On some recent developments in the theory of buildings

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Abstract. Buildings are cell complexes with so remarkable symmetry properties that many groups from important families act on them. We present some examples of results in Lie theory and geometric group theory obtained thanks to these highly transitive actions. The chosen examples are related to classical and less classical (often non-linear) group-theoretic situations.

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Introduction

Buildings are cell complexes with distinguished subcomplexes, called apartments, requested to satisfy strong incidence properties. The notion was invented by J. Tits about 50 years ago and quickly became useful in many group-theoretic situations [75]. By their very definition, buildings are expected to have many symmetries, and this is indeed the case quite often. Buildings are relevant to Lie theory since the geometry of apartments is described by means of Coxeter groups: apartments are so to speak generalized tilings, where a usual (spherical, Euclidean or hyperbolic) reflection group may be replaced by a more general Coxeter group. One consequence of the existence of sufficiently large automorphism groups is the fact that many buildings admit group actions with very strong transitivity properties, leading to a better understanding of the groups under consideration.

The beginning of the development of the theory is closely related to the theory of algebraic groups, more precisely to Borel-Tits' theory of isotropic reductive groups over arbitrary fields and to Bruhat-Tits' theory of reductive groups over non-archimedean valued fields. In the former theory the involved buildings are spherical (i.e., the apartments are spherical tilings) and the group action reflects the existence, on the rational points of the algebraic group, of a strong combinatorial

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structure called Tits system (or BN-pair). Roughly speaking, such a structure formalizes the existence of a Bruhat decomposition indexed by a Coxeter group (called the Weyl group of the Tits system) and, among other things, leads to a uniform way of proving projective simplicity of rational points of classical groups. In the latter theory, the involved buildings are Euclidean (i.e., the apartments are Euclidean tilings) and the Weyl group of the Tits system is an affine Coxeter group. The group action on the building is a crucial tool to understand the subtle structure of the rational points of the algebraic group. For instance, by passing to cell stabilizers, Bruhat-Tits buildings parametrize remarkable compact open subgroups obtained from forms of the group over the valuation ring of the local ground field.

These two situations (spherical and Euclidean buildings), which are related to algebraic groups (via Borel-Tits and Bruhat-Tits' theory), will be called classical in the sequel of the report.

What is probably less well known is the fact that some buildings in which apartments are modeled on neither spherical nor affine tilings have recently led to interesting group-theoretic situations. One strong connection with geometric group theory is given by the existence, for any building, of a distance such that the resulting metric space is complete, contractible and non-positively curved in some suitable sense due to M. Gromov. In this case, the general theme is to study isometry groups of non-classical buildings by analogy with classical Lie-theoretic situations. In this analogy, buildings are seen as analogues of non-compact Riemannian symmetric spaces and their automorphism groups are seen as generalizations of semisimple Lie groups.

From that viewpoint, Kac-Moody theory is very useful even though it may not be so clear at first glance. This theory is usually presented as an infinite-dimensional generalization of semisimple Lie algebras, with applications in representation theory. It turns out that there exist some constructions of groups integrating (possibly in a loose sense) Kac-Moody Lie algebras. For these groups, nice structures from algebraic geometry are usually lost, but the combinatorial structures such as Tits systems stay available and can be translated in terms of strongly transitive group actions on (usually exotic) buildings. The notion of a ground field still makes sense and the possibility to work over a finite ground field leads to intriguing finitely generated groups or non-discrete locally profinite groups, according to the version of Kac-Moody groups chosen to begin with. These groups shall be compared to arithmetic groups and to non-archimedean Lie groups in positive characteristic, respectively.

Of course, not all exotic buildings with interesting isometry groups come from Kac-Moody theory. In dimension 2 for instance, products of arbitrary semi-homogeneous trees provide a much wider class; among groups actings properly discontinuously and cocompactly on these spaces, M. Burger and Sh. Mozes could exhibit the first simple torsion-free finitely presented groups. Still, one of the mains ideas of this report is that Kac-Moody groups shall be investigated thoroughly since they are at good distance from the classical situation of Lie groups and their discrete subgroups. In some sense, it is a class of (discrete and profinite) groups which are new in the sense that striking new phenomena occur but on which we still have a very useful Lie-theoretic control (e.g. via infinite root systems).

