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# Integrability of induction cocycles for Kac-Moody groups

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**Abstract.** We prove that whenever a Kac-Moody group over a finite field is a lattice of its buildings, it has a fundamental domain with respect to which the induction cocycle is $L^p$ for any $p \in [1; +\infty)$. The proof uses elementary counting arguments for root group actions on buildings. The applications are the possibility to apply some lattice superrigidity, and the normal subgroup property for Kac-Moody lattices.

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## Introduction

Let $\Lambda$ be an infinite (possibly twisted) Kac-Moody group over a finite field. It acts diagonally on the product $X_- \times X_+$ of its twinned buildings. We may, and shall, assume that the action is faithful (the kernel of the action lies in the finite center of $\Lambda$). The $\Lambda$-action on a simple factor is not discrete, and we call *geometric completion* of positive (resp. negative) sign the closure $\overline{\Lambda}_+$ (resp. $\overline{\Lambda}_-$) of the image of $\Lambda$ in the action on the positive (resp. negative) building [RR03, 1.B].

If we set $G := \overline{\Lambda}_- \times \overline{\Lambda}_+$, then $\Lambda$ can be seen as a discrete subgroup of $G$ via the diagonal embedding. If we denote by $W(t)$ the growth series of the common Weyl group $W$ of $X_-$ and $X_+$, then the finiteness of $W(\frac{1}{q})$ implies that $\Lambda$ is a lattice of $G$ [CG99], [Rem99]. By construction the lattice $\Lambda$ is *irreducible*, i.e. its projections on the factors $\overline{\Lambda}_\pm$ are dense. Moreover the group $\Lambda$ is generated by finitely many finite subgroups, which provides a length function $\ell_\Lambda$. To any fundamental domain $X$ for $G/\Lambda$ is attached a cocycle $\alpha_X : G \times X \to \Lambda$ by: $\alpha_X(g, x) = \lambda \Leftrightarrow gx\lambda \in X$. This cocycle is useful to induce representations of lattices in Lie groups, and Y. Shalom's work shows that it is a powerful tool to

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prove deep rigidity results (where the ambient topological groups needn't be Lie groups) [Sha00a], [Sha00b]. Our main purpose is to prove the following.

**Theorem.** *Let $\Lambda$, $G$ and $W$ be as above. Then, there is a fundamental domain $D$ for $G/\Lambda$, which is a countable union of compact open subsets $\{D_w\}_{w \in W}$ and such that for any $p \in [1; +\infty)$ and any $g \in G$, we have:*

$$\int_D \ell_\Lambda\big(\alpha_D(g, d)\big)^p \mathrm{d}\mu(d) < +\infty,$$

*whenever the minimal order $q$ of the root groups satisfies $W(\frac{1}{q}) < +\infty$.*

In other words, for the cocycle $\alpha_D$ to be $L^p$, there is no further assumption on a Kac-Moody group $\Lambda$ over a finite field than being a lattice of its buildings. Note that for $p = 2$, the above integrability is nothing else than condition (1.5) of [Sha00a, 1.II p.14]. This enables us to deduce:

**Corollary.** *All the results valid under the hypothesis* **(0.1)** *in the above cited paper by Y. Shalom, are still valid when the uniform lattice is replaced by a Kac-Moody group over a finite field, provided the latter group is a lattice of its twinned buildings, e.g. when $W(\frac{1}{q}) < +\infty$.*

We note that the idea to replace the cocompactness of a closed subgroup by representation-theoretic conditions (and in particular by integrability conditions) appears in [Mar91, III.1]. The results alluded to in the Corollary contain a superrigidity theorem for irreducible lattices, an arithmeticity theorem, a superrigidity theorem for actions on trees... The square integrability is also one ingredient needed in a recent paper by N. Monod, providing a very general superrigidity theorem for actions of irreducible lattices on arbitrary complete CAT(0)-spaces [Mon04], see Subsect. 3.1 for further details. We note that the only so far available superrigidity theorem for Kac-Moody lattices was a commensurator superrigidity [Bon03], which is easier to obtain than a lattice superrigidity. Conversely, we can see Kac-Moody lattices as a substantial enrichment of the list of examples for which one really needs the new rigidity results (with respect to those previously proved in [Mar91]).

Still, the main application of the square-integrability we have in mind is the normal subgroup property for Kac-Moody lattices, a well-known property for irreducible lattices of higher-rank Lie groups over local fields [Mar91, VIII.2]. In our case, this is a joint work with U. Bader and Y. Shalom which uses a general result due to them about amenability of factor groups of irreducible lattices [BS03], and a result due to Y. Shalom about property (T) for the same quotients [Sha00a]. This provides the following (see [BS03, Theorem 1.5] and Theorem 21 of the present paper):

**Theorem (with U. Bader and Y. Shalom).** *Let $\Lambda$ be a Kac-Moody group over a finite field, with irreducible Weyl group. Assume it is a lattice of the product of its* *twinned buildings. Then any normal subgroup of $\Lambda$ either has finite index or lies in the finite center $Z(\Lambda)$.*

Using a result on just infinite groups due to J.S. Wilson, we can then prove (Corollary 22):

**Corollary.** *If $\Lambda$ is center-free, e.g. because it is adjoint, and if it is not residually finite, then $\Lambda$ is virtually simple.*

Note that many Kac-Moody groups over finite fields are not linear over any field [Rem03b], therefore are potentially non-residually finite, but no example of a non-residually finite Kac-Moody group has been given yet.

This paper is organized as follows. In the first section, we recall the basic combinatorial notions for groups with twin root data and their geometric completions, and we describe the fundamental domain $D$ in these terms. In the second section, we prove the above $L^p$-integrability. The proof uses a quantitative version of the fact that a fundamental domain for the diagonal action of a group with twin root datum on the product of its twinned buildings, is the product of a negative chamber by a suitable positive apartment. In the third section, we provide applications of the integrability. We first show that this enables us to apply the superrigidity results proved by Y. Shalom [Sha00a], and leads to ask whether N. Monod's more recent work [Mon04] can be applied. The second application is group-theoretic, it proves that Kac-Moody lattices enjoy the normal subgroup property.

