups. It can be seen as the result of efforts made by several people including V. Kac and D. Peterson [25]. The concrete definition consists in generalizing a theorem due to R. Steinberg, giving an explicit presentation of split semisimple groups [39]. The main difference is that the generalized root system and Weyl group are now infinite. J. Tits [44] later refined the group combinatorics of [42], but the underlying idea is still the same, and if we want to keep available the crucial notion of opposite parabolic subgroups, we need to make the group act on a pair of buildings (instead of a single one in the algebraic theory). This is the starting point of the theory of twin buildings.

The group $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$ is a very special case of a Kac–Moody group over the finite field $\mathbf{F}_q$, and is also an arithmetic group. This paper reports on work based on the analogy between Kac–Moody groups over finite fields and arithmetic groups over function fields. The first convincing argument is that such a group is often a lattice of its geometry (a product of Euclidean buildings in the arithmetic group case). In the general case, we get products of buildings which range through a wide new class allowing strictly negative curvature and dimension at least two for the same space. Then comes the problem of knowing whether classical properties for lattices in Lie groups are relevant or true. Some groups are known to enjoy Kazhdan's property (T) or other cohomology vanishing, as well as some cohomological finiteness properties. These are arguments supporting the above analogy, but since the buildings are new, it is natural to ask whether the groups themselves are original. The new results in this paper concern non-linearity of Kac–Moody lattices as one way to answer this question.

Theorem 1.1. *An infinite Kac–Moody lattice over the field* $\mathbf{F}_q$ (*of characteristic* p*, with* q *elements*) *cannot be linear over any field of characteristic* $\neq p$.

The first part of this article presents some basic facts from Bruhat–Tits theory to define a geometric framework for a fundamental existence result of lattices in positive characteristic, due to H. Behr and G. Harder. The second part introduces Kac–Moody groups, adopting as soon as possible the point of view of discrete groups. Group combinatorics is used to stick to the previously defined geometric framework, where Kac–Moody groups appear as lattices. The last part lists some known facts about Kac–Moody lattices and classical properties, and settles strong restrictions on linearity possibilities.

By a *local field* we mean a locally compact non-Archimedean local field. Unless explicitly stated (e.g., in Sect. 3.1), $\mathbf{K}$ denotes the local field $\mathbf{F}_q((t))$, the Laurent power series with poles at 0. It is a completion of the ring of Laurent polynomials $\mathbf{F}_q[t, t^{-1}]$. Its ring of integers $\mathbf{F}_q[[t]]$ is denoted by $\mathcal{O}$.

2 A Classical Arithmetic Situation and its Geometric Formulation

A classical theorem by H. Behr and G. Harder proves the existence of lattices in reductive groups over local fields of positive characteristic [4], [21]. This section starts from a special case of this situation and makes use of Bruhat–Tits theory to get a geometric interpretation of the group-theoretic result. This provides the framework which enables generalization in the context of Kac–Moody groups.

2.1 A Special Case of the Behr–Harder Theorem

Let us assume that we are given a connected semisimple group $\mathbf{G}$ defined over the finite field $\mathbf{F}_q$. We can thus define the countable groups

$$\Lambda := \mathbf{G}(\mathbf{F}_q[t, t^{-1}]) \quad \text{and} \quad \Gamma := \mathbf{G}(\mathbf{F}_q[t^{-1}])\,.$$

In the classical terminology, Λ (resp. Γ) is a $\{0;\infty\}$- (resp. $\{0;\}$-)arithmetic subgroup of $\mathbf{G}\big(\mathbf{F}_q(t)\big)$. We also introduce Lie groups over local fields, namely

$$G := \mathbf{G}\big(\mathbf{F}_q(\!(t)\!)\big) \quad \text{and} \quad G_- := \mathbf{G}\big(\mathbf{F}_q(\!(t^{-1})\!)\big)\,.$$

Given a topological group G, a lattice of G is a discrete subgroup H in G such that the homogeneous space G/H carries a finite G-invariant measure. When G/H is compact, the lattice H is called uniform. Here is the fundamental existence result of lattices in our case – see [28, Theorem I.3.2.4] for the statement in its full generality.

Theorem 2.1 (H. Behr–G. Harder, a simple case). *The group Λ, diagonally embedded in the product $G \times G_-$, is a non-uniform lattice. The group Γ itself is a non-uniform lattice of G.*

In the general case, the algebraic group need not be defined over a finite field. The ground field may be a global field of positive characteristic, i.e., the field of rational functions on a curve C defined over $\mathbf{F}_q$. The proof involves reduction theory, for which the language of vector bundles over C is useful.

2.2 Geometry via Bruhat–Tits Theory

Now we adopt a geometric point of view on the above situation. This is made possible by the fundamental work of F. Bruhat and J. Tits [41].

Generalities. The group G (resp. G_-) acts on a *Euclidean building* Δ (resp. Δ_-), that is a labelled simplicial complex covered by Euclidean tilings, the *apartments*, satisfying certain incidence axioms [12]. Simplices are called *facets*, and the maximal ones, which all have the same dimension, are the *chambers*. A codimension one simplex is a *panel*; the *thickness* at a panel Π is the number of chambers containing Π in their closure. Two chambers are *adjacent* if their closures intersect along a panel; a sequence of consecutively adjacent chambers is a *gallery*.