The structure of this report is as follows. In the first section we recall some basic facts in building theory; we give some examples of results for both classical and non-classical buildings. The second section is dedicated to Euclidean buildings via two themes: compactifications of buildings (joint works with Y. Guivarc'h, and with A. Thuillier and A. Werner) and cohomology of arithmetic groups; we present two techniques of compactification, one of them using a promising relationship with non-archimedean analytic geometry. The third section deals with applications of Kac-Moody theory to the construction of interesting discrete, or non-discrete locally compact, groups (joint work with P.-E. Caprace); we explain for instance that these groups provide infinitely many quasi-isometry classes of finitely presented simple groups. It also mentions pro-p groups arising from Kac-Moody theory (joint work with I. Capdeboscq).

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1. Building theory

In this section, we introduce the main subject matter of this report, namely the notion of a building. We briefly present two well-known families of buildings, that of spherical and of Euclidean buildings, and explain quickly how they are related to the theory of algebraic groups. We also mention other examples of buildings providing interesting spaces for geometric group theory.

1.1. Definition of a building. A general reference for buildings is [1]. In order to provide a definition, we first have to introduce the notion of a Coxeter complex.

- A *Coxeter group*, say W, is a group admitting a presentation: $W = \langle s \in S \mid (st)^{M_{s,t}} = 1 \rangle$ where $M = [M_{s,t}]s, t \in S$ is a *Coxeter matrix* (i.e., symmetric with 1's on the diagonal and other entries in $\mathbf{N}_{\geqslant 2} \cup \{\infty\}$).
- For any Coxeter system (W, S) there is a natural simplicial complex Σ on the maximal simplices of which W acts simply transitively: Σ is called the *Coxeter complex* of (W, S).

Example 1.1. Let us go the other way round and start with a Euclidean or hyperbolic polytope whose dihedral angles are integral submultiples of π. Then, by a theorem of Poincaré's [53, IV.H.11], the group W generated by the reflections in the codimension 1 faces of the fundamental tile, is a discrete subgroup of the full isometry group of the ambient space. In fact, W is a Coxeter group and the tiling is a useful geometric realization of its Coxeter complex Σ.

The reason why we introduced Coxeter complexes is that they are so to speak generalized tilings on which the distinguished slices of a building are modeled. We freely use the previous notation W and Σ.

Definition 1.2. A building of type (W, S) is a cellular complex, covered by subcomplexes all isomorphic to Σ, called the *apartments*, such that:

(i) any two cells, called the *facets*, are contained in a suitable apartment;

(ii) given any two apartments A and A', there is a cellular isomorphism $A \simeq A'$ fixing $A \cap A'$.

The group W is called the *Weyl group* of the building. When W is a Euclidean reflection group [13, V §3], one says that the building is *affine* or, equivalently here, *Euclidean*.

Example 1.3. A tree all of whose vertices have valency $\geqslant 2$ (resp. a product of such trees) is a building with W equal to the infinite dihedral group D_∞ (resp. with W equal to $D_\infty \times D_\infty$).

The above examples of trees are elementary, but they are the only ones with infinite Weyl group which can be reasonably drawn. They are elementary examples but it is enough to consider them in order to see one difficulty in producing interesting group-theoretic situations from buildings. Indeed, take a tree T in which any two distinct vertices have distinct valencies. Then $\mathrm{Aut}(T) = \{1\}$, which shows that one has to make further assumptions on a building in order to obtain sufficiently transitive group actions.

Let us finish with some motivation of metric nature for the axioms. Assume that the Coxeter complex Σ of the Weyl group W admits an interesting W-invariant distance. This implies that each apartment carries a good metric structure, and one would like to show that this metric can be seen as being induced from a metric on the building itself. The first axiom precisely says that for any two arbitrary points in the building a distance can be computed (by choosing an apartment containing them) and the second axiom (up to some work to define suitable retractions onto apartments) can be used to show that the distance computed this way doesn't actually depend on the choice of the apartment containing the points. We will see in 1.3 that this fits very well with nice non-positive curvature properties for Coxeter complexes associated with infinite Weyl groups.