It is a great pleasure to thank Uri Bader, Nicolas Monod and Yehuda Shalom. Their comments and encouragements were extremely useful in writing this paper.

## 1. Twin building, automorphism groups, fundamental domain and cocycle

We introduce some automorphism groups of twin buildings, and we show that their combinatorial properties enable to construct nice fundamental domains.

### *1.1. Twin building and automorphism groups*

Let $(W, S)$ be a Coxeter system [Bou81, IV.1]. Our starting point is a group $\Lambda$ admitting a twin root datum $\big(\Lambda, \{U_a\}_{a\in\Phi}, T\big)$ whose root groups $U_a$ are indexed by the root system $\Phi$ of $(W, S)$ [Tit87], [Rem02, 5.4.1]. We assume that all the root groups $U_a$, as well as the subgroup $T$, are finite and that the Weyl group $W$ is infinite. This is the context chosen in [RR03, Sect. 1] to define topological groups generalizing semisimple groups over local fields of positive characteristic. We will recall briefly the main notions, properties and references; further details and motivations are given in [Rem03a, §3].

The notion of a twin root datum was given in a paper by J. Tits [Tit92]. In the axioms, the Weyl group $W$ appears as the image of a quotient map $\nu : N \to W$

with kernel $T$, where $N$ is a subgroup of $\Lambda$. The group $T$ normalizes each root group $U_a$ and we have: $nU_an^{-1} = U_{\nu(n).a}$ for any root $a \in \Phi$ and any $n \in N$. It is also required that any element in $N$ lifting a reflection $s$ in the canonical generating set $S$ of $W$, lies in the finite subgroup $\langle U_{a_s}, U_{-a_s}, T\rangle$, where $a_s$ is the simple root attached to $s$. Each group $M_s := \langle U_{a_s}, U_{-a_s}, T\rangle$ may be seen as a generalized rank one finite group of Lie type (it is, strictly speaking, a finite group of Lie type when $\Lambda$ is a Kac-Moody group over a finite field). The root groups $\{U_a\}_{a\in\Phi}$ and $T$ are required to satisfy other additional properties for which we refer to [Rem02, 1.6]. At last, $\Lambda$ is generated by $T$ and the $U_a$'s, so in view of the previous remarks, the group $\Lambda$ is finitely generated.

**Definition 1.** (i) *We denote by $\ell_W$ the length function on $W$ associated to the canonical generating set $S$.*
(ii) *We denote by $\ell_\Lambda$ the length function on $\Lambda$ associated to the finite set of generators given by the union of the groups $M_s$, when $s$ ranges over $S$.*

In what follows, we are mainly interested in the geometric counterpart to this, which involves the structure of a *building* for which we adopt the viewpoint of chamber systems [Ron89, §2]. To $\Lambda$ is associated a twin building $(X_+, X_-, w^*)$ together with a twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$ and a pair of opposite chambers $\{c_-; c_+\}$ in $\Sigma$ [Tit92], [Abr97], [Rem02, §2]. By non-triviality and finiteness of the root groups, the buildings $X_\pm$ are thick and locally finite. «Thickness» means that for any *panel* (i.e. any codimension one simplex) $\Pi$, the set of chambers whose closure contains $\Pi$ has at least three elements. We denote by $B_+$ (resp. $B_-$) the fixator of the positive (resp. negative) chamber $c_+$ (resp. $c_-$) in $\Lambda$; it contains as a finite index subgroup the group $U_+$ (resp. $U_-$) generated by the root groups indexed by the positive (resp. negative) roots.

The codistance $w^*$ is a map $(X_- \times X_+) \sqcup (X_+ \times X_-) \to W$ defined thanks to the Birkhoff decomposition of $\Lambda$ [Abr97, §2]. The group of twin building automorphisms of $(X_+, X_-, w^*)$ is the subgroup $A$ of couples $(g_-, g_+) \in \mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)$ which satisfy $w^*(g_-.c_-, g_+.c_+) = w^*(c_-, c_+)$ for any couple of chambers $(c_-, c_+) \in X_- \times X_+$. We have: $\Lambda < A$.

The twin building $(X_+, X_-, w^*)$ has the *Moufang property* [Ron89, §6]: if we identify $\Phi$ with the set of twin roots in $\Sigma$, this roughly means that for any twin root $a \subset \Sigma$ and any chamber $c$ having a panel $\Pi$ in the wall $\partial a$, the root group $U_a$ fixes the half twin apartment $a \subset \Sigma$ and acts simply transitively on the chambers different from $c$ and containing $\Pi$. This explains why the local finiteness of the buildings $X_\pm$ amounts to the finiteness of the root groups.

We can now turn to topology. We denote by $\mathrm{Aut}(X_\pm)$ the group of all type-preserving building automorphisms of $X_\pm$. For the compact open topology, in which a fundamental system of neighborhoods of the identity is given by fixators of finite subsets of chambers, the group $\mathrm{Aut}(X_\pm)$ is locally compact.

**Definition 2.** *We denote by* $\overline{\Lambda}_\pm$ *the closure in* $\mathrm{Aut}(X_\pm)$ *of the image of the* $\Lambda$-*action on* $X_\pm$ *and we call it the* geometric completion *of* $\Lambda$ *of sign* $\pm$. *The fixator of the chamber* $c_\pm$ *in* $\overline{\Lambda}_\pm$ *is called the* standard Iwahori subgroup *of* $\overline{\Lambda}_\pm$ *and is denoted by* $\mathcal{B}_\pm$. *We denote by* $\widehat{U}_\pm$ *the closure of* $U_\pm$ *in* $\overline{\Lambda}_\pm$, *and for any* $w \in W$ *we introduce the group* $\widehat{U}_-^w := \widehat{U}_- \cap w^{-1}\widehat{U}_- w$.