Given the inclusion of a chamber c in an apartment $\mathbb{A}$, the stabilizer of $\mathbb{A}$ in G acts on $\mathbb{A}$ via the Coxeter group W generated by the inversions with respect to the panels of the chamber c. The definition of W does not depend on the choice of an apartment, and is an affine reflection group called the *Weyl group* of Δ (resp. Δ_-). Denoting by S the generating set of W for its Coxeter presentation, the labelling set of Δ is the power set of S. Panels are labelled by singletons $\{s\}$, hence a gallery gives rise to a word in W, just by writing the types of the panels successively crossed. If c and d are two chambers, the word associated to any minimal gallery from c to d gives the same element of W: the *W-distance* from c to d.

The G-action on $\Delta_{(-)}$ is *strongly transitive*, [36, Sect. 5]: the Weyl group W is transitive on chambers in an apartment, and G is transitive on pairs of chambers at fixed W-distance w. A facet fixator is called a *parahoric subgroup* of G, an *Iwahori subgroup* if the facet is a chamber. These are compact open subgroups of G.

As metric spaces, the buildings Δ and Δ_- are complete and non-positively curved: roughly speaking, geodesic triangles are at least as thin as Euclidean triangles. This is formalized by the notion of a CAT(0)-*space*, [11]. The latter property implies contractibility of Δ. By the Bruhat–Tits fixed point theorem [12, Chap. VI], a compact group of isometries of $\Delta_{(-)}$ must fix a point.

The SL_n case. We assume we are given a further arbitrary local field $\mathbf{F}$ of any characteristic, with ring of integers $\mathcal{V}$, uniformizer ϖ and finite residue field κ of characteristic l. The building $\mathcal{B}$ of $\mathrm{SL}_n(\mathbf{F})$ is known to be the set of ultrametric norms of $\mathbf{F}^n$ up to homothety, [13, Sect. 10]. Vertices of $\mathcal{B}$ are in one-to-one correspondence with $\mathbf{F}$-homothetic classes of $\mathcal{V}$-lattices in $\mathbf{F}^n$. The model of an apartment is the Euclidean tessellation of $\mathbf{R}^{n-1}$ by regular simplices.

By the Bruhat–Tits fixed point theorem, the maximal compact subgroups of $\mathrm{SL}_n(\mathbf{F})$ are the vertex stabilizers; consequently, by type-preservation and chamber-transitivity of $\mathrm{SL}_n(\mathbf{F})$ on $\mathcal{B}$, there are n conjugacy classes of maximal compact subgroups, parametrized by the vertices in the closure of a given chamber. All these subgroups are isomorphic to $\mathrm{SL}_n(\mathcal{V})$. Let us introduce $K := \ker\big(\mathrm{SL}_n(\mathcal{V}) \twoheadrightarrow \mathrm{SL}_n(\kappa)\big)$, the first congruence subgroup of $\mathrm{SL}_n(\mathcal{V})$. Denoting by M_n the $n \times n$ matrices, we see that K is a pro-l group since $K = \varprojlim_{j\geq 1}\{M \in \mathrm{id} + \varpi M_n(\mathcal{V}/\varpi^j) : \det M = 1\}$. Hence, $\mathrm{SL}_n(\mathcal{V})$ admits a finite index subgroup K where the only possible torsion is l-torsion. A simple matrix computation shows that when $\mathrm{char}(\mathbf{F}) = 0$, K is torsion-free.

Example 2.2.
1. For $G = \mathrm{SL}_3(\mathbf{K})$, the building Δ is two-dimensional. The apartments are tessellations of the Euclidean plane $\mathbf{R}^2$ by regular triangles, and small spheres centered at vertices – which may be seen as incidence graphs – are isomorphic to the projective plane $\mathbb{P}^2(\mathbf{F}_q)$.
2. The building of the rank one group $G = \mathrm{SU}_3(\mathbf{K})$ is a tree. Trees are the only hyperbolic metric spaces in the class of Euclidean buildings. A geometric consequence of the non-splitness of G is that it is not regular but only biregular, of valencies (=thicknesses) $1 + q$ and $1 + q^3$.

2.3 The Framework for Generalization

Given a locally finite cell complex X, its full automorphism group $\mathrm{Aut}(X)$ carries a natural structure of totally disconnected locally compact group. Hence, the problem of existence of lattices in $\mathrm{Aut}(X)$ makes sense. In view of the geometric definition of $\mathrm{Aut}(X)$ we will speak of *lattices of* X instead of lattices in $\mathrm{Aut}(X)$. Recall also that given an inclusion of groups $\Gamma < H$, the *commensurator* of Γ in H is the subgroup of H:

$$\mathrm{Comm}_H(\Gamma) := \{g \in H : \Gamma \cap g\Gamma g^{-1} \text{ is of finite index in } \Gamma \text{ and } g\Gamma g^{-1}\}\,.$$

The geometric setting we keep in mind for generalization consists of the following data. First, the geometries and the groups:

(Bui) Two isomorphic (locally finite) buildings $\Delta \simeq \Delta_-$;
(Top) Cocompact subgroups $G < \mathrm{Aut}(\Delta)$ and $G_- < \mathrm{Aut}(\Delta_-)$;
(Dis) A countable group Λ acting diagonally on $\Delta \times \Delta_-$.

These objects satisfy the following requirements.

(Lat) The group Λ is a lattice of $\Delta \times \Delta_-$;
(Den) The closure of the projection of Λ in $\mathrm{Aut}(\Delta_{(-)})$ is $G_{(-)}$;
(Com) For any point $x_{(-)} \in \Delta_{(-)}$, we have $\Lambda < \mathrm{Comm}_G\big(\Lambda(x_{(-)})\big)$.

Everything is defined in such a way that the groups in Sect. 2.1 satisfy the above axioms. Theorem 2.1 proves (Lat).