1.2. Spherical and Euclidean buildings. A building with a finite Weyl group is called *spherical*: this is because in that case the apartments are spherical tilings. The two families of affine and spherical buildings are very classical because they are closely related to the theory of algebraic groups.

First of all, it is well-known that if one starts with a reductive algebraic group, say G, over an arbitrary ground field, say k, then up to some isotropy condition on G with respect to k (namely, the existence of a non-central k-split torus) the group of rational points $G(k)$ admits a remarkable combinatorial structure called a *Tits system* (or also a BN-*pair*) [13, IV §2]. This is the main result of Borel-Tits' theory [10] and it can be reformulated as the fact that there exists a well-defined spherical building on which $G(k)$ acts *strongly transitively*, i.e. transitively on the inclusions of a chamber (i.e. a maximal facet) in an apartment.

Now, if the field k is endowed with a non-archimedean absolute value, under the above isotropy assumption (and further hypotheses on k when it is not locally compact), a similar statement says that there exists a well-defined Euclidean building on which $G(k)$ acts strongly transitively. This is one of the main results of Bruhat-Tits' theory but it doesn't exhaust the whole theory of reductive groups over valued fields [76] since one of the main tools (and objects of study at the same time) is given by forms of the group G over the valuation ring k° of k: see [19] for the building-theoretic part and [20] for the group scheme part of this deep theory.

In the spherical case, the theory of buildings may appear as a reformulation of some results proved by algebraic group-theoretic means. This is true for the statement formulated before, but quite not completely in the sense that the possibility to see the spherical building of a semisimple Lie group as the boundary at infinity of the associated symmetric space is a key step to prove Mostow's strong rigidity in differential geometry [55].

Moreover the structure of a Tits system with a finite Weyl group provides a uniform way to prove the projective simplicity of rational points of (suitable) simple isotropic algebraic groups, and reappeared recently in the theory of pseudo-reductive groups. The latter groups are algebraic groups whose study was initiated by J. Tits [81] and thoroughly made by B. Conrad, O. Gabber and G. Prasad [34]; a better understanding of these groups led to great progress in the theory of arbitrary algebraic groups in positive characteristic, with applications in number theory.

The theory of Euclidean buildings has a non-simplicial generalization which was already considered in Bruhat-Tits' work (it corresponds to the case when the valuation of the ground field k is not discrete). For geometric purposes, it was extended by B. Kleiner and B. Leeb in order to prove a strengthening of strong rigidity [48] stated by M. Gromov and called the rigidity of quasi-isometries. The non-simplicial buildings here are higher-dimensional analogues of real trees. In Thurston's approach to Teichmüller theory, the latter trees appear as degenerations (technically speaking: asymptotic cones) of hyperbolic spaces; therefore it is quite natural to see group actions on these (so to speak, branching everywhere) Euclidean buildings appear at the boundary of some compactifications of representations spaces [57]. We will see in 2.2 that these buildings also appear naturally when combining Bruhat-Tits' theory and V. Berkovich's approach to non-archimedean analytic geometry.

The classification of spherical buildings, initially formulated in J. Tits' lecture notes [74], has been simplified and extended by J. Tits and R. Weiss in the book [82]. The classification of Euclidean buildings was done by J. Tits too [77]; as for Mostow rigidity, the proof is based on the fact that the boundary at infinity of an affine building is a spherical building. This classification was then completed by R. Weiss in the book [83]. Loosely speaking, in higher rank a spherical or a Euclidean building is related to some (possibly twisted, or even suitably generalized) algebraic group.