These groups were introduced in [RR03, 1.B], where it was checked that $(\overline{\Lambda}_+, N, \widehat{U}_+, U_-, T, S)$ and $(\overline{\Lambda}_-, N, \widehat{U}_-, U_+, T, S)$ both satisfy the axioms of a *refined Tits system* whenever $\Lambda$ is an infinite Kac-Moody group over a finite field. This notion is due to V. Kac and D. Peterson [KP85] and its basic properties will be used in the sequel with appropriate references.

**Assumption 3.** *Until the end of the paper, the group* $\Lambda$ *is assumed to be an infinite Kac-Moody group over a finite field.*

All the results of the paper remain valid for groups with twin root data with infinite Weyl groups and finite root groups and whose geometric completions are refined Tits systems. This is the case when $T = \{1\}$, which is possible for some exotic Moufang twin buildings [AR03, Example 69].

### *1.2. Fundamental domain and cocycle*

We keep the twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$, the standard pair of opposite chambers $\{c_-; c_+\}$, and we now introduce a remarkable subset of $G$.

**Definition 4.** *For each* $w \in W$, *we denote by* $D_w$ *the subset* $\widehat{U}_-^w \times \mathcal{B}_+ w$ *of* $G$. *We denote by* $D$ *the disjoint union* $\bigsqcup_{w\in W} D_w$.

Here is the main property of $D$.

**Proposition 5.** *For any* $g = (g_-, g_+) \in G$, *there is a unique* $\lambda \in \Lambda$ *and a unique* $w \in W$ *such that* $g_-\lambda \in \widehat{U}_-^w$ *and* $g_+\lambda \in \mathcal{B}_+ w$, *i.e.* $D$ *is a fundamental domain for* $G/\Lambda$.

*Proof.* We use the right action on $X_- \times X_+$ defined by: $(d_-, d_+).(h_-, h_+) = (h_-^{-1}.d_-, h_+^{-1}.d_+)$ for any $(h_-, h_+) \in \mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)$ and any pair of chambers of opposite signs $\{d_-; d_+\}$. We argue on the $G$-transforms of the standard couple of chambers $(c_-, c_+)$. Since any pair of chambers is contained in a twin apartment [Abr97, Lemma 2 p.24] and since the diagonal $\Lambda$-action is transitive on the set of twin apartments [Abr97, Lemma 4 p.29], there exist $\delta \in \Lambda$ and $w \in W$ such that $(c_-, c_+).(g_-, g_+).\delta = (c_-, w^{-1}.c_+)$, so we have $g_-\delta \in \mathcal{B}_-$ and $g_+\delta \in \mathcal{B}_+ w$.

By standard properties of refined Tits systems [Rem02, 1.2.3], we have $\mathcal{B}_- = \widehat{U}_- \rtimes T = \widehat{U}_-^w.U_{-w}.T$, with $U_{-w} = \widehat{U}_- \cap w^{-1}U_+w = U_- \cap w^{-1}U_+w$ and $\widehat{U}_-^w = \widehat{U}_- \cap w^{-1}\widehat{U}_-w$. We can thus write: $g_-\delta = \hat{u}_-^w u_{-w} t$, with $\hat{u}_-^w \in \widehat{U}_-^w$,

$u_{-w} \in U_{-w}$ and $t \in T$. If we set $\lambda := \delta t^{-1}(u_{-w})^{-1} \in \Lambda$, we have that $g_-\lambda = g_-\delta t^{-1}(u_{-w})^{-1}$ lies in $\widehat{U}_-^w$. Moreover writing $g_+\delta = b_+w$ with $b_+ \in \mathcal{B}_+$, we get: $g_+\lambda = b_+(wt^{-1}w^{-1})(wu_{-w}w^{-1})w$. Since $T$ is normalized by $W$, we finally obtain: $g_+\lambda \in \mathcal{B}_+w$ by definition of $U_{-w}$.

This proves that $D$ contains a representative for each class of $G/\Lambda$. It remains to check that $\lambda \in \Lambda$ and $w \in W$ are uniquely determined by $(g_-, g_+)$ and the conditions $g_-\lambda \in \widehat{U}_-^w$ and $g_+\lambda \in \mathcal{B}_+w$. Assume there exist $\delta \in \Lambda$ and $z \in W$ such that $g_-\delta \in \widehat{U}_-^z$ and $g_+\delta \in \mathcal{B}_+z$. We have $\lambda^{-1}g_-^{-1}c_- = c_-$ and $\lambda^{-1}g_+^{-1}c_+ = w^{-1}c_+$; and since the diagonal $\Lambda$-action preserves the codistance $w^*$ between chambers of opposite signs, this gives $w^*(g_-^{-1}.c_-, g_+^{-1}.c_+) = w^*(\lambda^{-1}g_-^{-1}.c_-, \lambda^{-1}g_+^{-1}.c_+) = w^*(c_-, w^{-1}.c_+) = w^{-1}$. We can do the same computation with $\lambda$ replaced by $\delta$ and $w$ replaced by $z$, to get $w = z$.