3 Kac–Moody Theory and the Generalization

The initial purpose of Kac–Moody theory was to define infinite-dimensional analogues of split semisimple Lie algebras and split semisimple algebraic groups. The case of Lie algebras was the starting point; for groups, the goal is now reached over arbitrary fields, according to J. Tits [42]. We explain in this section that we can see Kac–Moody groups over finite fields as generalizations of the above special cases of arithmetic groups.

3.1 Kac–Moody Groups

This section is an introduction to Kac–Moody theory with a view toward discrete group theory. We show that the analogy with algebraic groups provides precise decompositions of a Kac–Moody group and some of its subgroups. This first leads to see them as discrete groups and then gives the tools needed to prove properties supporting the analogy (e.g., the axioms Sect. 2.3). In this subsection, $\mathbf{K}$ denotes an arbitrary field.

Presentations. Split semisimple Lie algebras and algebraic groups have been so intensively studied that they can now be defined by generators and relations: for Lie algebras, this is the *Serre presentation*. This presentation is completely determined by a characteristic 0 field $\mathbf{K}$ and a so-called Cartan matrix $A = [A_{s,t}]_{s,t\in S}$, where S is a finite index set. Relaxing some conditions on A, we get the definition of a *generalized Cartan matrix*. It is an integral matrix $A = [A_{s,t}]_{s,t\in S}$ such that $A_{s,s} = 2$ for any s, $A_{s,t} \leq 0$ for $s \neq t$ and $A_{s,t} = 0 \Leftrightarrow A_{t,s} = 0$. A *Kac–Moody algebra* is defined by a Serre-type presentation, where the Cartan matrix is replaced by a generalized one, [24, Sect. 1]. According to Steinberg, a split semisimple algebraic group admits a presentation, too. The needed data are the same as for Lie algebras, but the field $\mathbf{K}$ here may be of arbitrary characteristic.

From spherical to twin combinatorics. J. Tits showed that such a definition for a generalized Cartan matrix leads to a non-trivial group, which besides admits nice group combinatorics [42]. A *BN-pair* is a combinatorial structure for groups which formalizes properties of isotropic reductive groups over arbitrary fields [6, 14.15 and 21.15]. For instance, a group endowed with a BN-pair admits a Bruhat decomposition by a general abstract argument, [8, Chap. IV.2]. The structure of BN-pair should be seen as a starting point, to be refined according to the group-theoretic situation under consideration. The refinement adjusted to semisimple groups over local fields is powerful enough to imply the Cartan and the Iwasawa decompositions, [41, Sect. 3]. In the Kac–Moody case, the structure involved is that of *twin root datum*, [44].

Let us consider the group $\Lambda = \Lambda(A, \mathbf{K})$ defined by the generalized Cartan matrix A and the field $\mathbf{K}$, [42]. The dictionary between BN-pairs and buildings is well known, [36, Sect. 5]: a group acting strongly transitively (Sect. 2.2) on a building admits a BN-pair and there is a standard way to construct a building from a BN-pair. In the Kac–Moody case, Λ admits two (twin) BN-pairs which cannot be deduced from one another by conjugation. Geometrically, this provides two buildings Δ and Δ_-. The presentation of Λ distinguishes the inclusion $c \subset \mathbb{A}$ (resp. $c_- \subset \mathbb{A}_-$) of a chamber in an apartment of Δ (resp. Δ_-). The chamber fixators $\Lambda(c)$ and $\Lambda(c_-)$ are the Borel subgroups of the BN-pairs.

Now we can wonder what makes the groups $\Lambda(A, \mathbf{K})$ usually infinite-dimensional. The answer is that the Coxeter systems (W, S) attached to the BN-pairs are isomorphic but the so-obtained Coxeter group W is generally infinite. We thus get two Bruhat decompositions indexed by the same Coxeter group but not carrying the same information. Since W acts simply transitively on the chambers of $\mathbb{A}$ (resp. $\mathbb{A}_-$), the apartments are infinite, too. And so is the (real) *root system*. To define this set, we introduce the free $\mathbf{Z}$-module over the symbols a_s, indexed by S. This module admits a W-action by a rule involving the matrix A, namely: $s.a_t = a_t - A_{st}a_s$. The roots are the elements wa_s for $w \in W$ and $s \in S$, a simple root just being one of the a_s's. The set of roots Φ keeps the nice properties of a finite root systems, [8, Chap. 5]. In particular, a root a has either all positive or all negative coordinates in the basis $\{a_s\}_{s \in S}$, the height of a still being the sum of them. Geometrically, we define the ***walls*** as the codimension one subcomplexes of $\mathbb{A}_{(-)}$ fixed by a given reflection in W. A root can be seen as a half apartment defined by a wall.

Though the root system involved is infinite, there is a strong analogy with the finite-dimensional case. Indeed, to each root a is attached a *root group* U_a. In the split case we are considering now, U_a is isomorphic to the additive group of $\mathbf{K}$. The positive (resp. negative) Borel subgroup i.e., the chamber fixator $\Lambda(c)$ (resp. $\Lambda(c_-)$), is the semi-direct product of a group abstractly isomorphic to a split torus with the normal subgroup U (resp. U_-) generated by all the positive (resp. negative) root groups. Let us choose w an element of the Weyl group W, of length $\ell(w)$ with respect to the generating system S. Then the fixator in U of the pair of chambers $\{c; w.c_-\}$ is $U \cap wU_-w^{-1}$. Set-theoretically, it is in bijection with the product in any order of the root groups indexed by the $\ell(w)$ roots containing c but not containing $w.c$, [42, 4.8]. This result is the abstract analogue of [6, 21.9].