1.3. Some more buildings. During the last decades, some buildings of non-classical type (i.e. neither of spherical nor of Euclidean type) have become more and more interesting to study from various perspectives. For instance, the possibility to construct buildings in which the apartments are isomorphic to tilings in real hyperbolic spaces was the opportunity to obtain interesting contractible spaces of negative curvature for geometric group theory. These spaces led to important instances of Mostow rigidity [14] and quasi-isometric rigidity [15] in the setting of singular spaces.

Let us consider now the natural question of classifying buildings. As mentioned in 1.2, the classification of classical buildings is achieved in higher rank. Up to using the notion of a boundary at infinity [16, §II.8], it eventually amounts to classifying the spherical ones. In the classification of the latter buildings there are two key ingredients, namely the longest element in the (finite) Weyl group and a property, called the *Moufang property*, ensuring that the building has sufficiently many automorphisms [1, §7]. One important step consists in proving that a spherical building of rank $\geqslant 3$ automatically enjoys the Moufang property.

When dealing with buildings with infinite Weyl group, say W, the Moufang property often has to be taken as a hypothesis, and of course the existence of a longest element in W completely fails. In the attempt to classify non-affine buildings with infinite Weyl group, J. Tits had the idea to propose the hypothesis, as a substitute for the longest element in W, that the buildings under consideration admit a second *twin* building related to the previous one by a suitable opposition relation between the chambers [80]. The most important examples of Moufang twin buildings are provided by Kac-Moody groups as presented in 3.1, but there are other examples [2].

From the point of view of geometric group theory, an important reason why buildings sometimes play an interesting role as test spaces is probably the following result, due to G. Moussong and M. Davis [35].

Theorem 1.4. *Any building X admits a distance for which X is a complete, geodesic,* CAT(0)*-space.*

The CAT(0)-property is an important non-positive curvature property: roughly speaking, assuming that the space is geodesic (i.e. that any two points are always connected by a geodesic segment), it says that geodesic triangles are at least as thin as in the Euclidean plane; a CAT(0)-space is automatically contractible. This property is fundamental in the sense that it is formulated in an elementary way but it has very deep consequences [16, Part II]. For instance, it implies that an isometric group action with a bounded orbit (e.g. because the group is compact) has a fixed point: it is a generalization of the so-called Bruhat-Tits fixed point lemma. This result was initially used for Galois actions in a context of descent of the ground field for algebraic groups, but it has today a much broader spectrum of applications.

In view of 1.2, it is natural to see buildings with infinite Weyl groups as generalizations of Riemannian symmetric spaces. More generally, this can be done for all CAT(0)-spaces, but it follows from remarkable papers by P.-E. Caprace and

N. Monod that buildings (together with symmetric spaces) often play a prominent role in a metric space situation that might seem more general at first glance (see [27] for structure theory and [26] for discrete group actions). The main properties of semisimple Lie groups and of their discrete subgroups become therefore challenging questions for more general, sufficiently large, isometry groups of non-classical buildings with infinite Weyl groups. Among these questions, we have of course the problem of simplicity of isometry groups and the problem of rigidity of their natural actions (loosely speaking, a group action on a metric space is said to be *rigid* if there is no non-degenerate action of the group on reasonably different metric spaces). An additional question is, in some sense, a more basic one which detects to what extent the situation under consideration is new: it consists in deciding whether the isometry group of a metric space (or some subgroup of it) is linear or not, i.e. is a matrix group for some suitable dimension and field. There exist very useful sufficient conditions for linearity concerning groups acting on CAT(0) cell complexes [44], and some simplicity results for automorphism groups of exotic buildings [43]. In Section 3, the three questions of linearity, rigidity and simplicity are discussed for groups acting on Kac-Moody buildings.

2. Classical buildings

Let us go back to classical buildings for a while, and more precisely to Euclidean ones. The latter spaces are often presented as non-archimedean analogues of Riemannian symmetric spaces of the non-compact type associated to real semisimple Lie groups (of positive rank). This leads to natural questions, usually more precise than the questions mentioned in 1.3 (where the analogy is looser since it compares symmetric spaces and arbitrary buildings with infinite Weyl groups). This section discusses compactifications of Bruhat-Tits buildings and cohomology of arithmetic groups in positive characteristic. The first point will be the opportunity to mention a new approach to Bruhat-Tits' theory that uses non-archimedean analytic geometry in the sense of V. Berkovich.