It remains to compute: $g_-\delta = (g_-\lambda)(\lambda^{-1}\delta)$, which implies $\lambda^{-1}\delta \in \widehat{U}_-^w$. Moreover we have: $\lambda^{-1}g_+^{-1}.c_+ = w^{-1}.c_+$, but also: $\lambda^{-1}g_+^{-1}.c_+ = (\lambda^{-1}\delta).(\delta^{-1}g_+^{-1}.c_+) = (\lambda^{-1}\delta).(w^{-1}.c_+)$. This shows that $\lambda^{-1}\delta$ fixes the chamber $w^{-1}.c_+$, hence belongs to $w^{-1}\mathcal{B}_+w$. We have: $\lambda^{-1}\delta \in \widehat{U}_-^w \cap w^{-1}\mathcal{B}_+w$, which provides: $w(\lambda^{-1}\delta)w^{-1} \in \mathcal{B}_+ \cap w\widehat{U}_-^w w^{-1} \cap \Lambda < \mathcal{B}_+ \cap \widehat{U}_- \cap \Lambda = \mathcal{B}_+ \cap U_- = \{1\}$ [KP85, Axiom (RT3)]. We finally obtain: $\lambda = \delta$. □

This enables to recover a basic result on the existence of lattices for Kac-Moody buildings [Rem99]. We normalize the right Haar measure $\mu_\pm$ on $\mathrm{Aut}(X_\pm)$ so that $\mu_\pm(\widehat{U}_\pm) = 1$, and set $\mu := \mu_- \otimes \mu_+$.

**Corollary 6.** *The $\mu$-volume of $D_w$ is $\dfrac{|T|}{|U_{-w}|}$; the group $\Lambda$ is a lattice of $G$ whenever $\displaystyle\sum_{w\in W} \frac{1}{|U_{-w}|}$ converges. The latter condition is fulfilled when the ground field $\mathbf{F}_q$ satisfies $W(\frac{1}{q}) < +\infty$.*

*Proof.* We have: $\mathrm{Vol}(D_w, \mu) = \mu_-(\widehat{U}_-^w) \cdot \mu_+(\mathcal{B}_+w) = \dfrac{1}{|U_{-w}|} \cdot \mu_-(\widehat{U}_-) \cdot \mu_+(\mathcal{B}_+)$. The second equality follows from $\widehat{U}_- = \widehat{U}_-^w \cdot U_{-w}$ [Rem02, 1.2.3], and the first assertion follows from $\mathcal{B}_+ = T \ltimes \widehat{U}_+$ [loc. cit.]. The second assertion is then obvious, and the third one follows from the existence of a bijection between $U_{-w}$ and the product of $\ell_W(w)$ suitable root groups, all having at least $q$ elements (see also the proof of Lemma 16 for details). □

We can now introduce the cocycle we are interested in.

**Definition 7.** *We define $\alpha = \alpha_D : G \times D \to \Lambda$ by setting $\alpha\big((g_-, g_+), (u_-^w, b_+w)\big) = \lambda$ if and only if $g_-u_-^w\lambda \in \widehat{U}_-^z$ and $g_+b_+w\lambda \in \mathcal{B}_+z$ for some $z \in W$. In other words, denoting by $d$ the element $(u_-^w, b_+w)$ of $D_w$, we set $\alpha(g, d) = \gamma$ if and only if $gd\gamma \in D$.*

*Remark 8.* In terms of this cocycle, the first two paragraphs of the proof of Proposition 5 show that, given $g = (g_-, g_+) \in G$, an element $\delta \in \Lambda$ such that $\delta^{-1} g_-^{-1}.c_- = c_-$ and $\delta^{-1} g_+^{-1}.c_+ = w^{-1}.c_+$ for some $w \in W$, is an approximation modulo $TU_{-w}$ of $\alpha(g, 1_G)$. This is used to prove Lemma 16.

*Remark 9.* What we call «cocycle» leads to the relation: $\alpha(gh, x) = \alpha(h, x)\alpha(g, h.x)$; in order to have a true cocycle, i.e. $\alpha(gh, x) = \alpha(g, h.x)\alpha(h, x)$, we should set $\alpha(g, d) = \gamma^{-1}$ if and only if $gd\gamma \in D$. Since we want to apply Y. Shalom's results, we adopted his «anticocycle» convention.

## 2. Integrability

We can now proceed to the proof of the integrability of the length of the cocycle. We first prove some geometric inequalities, basically following from counting root group actions. Then we use these inequalities to show that the integrability of the cocycle only depends on the convergence of a power series which is very close to the one computing the covolume in Corollary 6.

We keep the infinite Kac-Moody group $\Lambda$ over $\mathbf{F}_q$ with twin root datum $\big(\Lambda, \{U_a\}_{a\in\Phi}, H\big)$. We will still use $(X_+, X_-, w^*)$ its thick, locally finite, Moufang twin building, as well as the twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$ and the standard pair of opposite chambers $\{c_-; c_+\}$ in $\Sigma$.

### 2.1. Geometric inequalities

The following result is a quantitative version of the fact that the diagonal $\Lambda$-action on pairs of chambers of opposite signs admits $\{c_-\} \sqcup \Sigma_+$ as fundamental domain.

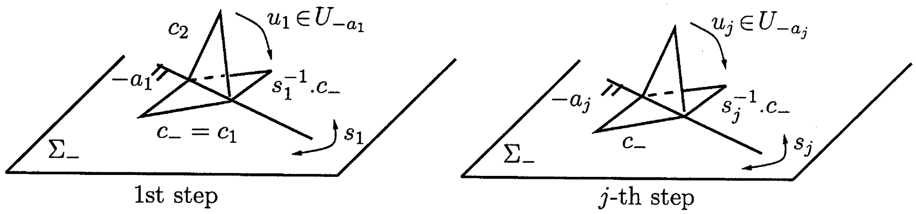
**Proposition 10.** *Let $\{d_-, d_+\}$ be a pair of chambers of opposite signs. We introduce the combinatorial distances $L_- := \mathrm{dist}(c_-, d_-)$ and $L_+ := \mathrm{dist}(c_+, d_+)$, and their sum $L := L_- + L_+$. Then, there exist $\lambda \in \Lambda$ and $w \in W$ with: $\lambda^{-1}.d_- = c_-$ and $\lambda^{-1}.d_+ = w^{-1}.c_+$, whose lengths satisfy: $\ell_\Lambda(\lambda) \leq 2L^2 + 3L$ and $\ell_W(w) \leq L$.*