Relative theory. What we described so far is split Kac–Moody theory. A natural question then is to define non-split groups. Once again, the model for this is the case of semisimple algebraic groups, that is Borel–Tits theory, [7], [6, Chap. V]. The main difficulty is that the algebro-geometric structure is lost while passing from algebraic to Kac–Moody groups. The substitutes for this are an infinite-dimensional adjoint representation, [32, 9], and the action of the group on its twin building. A typical argument consists in combining the Bruhat–Tits fixed point theorem to get a relevant algebraic subgroup and then to use classical arguments. This is how conjugation of Cartan subgroups is proved [32, 10.4]. These arguments apply to Galois group actions, to define a new class of Kac–Moody groups, the *almost split Kac–Moody groups.* The main result is a descent theorem asserting that the $\mathbf{K}$-rational points of an almost split Kac–Moody group still admit the structure of twin root datum, the buildings so-obtained appearing as Galois-fixed points in the original split ones [32, Sect. 12]. Hence, the above combinatorial properties stay true for almost split Kac–Moody groups up to minor changes. Root groups are $\mathbf{K}$-points of root groups in algebraic groups.

We specialize now our situation to the case where $\mathbf{K}$ is the finite field $\mathbf{F}_q$. We get then root groups of cardinality q or q^3. In view of the above description of the groups U_w, we also get the following lemma, useful to study linearity properties in Sect. 4.2

Lemma 3.1. *An almost split Kac–Moody group over* $\mathbf{F}_q$ *contains arbitrarily large* p*-groups.*

Being generated by its standard Cartan subgroup and the simple positive and negative root groups, a Kac–Moody group over a finite field is always finitely generated. Since we use the non-positively curved realizations for the buildings [16], only spherical facets appear and their links are buildings of finite groups of Lie type, [32, 6.2.3]. In particular, the buildings are then locally finite.

Example 3.2. 1. Values of Chevalley groups on rings of Laurent polynomials are *Kac–Moody groups of affine type.* For instance, $\Lambda := \mathrm{SL}_n(\mathbf{K}[t, t^{-1}])$ is Kac–Moody and operates diagonally on the product of the Bruhat–Tits buildings of $\mathbf{SL}_n\big(\mathbf{F}_q(\!(t)\!)\big)$ and $\mathbf{SL}_n\big(\mathbf{F}_q(\!(t^{-1})\!)\big)$. A vertex fixator in Λ is the intersection of a maximal compact subgroup with Λ, hence is isomorphic to $\mathbf{SL}_n\big(\mathbf{F}_q[t^{-1}]\big)$. This is a step toward the axioms in Sect. 2.3.

2. Let us turn now to the case of an infinite rank 2 Kac–Moody group, defined by the generalized Cartan matrix $\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ with $mn \geq 4$. The buildings involved are trees and an apartment is the real line tessellated by the integers. We are interested in an abstract group-theoretic consequence of commutator relations between root groups. These relations were determined up to sign by J. Morita [30]. The set of real roots in an infinite rank 2 Kac–Moody root system can roughly be seen as the half-lines defined by the vertices in an apartment. The roots split naturally into two halves, those in the same half being oriented the same way. Let ϕ be one of these halves, and consider the group U_ϕ generated by the root groups indexed by them. A more detailed description of root systems [24, Exercises p.75–76] and [26, Sect. 4], combined with the commutator relations, shows that basically two cases occur. Either m and n are both ≥ 2 and then all root groups indexed by ϕ commute, in which case U_ϕ is the additive group of the direct sum of the root groups. Or one off-diagonal coefficient is 1 and the group U_ϕ is metabelian. In any case, if we are working over the finite field $\mathbf{F}_q$, the group U_ϕ is p-torsion with a uniform bound on the orders. This will be a useful fact in Sect. 4.2.

Remark 3.3. The fact that there exist big abelian subgroups in rank 2 Kac–Moody groups was first remarked by J. Morita [31, Sect. 3, Ex. 6].

3.2 Kac–Moody Lattices

We are now in position to check the axioms of the geometric framework in Sect. 2.3. As suggested by the notation, we make an almost split Kac–Moody group $\Lambda(A,q)$ over $\mathbf{F}_q$ play the role of the group Λ. The isomorphism between the associated buildings comes from opposition of the signs of the roots, via the BN-pairs defining them. Recall that Δ is the Moussong–Davis realization of an abstract building [16], hence only the spherical facets appear. The link of such a facet is the building of a finite group of Lie type, [32, 6.2.3]. This proves (Bui). According to the definition of Λ by the Steinberg presentation [42], it follows from the finiteness of $\mathbf{F}_q$ that Λ is finitely generated, which proves (Dis). In this context, we have not defined G and G_- yet. In order to satisfy (Den), the relevant topological groups are not the full automorphism groups. Instead, we define G and G_- as the closures $\overline{\Lambda}^{\mathrm{Aut}(\Delta)}$ and $\overline{\Lambda}^{\mathrm{Aut}(\Delta_-)}$ respectively. Since Λ is transitive on chambers for its action on each building, G and G_- are cocompact; hence (Top). The following lemma just says that (Com) is satisfied. Its proof is very easy once a refined Bruhat decomposition taking into account the existence of root groups has been introduced, [35].

Lemma 3.4. *For any point* $x_{(-)} \in \Delta_{(-)}$ *we have:* $\Lambda < \mathrm{Comm}_G\big(\Lambda(x_{(-)})\big)$.