2.1. Group-theoretic compactifications. There are many reasons to wish to compactify equivariantly symmetric spaces and Bruhat-Tits buildings associated to semisimple groups. Some of them are related to the computation of the cohomology of discrete subgroups of Lie groups, some other reasons are related to random walks on Lie groups and related geometries. We refer to the books [41] and [8] for more details and discuss here a partial compactification procedure that has the advantage to be generalized to arbitrary buildings.

The starting point of this procedure is the (maybe surprising at first glance) fact that for any locally group H, the set $\mathscr{S}_H$ of closed subgroups in H has a natural topology which is compact [12, §5]: it is called the *Chabauty topology* (hint: identify closed subgroups with homothety classes of measures on the ambient group satisfying suitable invariance properties for the action of their support). The idea to use this fact in order to compactify Riemannian symmetric spaces (with underlying real Lie groups) is due to Y. Guivarc'h. It was generalized to the case of Bruhat-Tits buildings (with underlying non-archimedean Lie groups) in [42].

Let k be a locally compact local field, archimedean or not to begin with, and let G be a (simply connected) semisimple algebraic group over k. We let X be the symmetric space associated to $G(k)$ in the case when k is archimedean, or the Bruhat-Tits building of $G(k)$ if k is totally disconnected (1.2). In the first case, we have $X = G(k)/K$ where K is a maximal compact subgroup; in the second case, the $G(k)$-action on X admits any chamber as fundamental domain and the vertices in the closure of a given chamber parametrize the conjugacy classes of maximal compact subgroups (this follows from the Bruhat-Tits fixed point lemma of 1.3). It is a classical fact that the root system of a semisimple Lie group can be seen as a finite set of half-spaces in any maximal flat subspace A of X: see [4] in the real case; it is so by construction in the non-archimedean case, where A turns out to be an apartment [76]. Up to making a better choice in the second case, a maximal compact subgroup K in $G(k)$ always admits a fundamental domain given by a closed Weyl chamber in A, whose codimension 1 faces are called here *sector panels*; this is the geometric version of the Cartan decomposition of $G(k)$.

Now we restrict our attention to the case when k is non-archimedean and let $\{v_n\}_{n\geqslant 1}$ be a sequence of vertices in some closed Weyl chamber, say $\overline{\mathscr{Q}}$. By passing to stabilizers in $G(k)$ we obtain a sequence of maximal compact subgroups $\{K_{v_n}\}_{n\geqslant 1}$. If we further assume that for each sector panel Π of $\overline{\mathscr{Q}}$, the distance $d_X(v_n,\Pi)$ has a (possibly infinite) limit as $n \to +\infty$, then $\{K_{v_n}\}_{n\geqslant 1}$ converges in $\mathscr{S}_{G(k)}$. The limit group D is Zariski dense in some parabolic k-subgroup Q fixing a face of the chamber $\partial_\infty\overline{\mathscr{Q}}$ in the spherical building at infinity of X. Moreover D can be written as a semi-direct product $K \ltimes \mathscr{R}_u(Q)(k)$, where K is an explicit maximal compact subgroup of some reductive Levi factor of Q and $\mathscr{R}_u(Q)$ is the unipotent radical of Q. This convergence, proved by measure-theoretic means in the vein of ideas due to H. Furstenberg, is true in the archimedean case with vertices replaced by arbitrary points. It is the key fact to define a compact space $\overline{V}_X^{\mathrm{gp}}$ with a natural $G(k)$-action in any of the two cases.

Definition 2.1. The *group-theoretic compactification* of X is the closure of the set of maximal compact subgroups in $\mathscr{S}_{G(k)}$. In other words, it is the closure of the image of the $G(k)$-equivariant map $x \mapsto \mathrm{Stab}_{G(k)}(x)$ from X to $\mathscr{S}_{G(k)}$, which has to be restricted to the set V_X of vertices in X when X is a building (i.e., when k is ultrametric).