*Proof.* The proof is divided into three steps.
Step 1: negative chambers (see Picture 11). Let $c_- = c_1, c_2, \ldots c_i, c_{i+1}, \ldots c_n = d_-$ be a minimal gallery from $c_-$ to $d_-$, such that $\{c_i; c_{i+1}\}$ is a pair of $s_i$-adjacent chambers for some $s_i \in S$. We have $n = L_- + 1$. For each $i$, we denote by $a_i$ the simple root attached to the reflection $s_i$. By minimality we have $c_1 \neq c_2$, so the Moufang property [Ron89, 6.4] implies that there is a unique $u_1 \in U_{-a_1}$ such that $s_1 u_1.c_2 = c_1$. Then $s_1 u_1.c_2, \ldots s_1 u_1.c_i, s_1 u_1.c_{i+1}, \ldots s_1 u_1.c_n$ is a minimal gallery from $c_-$. Therefore, there is a unique $u_2 \in U_{-a_2}$ such that $s_2 u_2 s_1 u_1.c_3 = c_-$. We iterate the procedure. After the $(j-1)$-th step, we have: $s_{j-1} u_{j-1} \ldots s_2 u_2 s_1 u_1.c_j = c_-$, and we are interested in the chamber $s_{j-1} u_{j-1} \ldots s_2 u_2 s_1 u_1.c_{j+1}$. It is a chamber $s_j$-adjacent to $c_-$ and different from $c_-$ in $X_-$, so by the Moufang property there

is a unique $u_j \in U_{-a_j}$ such that $u_j s_{j-1} u_{j-1} \ldots s_2 u_2 s_1 u_1 . c_{j+1}$ is the unique chamber $s_j^{-1}.c_-$ to be $s_j$-adjacent to $c_-$ and different from $c_-$ in $\Sigma_-$. We eventually obtain an element $\delta = (s_{n-1}u_{n-1}) \ldots (s_2 u_2)(s_1 u_1)$ such that

$$\delta.d_- = c_- \qquad \text{and} \qquad \ell_\Lambda(\delta) \leq 2L_- \leq 2L.$$

**Picture 11.**




Step 2: signs of roots. We set $z := s_{n-1} \ldots s_2 s_1$.

**Claim 12.** *We have:* $\delta \in U_+ z$.

*Proof of the claim.* Let us write:

$$\delta = (s_{n-1}u_{n-1}s_{n-1}^{-1}).(s_{n-1}s_{n-2}u_{n-2}s_{n-2}^{-1}s_{n-1}^{-1}) \ldots (s_{n-1} \ldots s_j u_j s_j^{-1} \ldots s_{n-1}^{-1})$$
$$\ldots (s_{n-1} \ldots s_1 u_1 s_1^{-1} \ldots s_{n-1}^{-1}) z.$$

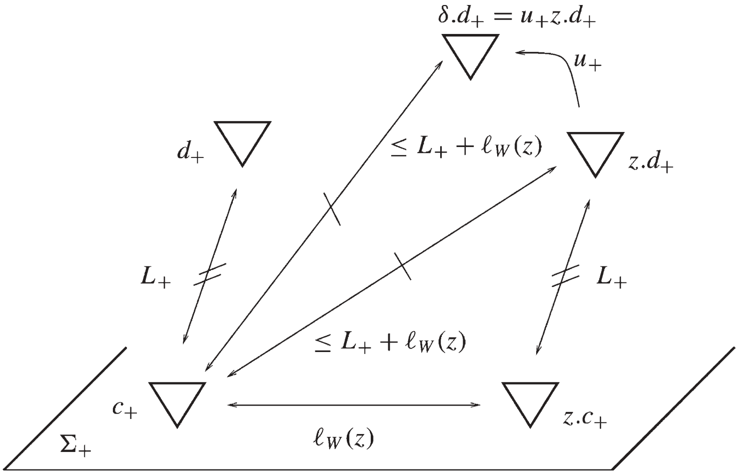
For each $j$, we set $\delta_j = s_{n-1} \ldots s_j u_j s_j^{-1} \ldots s_{n-1}^{-1} \in U_{s_{n-1} \ldots s_{j+1}.a_j}$. It is enough to show that each factor $\delta_j$ lies in a positive root group. Recall the combinatorial definition of the root system $\Phi$ [Tit87, 5.1]: the simple root attached to the canonical reflection $s_j \in S$ is $a_j = \{w \in W \mid \ell_W(s_j w) > \ell_W(w)\}$, and its opposite is $-a_j = \{w \in W \mid \ell_W(s_j w) < \ell_W(w)\}$; an arbitrary root $w.a_s$ is defined by translation of a simple root by a suitable element of the Weyl group $W$. In this description, a root is positive if and only if it contains $1_W$. Now the fact that $s_{n-1} \ldots s_2 s_1$ is a reduced word implies that $s_{n-1} \ldots s_{j+1}.a_j$ is a positive root for any $j$. □

We can therefore write $\delta = u_+ z$ for some $u_+ \in U_+$, and because $u_+$ fixes $c_+$ we have:

$$\begin{aligned} \mathrm{dist}(c_+, \delta.d_+) &= \mathrm{dist}(c_+, z.d_+) \\ &\leq \mathrm{dist}(c_+, z.c_+) + \mathrm{dist}(z.c_+, z.d_+) \\ &= \ell_W(z) + L_+ = L, \end{aligned}$$

see also Picture 13.