The remaining axiom is (Lat), for which we have to make an assumption on q. More precisely, we have [15], [33]:

Theorem 3.5. *Let Λ be an almost split Kac–Moody group over the finite field $\mathbf{F}_q$, with infinite Weyl group W . Denote by $\sum_{n\geq 0} d_n t^n$ the growth series of W, and assume $\sum_{n\geq 0} d_n/q^n < \infty$. Then, Λ is a lattice of $\Delta_+ \times \Delta_-$ for its diagonal action, and for any point $x_{(-)} \in \Delta_{(-)}$, $\Lambda(x_{(-)})$ is a lattice of Δ_+. These lattices are not uniform.*

Remark 3.6. 1. The proof of this theorem is another illustration of the fact that combinatorial properties derived by analogy with the algebraic group case are precise enough to prove results on the discrete group side. A Levi type decomposition [33] implies that given a point in each building, the fixator of the pair is finite: this is the discreteness part. The finiteness of the covolume comes from another abstract decomposition where the groups U_w come into play [33].
2. The condition $\sum_{n\geq 0} d_n/q^n < \infty$ is sharp in the split case.
3. The $\Lambda(x_-)$-action on the building Δ admits a fundamental domain, explicitly described as an intersection of roots (seen as half-spaces) in an apartment. This fact was noticed by J. Tits – see [1, Sect. 3].

From now on, we assume that q is large enough to satisfy (Lat).

3.3 New Buildings and Automorphism Groups

Given an arbitrary building or a more general cell-complex, it is not clear that its full automorphism group is big enough (to be seen as a generalized Lie group, for instance). Since the Kac–Moody buildings are obtained via BN-pairs, their automorphism groups are by definition strongly transitive. There is another construction procedure, consisting in defining a complex of groups [11] and taking its covering space and fundamental group. This provides in one stroke a cell-complex and an automorphism group. Local criteria on a complex of groups to decide whether its covering space is a building or not were determined by D. Gaboriau and F. Paulin [18].

As already mentioned, no Euclidean building of dimension at least two can be a hyperbolic metric space. Hence the new class of *hyperbolic buildings* – where apartments are hyperbolic tilings instead of Euclidean ones – is interesting at least because it provides CAT(-1)-spaces with nice incidence properties. These spaces, their full automorphism groups as well as their lattices were studied by several authors [18], [20], [9]. The article by M. Bourdon and H. Pajot in this book is dedicated to a nice two-dimensional case of these buildings [10], where apartments are tilings of the hyperbolic plane by regular right-angled r-gons, and small spheres at vertices are complete bipartite graphs.

Some CAT(-1) buildings do come from Kac–Moody groups over finite fields [34]. The latter two-dimensional examples are Kac–Moody as soon as the thickness is constant and equal to $1 + q$, q a prime power. Still, another striking result about hyperbolic buildings is the existence of an uncountable family of buildings with the same incidence structure at vertices and same shape of apartments [20]. Hence, the Kac–Moody construction procedure is far from being exhaustive. A definition of additional invariants by F. Haglund is a step toward classification in suitable cases [19].

4 Questions Arising from the Generalization

The question we are considering now is whether classical properties of lattices in Lie groups are relevant to this new class of discrete groups. We will see that some positive answers have already been given. Then we discuss the more basic problem of linearity over arbitrary fields, which says to what extent Kac–Moody lattices are new. The last subsection deals with an abstract combinatorial approach which allows to go further in the generalization.

4.1 Classical Properties of Lattices

In order to present positive results concerning classical properties, we proceed from the widest class to more particular classes of buildings and groups.

Residual finiteness. As noted by M. Burger, a lattice $\Lambda(x_{(-)})$ is always residually finite. This can be seen thanks to the geometric definition of $\Lambda(x_{(-)})$, as the fixator in Λ of a point $x_{(-)} \in \Delta_-$. Taking fixators of balls of increasing radius $n \in \mathbf{N}$ around $x_{(-)}$ provides a family of normal subgroups $\Lambda_n(x_{(-)}) \lhd \Lambda(x_{(-)})$ with trivial intersection. The groups $\Lambda_n(x_{(-)})$ are analogues of congruence subgroups.

Commensurators. strictly speaking, the main property is axiom (Com) of Sect. 2.3. Still, recall that in Sect. 3.2, axiom (Den) imposed to define the topological groups G and G_- as closures of the countable group Λ. Forgetting for a while discrete groups, this calls for arguments supporting the analogy between G and a semisimple group over a positive characteristic local field. Actually, G admits a refined Tits system structure (a notion defined in [25]) and for any spherical facet F, the subgroup $G_F = \mathrm{Fix}_G(F)$ decomposes as $G_F = M(F) \ltimes U_F$, where $M(F)$ is a finite group of Lie type and U_F is a pro-p group [35]. The group U_F contains arbitrarily large p-groups.

For the next properties, we restrict our attention to special cases of Kac–Moody groups. Namely, we consider the groups Λ whose associated buildings Δ and Δ_- are hyperbolic in the sense of [18] (see Sect. 3.3; in particular, they are CAT(-1)-spaces, [11]).

Amenability. According to M. Burger and S. Mozes [14], if a CAT(-1)-space X admits a closed cocompact automorphism group G, then for any boundary point $\xi \in \partial_\infty X$ the fixator $\mathrm{Fix}_G(\xi)$ is amenable. This applies to our groups $\overline{\Lambda}^{\mathrm{Aut}(\Delta)}$ and $\overline{\Lambda}^{\mathrm{Aut}(\Delta_-)}$ as above. The comparison with symmetric spaces and Bruhat–Tits buildings suggests considering the fixators of boundary points as generalized parabolic subgroups of G.

Kazhdan's property (T). The first investigation concerning property (T) for groups acting on exotic geometries is due to A. Żuk [45]. The idea is to extend the use of Garland's combinatorial Laplace operator to this situation, since what is mainly involved is the local structure of the buildings. The results obtained are cohomology vanishing theorems which are stronger than (T). Recall that property (T) amounts to vanishing of 1-cohomology with arbitrary unitary representations as coefficients. The following stronger result is independently due to J. Dymara–T. Januszkiewicz [17] and L. Carbone–H. Garland [15]. A *compact hyperbolic* Coxeter group is one coming from the tessellation of a hyperbolic space by a compact simplex.

Theorem 4.1. *Let Λ be a Kac–Moody group over $\mathbf{F}_q$ with compact hyperbolic Weyl group. Then for q large enough and $1 \leq k \leq n-1$, the continuous cohomology groups $H^k_{\mathrm{ct}}\big(\mathrm{Aut}(\Delta), \rho\big)$ with coefficients in any unitary representation ρ, vanish.*

Cohomological finiteness. Finiteness properties of arithmetic groups in positive characteristic is a difficult subject, [12, Chap. VII]. Results for Kac–Moody groups were obtained by P. Abramenko, [1], [2]. They involve both the size of the ground field $\mathbf{F}_q$ and the coefficients of the generalized Cartan matrix A. Definition of these properties – F_n or FP_n – involves cohomology or actions on complexes, but for the first two degrees they are closely related to finite generation and finite presentability.

Theorem 4.2. *Let $\Lambda(A, q)$ be a split Kac–Moody group, and Γ be a chamber fixator.*

(i) If A contains a rank two submatrix which is not Cartan, then Γ is not finitely generated (for arbitrary q);
(ii) If $q > 3$ and if all rank two, but not all rank three, submatrices are Cartan, then Γ is finitely generated but not finitely presentable;
(iii) If $q > 13$ and if all rank three submatrices are Cartan, then Γ is finitely presentable.

This theorem is due to P. Abramenko, who also determined generalized Cartan matrices whose lattices (for one building) are not finitely generated over tiny fields, while they have (T) for a larger q. Hence, the condition on thickness is necessary for property (T), while it is not in the classical case of simple groups over local fields (the rank being then the only criterion [28, III.5.6]).

4.2 Non-Linearity

From Sect. 3.3, it is clear that Kac–Moody buildings form a wide new class of geometries. Dealing with linearity is an attempt to decide to what extent the groups themselves are new. The first result involves almost split Kac–Moody groups $\Lambda(A, q)$, that is lattices of products of buildings.

Proposition 4.3. *Let Λ be an almost split Kac–Moody group over $\mathbf{F}_q$ with infinite Weyl group. Then:*

(i) A chamber fixator in Λ cannot be linear over any local field of characteristic $\neq p$;
(ii) The group Λ cannot be linear over any field of characteristic $\neq p$.

Proof. Let us pick a chamber fixator $\Gamma := \mathrm{Fix}_\Lambda(c)$. We assume with no loss of generality that the chamber c is the standard positive one. Hence, Γ is the Borel subgroup of the positive Tits system of Λ.

We first show that (i) implies (ii). Indeed, the group Λ inherits the non-linearity properties of its subgroup Γ. But since Λ is finitely generated, if it were linear over $\mathbf{F}$ of characteristic $l \neq p$, it would be contained in a group $\mathrm{GL}_n(\mathbf{F}_0)$, where $\mathbf{F}_0$ is finitely generated as a field, and of characteristic l. Such a field $\mathbf{F}$ can always be embedded in a local field of characteristic l: contradiction with (i).

We suppose now we are given a local field $\mathbf{F}$ of characteristic $l \neq p$. In order to prove non-linearity of Γ over $\mathbf{F}$, it is enough to show that no abstract homomorphism $\Gamma \to \mathrm{SL}_n(\mathbf{F})$ can be injective. Let us fix such a homomorphism ι and consider the Bruhat–Tits building $\mathcal{B}$ of $\mathrm{SL}_n(\mathbf{F})$. We adopt all the notations of Sect. 2.2. The first congruence subgroup K of $\mathrm{SL}_n(\mathcal{V})$ is a subgroup of finite index m. It is torsion-free or its only torsion elements are l-torsion according to whether l is 0 or a prime number. According to Sect. 3.1, we can choose a p-subgroup $U_w < \Gamma$ of cardinality $> m$: choose w in the infinite Weyl group W such that $q^{\ell(w)} > m$. By non-positive curvature and type-preservation, $\iota(V)$ must fix a vertex x in $\mathcal{B}$. The maximal parahoric subgroup $\mathrm{SL}_n(\mathcal{V})$ breaks into m classes modulo K, so we get two different elements v and v' whose images under ι lie in the same class. Hence, the p-torsion element $v^{-1}v'$ must be sent in the torsion-free or pro-l group K. Consequently, $\iota(v^{-1}v') = 1$ and the homomorphism $\iota|_V$ is already non-injective. □

Remark 4.4. 1. The assumption on W is natural since when W is finite, the matrix A must be a Cartan matrix, in which case Λ is a finite group of Lie type.

2. The trick of embedding a finitely generated field into a local field appears in [40]. The idea of applying it to the proof of a non-linearity result comes from [27]. In the latter reference was also found the idea of using torsion properties of congruence subgroups.