**Picture 13.**

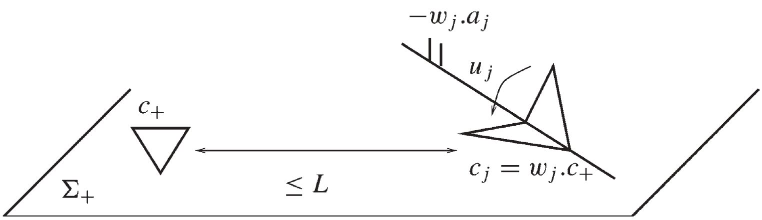



Step 3: negative root groups acting on the positive side.

**Claim 14.** *There is an element $u_- \in U_-$ such that $u_-u_+z.d_+ \in \Sigma_+$ and $\ell_\Lambda(u_-) \leq L(2L+1)$.*

*Proof of the claim (*see Picture 15*).* First, if $u_+z.d_+ \in \Sigma_+$, there is nothing to do because $u_+z.d_+$ is a chamber at combinatorial distance $\leq L$ from $c_+$, and as such can be written $w^{-1}.c_+$ for $w \in W$ with $\ell_W(w) \leq L$. If the combinatorial distance $\mathrm{dist}(\Sigma_+, u_+z.d_+)$ from $\Sigma_+$ to $u_+z.d_+$ is positive, then we choose a gallery $c_1 = c_+,\ c_2,\ \dots c_n = u_+z.d_+$ of length $\leq L$, and we denote by $j$ the smallest index such that for any $i > j$ we have $c_i \notin \Sigma_+$. The chambers $c_j$ and $c_{j+1}$ are $s_j$-adjacent, and there is a unique $w_j \in W$ such that $c_j = w_j.c_+$ and $\ell_W(w_j) = \mathrm{dist}(c_+, c_j) \leq L$. By the Moufang property, there is $u_j \in w_jU_{-a_j}{w_j}^{-1}$ such that $u_j.c_{j+1} = c_j$. Note that $\ell_\Lambda(u_j) \leq \ell_W(w_j) + 1 + \ell_W(w_j) \leq 2L+1$. We obtain a new gallery $c_1 = c_+, c_2, \dots c_j,\ u_j.c_{j+2},\ \dots u_ju_+z.d_+$, with $\mathrm{dist}(\Sigma_+, u_ju_+z.d_+) \leq \mathrm{dist}(\Sigma_+, u_+z.d_+) - 1$. Iterating the use of a suitable element $u_i$ of length $\leq 2L+1$ in a negative root group $w_iU_{-a_i}{w_i}^{-1}$ $(i > j)$, we obtain a sequence of at most $L-j$ elements of $U_-$ of length $\leq 2L+1$ whose product, say $u_-$, sends $u_+z.d_+$ to a chamber in $\Sigma_+$, at distance $\leq L$ from $c_+$. □

**Picture 15.**



*Proof of Proposition, conclusion.* The chamber $u_-u_+z.d_+$ is at combinatorial distance $\leq L$ from $c_+$, so it can be written $w^{-1}.c_+$ with $\ell_W(w) \leq L$. Therefore, setting $\lambda^{-1} := u_-u_+z$, we have:

$$\ell_\Lambda(\lambda) \leq 2L + L(2L+1) = 2L^2 + 3L \qquad \text{and} \qquad \ell_W(w) \leq L,$$

as well as:

$$\lambda^{-1}.d_- = (u_-u_+z).d_- = u_-.c_- = c_- \qquad \text{and} \qquad \lambda^{-1}.d_+ = w^{-1}.c_+.$$

□

### 2.2. *Computation*

We can now turn to the proof of the main result, i.e. the finiteness of the integral:

$$\int_D \ell_\Lambda\big(\alpha(g,d)\big)^p \mathrm{d}\mu(d),$$

for any $p \in [1; +\infty)$ and any $g \in G$.

First, since $1_G$ belongs to the fundamental domain $D$ and since $\lambda = \alpha(g,d) \Leftrightarrow gd\lambda \in D$, we have: $\alpha(g,d) = \alpha(gd, 1_G)$ for any $g \in G$ and any $d \in D$. Therefore it is enough to show:

$$\int_D \ell_\Lambda\big(\alpha(gd, 1_G)\big)^p \mathrm{d}\mu(d) < +\infty,$$

for any $p \in [1; +\infty)$ and any $g \in G$. We start with two lemmas.

**Lemma 16.** *Let $h = (h_-, h_+) \in G$. Let us set: $L_+(h) = \mathrm{dist}(c_+, h_+^{-1}.c_+)$, $L_-(h) = \mathrm{dist}(c_-, h_-^{-1}.c_-)$ and $L(h) = L_+(h) + L_-(h)$. Then we have: $\ell_\Lambda\big(\alpha(h, 1_G)\big) \leq P\big(L(h)\big)$, with $P(X) = 3X^2 + 3X + 1$.*

*Proof.* We take $\lambda \in \Lambda$ and $w \in W$ given by Proposition 10 and the choices $d_- = h_-^{-1}.c_-$ and $d_+ = h_+^{-1}.c_+$. We have $\alpha(h, 1_G)\lambda^{-1} \in TU_{-w}$ by Remark 8. From standard facts on twin root data [Rem02, Lemma 1.5.2], the group $U_{-w}$ is in bijection with a product $\prod_\beta U_\beta$, where $\beta$ runs in a suitable order over the $\ell_W(w)$ negative roots such that $w.\beta > 0$. If $w = s_n ... s_2 s_1$ is a reduced word, i.e. if $n = \ell_W(w)$, the roots $\beta$ under consideration are $-a_1, -s_1.a_2, ... -s_1 s_2 ... s_{n-1}.a_n$, where $a_i$ is the simple root attached to the canonical reflection $s_i$. Therefore we have:

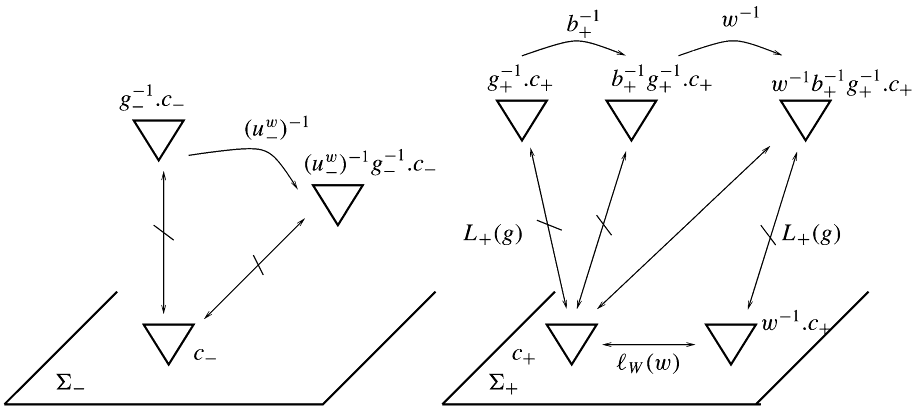
$$\ell_\Lambda\big(\alpha(h, 1_G)\lambda^{-1}\big) \leq 1 + \sum_{j=0}^{\ell_W(w)-1} (2j+1) = 1 + \ell_W(w)^2 \leq 1 + L^2$$

(the first term comes from the factor in $T$), and we finally obtain the required inequality. □

**Lemma 17.** *Let $g = (g_-, g_+) \in G$. Let $d = (u_-^w, b_+ w)$ be in the subset $D_w$ of $D$. Then we have: $\ell_\Lambda\big(\alpha(gd, 1_G)\big) \le Q\big(\ell_W(w)\big)$, with $Q(X) = 3X^2 + (6L(g) + 3)X + (3L(g)^2 + 3L(g) + 1)$.*

*Proof (*see Picture 18*).* Since $u_-^w$ belongs to $\widehat{U}_-^w$, hence to $\mathcal{B}_-$, it fixes $c_-$ so that we have the equality: $L_-(gd) = \mathrm{dist}(c_-, (u_-^w)^{-1} g_-^{-1}.c_-) = \mathrm{dist}(u_-^w.c_-, g_-^{-1}.c_-) = L_-(g)$. We have $b_+ \in \mathcal{B}_+$, so $L_+(gd) = \mathrm{dist}(c_+, w^{-1} b_+^{-1} g_+^{-1}.c_+) \le L_+(g) + \ell_W(w)$. This finally implies $L(gd) \le L(g) + \ell_W(w)$, and it remains to apply the previous lemma. $\square$

**Picture 18.**




At last, we can go back to the integral.

*Proof of the main theorem.* We can now compute:

$$\begin{aligned}
\int_D \ell_\Lambda\big(\alpha(gd, 1_G)\big)^p \mathrm{d}\mu(d) &= \sum_{w \in W} \Big( \int_{D_w} \ell_\Lambda\big(\alpha(gd, 1_G)\big)^p \mathrm{d}\mu(d) \Big) \\
&\le \sum_{w \in W} \Big( \int_{D_w} Q\big(\ell_W(w)\big)^p \mathrm{d}\mu(d) \Big) \\
&= \sum_{w \in W} \Big( Q\big(\ell_W(w)\big)^p \cdot \mathrm{Vol}(D_w, \mu) \Big) \\
&\le |T| \cdot \sum_{w \in W} \Big( \frac{Q\big(\ell_W(w)\big)^p}{q^{\ell_W(w)}} \Big) \\
&= |T| \cdot \sum_{n \in \mathbf{N}} c_n \frac{Q(n)^p}{q^n}.
\end{aligned}$$

The first equality follows from $D = \bigsqcup_{w \in W} D_w$. The first inequality follows from Lemma 17. The second inequality follows from the equality $\mu(D_w) = \frac{|T|}{|U_{-w}|}$ (Corollary 6) and the existence of a bijection between $U_{-w}$ and the product of $\ell_W(w)$ root groups, all of order at least $q$. The last equality follows from rearranging the elements of the Weyl group with respect their length, which makes appear the growth series $W(t) = \sum_{n \in \mathbf{N}} c_n t^n$. $\square$

## 3. Applications and questions

We mention the two main applications of the square-integrability of the cocycle with respect to the fundamental domain provided by our main result. The point is that this property enables to use induction for 1-cohomology with unitary coefficients (in the spirit of [Sha00a]) and more generally in the context of actions on CAT(0)-spaces (as introduced in [Mon04]). We first deal with applications to rigidity theory, and then state a normal subgroup theorem for Kac-Moody lattices.

### *3.1. Lattice superrigidity*

The square-integrability condition:

$$\int_D \ell_\Lambda\big(\alpha_D(g, d)\big)^2 \mathrm{d}\mu(d) < +\infty$$

is a sufficient condition to induce the first (reduced) cohomology of the unitary representations of the irreducible lattice $\Lambda$ to the first (reduced) continuous cohomology of the ambient topological group $G = \overline{\Lambda}_- \times \overline{\Lambda}_+$ [Sha00a, Proposition 1.11]. Induction in cohomology is the starting point of Y. Shalom's proof of the series of results alluded to in the first corollary of the introduction, i.e. all the results in [loc. cit.] for which the hypotheses are **(0.1)**. The datum of a uniform irreducible lattice $\Gamma$ in a compactly generated product of topological groups $G = G_1 \times \ldots \times G_n$ has to be replaced by the datum of a Kac-Moody lattice $\Lambda$ in the product of its two geometric completions $\overline{\Lambda}_- \times \overline{\Lambda}_+$. The fact that the groups $\overline{\Lambda}_\pm$ are compactly generated follows easily from a suitable Bruhat decomposition [Rem03b, Corollary 1.B.1]. The so-obtained results are (see [Sha00a, Introduction]): property (T) for proper quotients (Theorem 0.1), superrigidity of homomorphisms to groups containing some rank one lattices (Theorem 0.3), arithmeticity of some images (Theorem 0.5), some strong rigidity (Theorem 0.6), superrigidity of actions on trees (Theorem 0.7) and of characters (Theorem 0.8).