3. The problem in equal characteristic p is more delicate, since for example $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$ is a Kac–Moody group of affine type, and is obviously linear. Hence, the problem is rather the following:

Question 4.5. Does there exist a generalized Cartan matrix A such that the group $\Lambda(A, q)$ is not linear in characteristic p either?

The argument for the implication (i) $\Rightarrow$ (ii) in the above proof also shows that a chamber fixator Γ in Λ cannot be linear over any field of characteristic $\neq p$, provided we know that Γ is finitely generated. But this is not always the case (consider the case of a non-uniform lattice of a tree arising from an infinite rank 2 Kac–Moody group). In order to obtain a more general statement, we need the technical points on split rank 2 Kac–Moody groups presented in the second example of 2.1.

Theorem 4.6. *Let Λ be a split Kac–Moody group over $\mathbf{F}_q$ with infinite Weyl group W. Then a chamber fixator in Λ cannot be linear over any field of characteristic $\neq p$.*

Proof. We fix the inclusion of a chamber c in an apartment $\mathbb{A}$. Up to modifying the generalized Cartan matrix, we assume with no loss of generality that the chamber is the standard positive one. Hence the group we are regarding is the positive Borel subgroup Γ. Let us start with a simple root a_s, whose associated reflection is denoted by s. Then, as W is infinite, there exists a reflection r such that s and r generate an infinite dihedral group, [23, Proposition 8.1, p. 309]. The reflection r fixes the wall ∂b of a root b. Let us consider the set of roots of the form $b + ka_s$, $k \in \mathbf{Z}$. We denote by a the root of minimal height in this set, by r_a the associated reflection and by W' the group $\langle s, r_a \rangle$. Then, according to [29, Proposition 8.1], $\{a_s; a\}$ is the basis of an infinite rank 2 Kac–Moody root system, whose set of (real) roots is $\Phi := W'a_s \cup W'a$.

Setting $t := r_a s$ and $\phi := \bigcup_{n \in \mathbf{Z}} t^n\{a_s; -a\}$, we know by Sect. 3.1 that the group U_ϕ generated by the root groups indexed by ϕ is infinite, p-torsion with uniform bound on the orders. Since $\{a_s; a\}$ is the basis of an infinite root system, we must have $a_s \subset -a$ or $-a \subset a_s$, [32, 5.4] (in the technical Kac–Moody terminology, the pair $\{a_s; a\}$ is *not* prenilpotent). Hence, there is an infinite number of positive roots in ϕ, and $V := \Gamma \cap U_\phi$ is also infinite of finite exponent.

Let us now pick a field $\mathbf{F}$. By an argument involving Zariski closure [28, Lemma VIII.3.7], the image of $V < \Gamma$ by an abstract homomorphism ι to $\mathrm{GL}_n(\mathbf{F})$ contains a finite index unipotent subgroup. A unipotent group in characteristic 0 is torsion-free, whereas in prime characteristic l, any unipotent element is l-torsion. Indeed, if $g \in \mathrm{GL}_n(\mathbf{F})$ is unipotent, then $(g - \mathrm{id})^{l^k} = 0$ for some k, and $g^{l^k} = \big(\mathrm{id} + (g - \mathrm{id})\big)^{l^k} = \mathrm{id}$. This shows that if ι is injective, then the field $\mathbf{F}$ has to be of characteristic p, which proves the proposition. $\square$

Remark 4.7. 1. The argument of the first paragraph of the proof (giving the basis of a rank two system) is used in [5] in order to make an abstract study of root strings.

2. A deeper knowledge of commutation relations between root groups in almost split groups would probably lead to the same restriction for (even non-finitely generated) chamber fixators in that case. This question is relevant to the abstract theory of infinite root systems, [3], [22].

4.3 A Further Combinatorial Generalization

As explained in Sect. 3.1, the tools developed to study Kac–Moody groups are combinatorial. More precisely, J. Tits in [44] elaborated on the notion of a (twin) BN-pair, and proposed a list of axioms sharply refining it: these axioms define the structure of a twin root datum (see Sect. 3.1). Moreover in [43] J. Tits sketched a theoretical construction of twin root data whose associated geometries are twin trees, a special case of twin buildings whose study is initiated in [37], [38]. The construction in [43] is so flexible that it provides trees of valencies $1+q$ and $1+q'$ for q and q' powers of different primes. These trees cannot arise from any Kac–Moody group. Hence, the abstract framework of twin root data, though adjusted to the Kac–Moody situation, strictly contains it.

In [35] non Kac–Moody twin root data are concretely constructed, whose buildings are two-dimensional hyperbolic. The rank of the buildings is arbitrarily large, and things can be done in such a way that two panels of distinct types have thicknesses of distinct characteristics: this is thus a generalization in arbitrary rank of J. Tits' work on twin trees. If we turn back to linearity properties, a corollary of [35] is the existence of lattices of Kac–Moody type which cannot be linear over any field.

References

1. P. Abramenko: Twin buildings and applications to S-arithmetic groups. Lecture Notes in Mathematics **1641**, Springer-Verlag, Berlin, 1996
2. P. Abramenko, to appear in a proceedings volume of the conference Geometric and Combinatorial Group Theory (Bielefeld, August 1999), London Math. Soc. Lecture Notes Ser.
3. N. Bardy: Systèmes de racines infinis. Mém. Soc. Math. Fr. **65**, 1996
4. H. Behr: Endliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkörpern. Invent. Math. **7** (1969) 1–32
5. Y. Billig, A. Pianzola: Root strings with two consecutive real roots. Tôhoku Math. J. **47** (1995) 391–403
6. A. Borel: Linear algebraic groups. Graduate Texts in Math. **126**, Springer-Verlag, New York, 1990
7. A. Borel, J. Tits: Groupes réductifs. Inst. Hautes Études Sci. Publ. Math. **27** (1965) 55–151

8. N. Bourbaki. Groupes et algèbres de Lie IV–VI, Masson, Paris, 1981
9. M. Bourdon: Immeubles hyperboliques, dimension conforme et rigidité de Mostow. Geom. Funct. Anal. **7** (1997) 245–268
10. M. Bourdon, H. Pajot: Quasi-conformal geometry and hyperbolic geometry. In Rigidity in Geometry and Dynamics, M. Burger, A. Iozzi eds., Springer-Verlag, 2002
11. M. Bridson, A. Hæfliger: Metric spaces of nonpositive curvature. Springer-Verlag, Berlin, 1999
12. K. Brown: Buildings. Springer-Verlag, New York, 1989
13. F. Bruhat, J. Tits: Groupes réductifs sur un corps local I. Données radicielles valuées. Inst. Hautes Études Sci. Publ. Math. **41** (1972) 5–251
14. M. Burger, S. Mozes: CAT(−1)-spaces, divergence groups and their commensurators. J. Amer. Math. Soc. **9** (1996) 57–93
15. L. Carbone, H. Garland: Lattices in Kac–Moody groups. Math. Res. Lett. **6** (1999) 439–447
16. M. Davis: Buildings are CAT(0). In Geometry and Cohomology in Group Theory, P. H. Kropholler, G. A. Niblo, R. Stöhr eds., London Math. Soc. Lecture Notes Ser. **252** (1997) 108–123
17. J. Dymara, T. Januszkiewicz: New Kazhdan groups. Geom. Dedicata **80** (2000) 311–317
18. D. Gaboriau, F. Paulin: Sur les immeubles hyperboliques. To appear in Geom. Dedicata
19. F. Haglund: Existence, unicité et homogénéité de certains immeubles hyperboliques. Preprint Univ. Paris 11 Orsay, 1999
20. F. Haglund, F. Paulin: Simplicité de groupes d'automorphismes d'espaces à courbure négative. The Epstein birthday schrift. Geom. Topol. Monogr. 1, Coventry (1998) 181–248
21. G. Harder: Minkowskische Reduktionstheorie über Funktionenkörpern. Invent. Math. **7** (1969) 33–54
22. J.-Y. Hée: Systèmes de racines sur un anneau commutatif totalement ordonné. Geom. Dedicata **37** (1991) 65–102
23. J.-Y. Hée: Sur la torsion de Steinerg–Ree des groupes de Chevalley et des groupes de Kac–Moody. Thèse d'État Univ. Paris 11 Orsay, 1993
24. V. Kac: Infinite dimensional Lie algebras (third edition). Cambridge Univ. Press, 1990
25. V. Kac, D. Peterson: Defining relations for certain infinite-dimensional groups. Astérisque Hors-Série (1984) 165–208
26. J. Lepowski, R. Moody: Hyperbolic Lie algebras and quasi-regular cusps on Hilbert modular surfaces. Math. Ann. **245** (1979) 63–88
27. A. Lubotzky, S. Mozes, R. J. Zimmer: Superrigidity for the commensurability group of tree lattices. Comment. Math. Helv. **69** (1994) 523–548
28. G. A. Margulis: Discrete subgroups of semisimple Lie groups. Springer-Verlag, Berlin, 1991
29. R. Moody, A. Pianzola: On infinite root systems. Trans. Amer. Math. Soc. **315** (1989) 661–696
30. J. Morita: Commutator relations in Kac–Moody groups. Proc. Japan Acad. Ser. A Math. Sci. **63** (1987) 21–22
31. J. Morita: Root strings with three or four real roots in Kac–Moody root systems. Tôhoku Math. J. **40** (1988) 645–650

32. B. Rémy: Groupes de Kac–Moody déployés et presque déployés. To appear in Astérisque, Société Mathématique de France
33. B. Rémy: Construction de réseaux en théorie de Kac–Moody. C. R. Acad. Sci. Paris **329** Série I (1999) 475–478
34. B. Rémy: Immeubles de Kac–Moody hyperboliques. Groupes non isomorphes de même immeuble, to appear in Geom. Dedicata
35. B. Rémy, M. Ronan: Topological groups of Kac–Moody type, Fuchsian buildings and their lattices. In preparation
36. M. Ronan: Lectures on buildings. Academic Press, Boston, MA, 1989
37. M. Ronan, J. Tits: Twin trees. I. Invent. Math. **116** (1994) 463–479
38. M. Ronan, J. Tits: Twin trees. II. Local structure and a universal construction. Israel J. Math. **109** (1999) 349–377
39. R. Steinberg: Lectures on Chevalley groups. Yale mimeographed notes, 1968
40. J. Tits: Free groups in linear groups. J. Algebra **20** (1972) 250–270
41. J. Tits: Reductive groups over local fields. In Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math. Amer. Math. Soc. **33** 1 (1979) 29–69
42. J. Tits: Uniqueness and presentation of Kac–Moody groups over fields. J. Algebra **105** (1987) 542–573
43. J. Tits: Théorie des groupes. Résumé de cours, annuaire Collège de France (1989) 81–95
44. J. Tits: Twin buildings and groups of Kac–Moody type. London Math. Soc. Lecture Notes Ser. **165** (1992) 249–286
45. A. Żuk: La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres. C. R. Acad. Sci. Paris **323** (1996) 453–458