The idea to replace a cocompactness by an integrability assumption appears in [Mar91, III.1] and in [Sha00a]. The proof of the square-integrability of induction cocycles for non-uniform irreducible lattices in products of algebraic groups over local fields [Sha00a, §2] uses reduction theory for $S$-arithmetic groups (as summed up in [Mar91, VIII.1]), and the relation between word and Riemannian metrics for lattices in higher-rank semisimple Lie groups [LMR01]. Thus, from this point of view, Sect. 2 is a light substitute for the latter two deep results, in the Kac-Moody case. (The intersection between Kac-Moody groups over finite fields and $S$-arithmetic groups is non-empty; it contains the affine Kac-Moody type, corresponding to the points over $\mathbf{F}_q[t, t^{-1}]$ of simply connected Chevalley groups, i.e. $\{0; \infty\}$-arithmetic groups.)

We conclude this subsection by a potential generalization of the previously quoted results to superrigidity results for actions on CAT(0)-spaces. In [Mon04], such actions of uniform irreducible lattices are induced to actions of the ambient product of topological groups on much bigger (non-proper) CAT(0)-spaces. This, together with geometric splitting theorems in order to suitably generalize the Zariski density assumption, leads to the rigidity theorems.

**Question 19.** *To what extent can irreducible uniform lattices be replaced by Kac-Moody non-uniform lattices in N. Monod's work?*

Under the square-integrability condition, there is no further obstruction to induce actions in the context of metric spaces. Still, uniformness of the lattice is used elsewhere in [Mon04], namely to control the behaviour under induction of *evanescence*, a notion introduced to generalize the existence of a fixed point in $\partial_\infty X$ for actions on non proper CAT(0)-spaces. The results of [loc. cit.] are valid when the irreducible uniform lattices are replaced by irreducible square-integrable and weakly cocompact lattices. Recall that a lattice $\Gamma$ in $G$ is called *weakly cocompact* if the orthogonal complement $L^2_0(G/\Gamma)$ of the constant functions in $L^2(G/\Gamma)$ doesn't weakly contain the trivial one-dimensional representation [Mar91, p.111]. Weak cocompactness is fulfilled when the lattice has property (T), and the latter property is shown to often hold for Kac-Moody lattices [DJ02]. For instance, applying [Mon04, Theorem 6] gives:

**Corollary 20.** *Let $\Lambda < \overline{\Lambda_-} \times \overline{\Lambda_+}$ be a Kac-Moody lattice, which is assumed to be Kazhdan. Let $H < \mathrm{Isom}(X)$ be a closed subgroup, where $X$ is any complete* CAT(0)*-space. Let $\tau : \Lambda \to H$ be a homomorphism with reduced unbounded image. Then $\tau$ extends to a continuous homomorphism $\tilde{\tau} : \overline{\Lambda_-} \times \overline{\Lambda_+} \to H$.*

We refer to [loc. cit., Appendix B] for a detailed proof of the fact that the main results of this article can be applied to Kac-Moody lattices with property (T).

### *3.2. Normal subgroup theorem*

The main application of our main theorem is group-theoretic; it is the following statement, proved with U. Bader and Y. Shalom.

**Theorem 21.** *Let $\Lambda$ be a Kac-Moody group over a finite field, with irreducible Weyl group. Assume it is a lattice of the product of its twinned buildings. Then any normal subgroup of $\Lambda$ either has finite index or lies in the finite center $Z(\Lambda)$.*

*Reference.* The details of the proof are provided in [BS03, §4 p.27]. The strategy at large scale is the same as Margulis': in order to prove that a factor group is finite, one proves that it is both amenable and Kazhdan. The amenability half follows from [loc. cit., Theorem 1.3] and it doesn't use any uniformness or square-integrability assumption. The property (T) half is [Sha00a, Theorem 0.1], and it does need the square-integrability property, referred to as **S-I** in [BS03]. □

From our point view, the normal subgroup property has the following interesting consequence, which reduces (virtual) simplicity to non-residual finiteness for finitely generated Kac-Mody groups with irreducible Weyl groups.

**Corollary 22.** *If $\Lambda$ is center-free, e.g. because it is adjoint, and if it is not residually finite, then $\Lambda$ is virtually simple.*

*Proof.* The group $\Lambda$ is just infinite, i.e. all its proper quotients are finite. Set $N := \bigcap_{[\Lambda:\Delta]<\infty} \Delta$. Since $\Lambda$ is not residually finite we have $N \neq \{1\}$, and since $\Lambda$ is just infinite we have $[\Lambda : N] < \infty$. By [Wil71, Proposition 1], the group $N$ is the direct product of finitely many pairwise isomorphic simple groups. Therefore it is enough to show that there is only one factor in this product. Assume there are two, say $H$ and $G$. Topological simplicity of $\overline{\Lambda}$ [Rem03b, 2.A.1] first implies $\overline{N} = \overline{\Lambda}$ because $N \lhd \Lambda$, and then $\overline{G} = \overline{H} = \overline{\Lambda}$. Let us pick two non-commuting elements $g$ and $h$ in $\overline{\Lambda}$ and write them $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$, with $g_n \in G$ and $h_n \in H$ for each $n \geq 1$. We get a contradiction when writing: $[g, h] = \lim_{n\to\infty} [g_n, h_n] = 1$. □

It is not known whether non-residually finite finitely generated Kac-Moody groups exist. On the other hand, partial non-linearity results are available [Rem03b] and can probably be extended to wider classes of Kac-Moody groups.

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