The next step then is to understand the geometry of $\overline{V}_X^{\mathrm{gp}}$ in Lie-theoretic terms. It turns out that, as in [72] for symmetric spaces, the group-theoretic compactification of the Bruhat-Tits building of the maximal semisimple quotient of each parabolic k-subgroup of G appears in the boundary [42, Theorem 16].

Theorem 2.2. *For any proper parabolic k-subgroup Q with radical $\mathscr{R}(Q)$, the group-theoretic compactification of the Bruhat-Tits building of $Q/\mathscr{R}(Q)$ lies in the boundary of $\overline{V}_X^{\mathrm{gp}}$. We let P be a minimal parabolic k-subgroup of G and we set $D_\varnothing = K \ltimes \mathscr{R}_u(P)_F$, where K is the maximal compact subgroup of some reductive*

Levi factor of P. Then the conjugacy class of $D_\varnothing$ is $G(k)$-equivariantly homeomorphic to the maximal Furstenberg boundary $\mathscr{F}$ of $G(k)$, and it is the only closed $G(k)$-orbit in $\overline{V}_X^{\mathrm{gp}}$. In fact, for any closed subgroup $D \in \overline{V}_X^{\mathrm{gp}}$ there is a sequence $\{g_n\}_{n\geqslant 1}$ in $G(k)$ such that $\lim_{n\to+\infty} g_n D {g_n}^{-1}$ exists and belongs to $\mathscr{F}$.

We have thus a description of a compactification of the set vertices of a Bruhat-Tits building which looks like the description of the compactification of a moduli space, together with some basic statements on the dynamics of the group action on the boundary (the theory of Furstenberg boundaries is presented for instance in [51]).

Before explaining in 2.2 what can be done to compactify the full building X instead of V_X, let us finish by saying that $\overline{V}_X^{\mathrm{gp}}$ can be used to give a geometric classification (up to finite index) of remarkable closed subgroups in $G(k)$: the boundary of $\overline{V}_X^{\mathrm{gp}}$, seen as a subset of $\mathscr{S}_{G(k)}$, as well as the family of the normalizers of the groups in this boundary, can be characterized by means of dynamical notions (distality and amenability).

Remark 2.3. The results mentioned here are contained in [42] but many of them were generalized since then to arbitrary locally finite buildings by P.-E. Caprace and J. Lécureux [25]. Moreover J. Lécureux proved that the group action on the boundary is amenable, leading to positive answers to the Baum-Connes conjecture for interesting classes of groups [50].

2.2. Compactifications using analytic geometry. This subsection presents joint work with A. Thuillier and A. Werner.

There are two main problems with the compactification procedure described in 2.1. The first one is that $\overline{V}_X^{\mathrm{gp}}$ is only a compactification of the set of vertices in X. The second one is the fact that, if one has in mind the compactifications of symmetric spaces as defined by I. Satake [72] or by H. Furstenberg [40], the outcome should be a finite family of compact spaces. The group-theoretic compactification corresponds to the maximal Satake-Furstenberg one. The main idea in the papers [69] and [70], which allows one to overcome these two difficulties, is to combine Bruhat-Tits' theory of semisimple groups over valued fields and Berkovich's theory of analytic spaces over complete non-archimedean fields.

Berkovich geometry [6] is a version of analytic geometry over complete non-archimedean valued fields in which the spaces have nice local connectivity properties. This is surprising because local fields have a totally disconnected topology, but this good local behaviour is due to the fact that many points (of analytic nature) are added to the points given by algebraic considerations. In algebraic geometry the building blocks are algebraic spectra $\mathrm{Spec}(A)$ consisting of prime ideals of commutative rings A endowed with the Zariski topology, while in Berkovich geometry they are analytic spectra $\mathscr{M}(A)$ of Banach k-algebras, consisting of multiplicative bounded seminorms $A \to \mathbf{R}_+$. More precisely, let A be a Banach ring i.e., a commutative unit ring endowed with a Banach norm $\|\cdot\|_A$ that is submultiplicative. The *analytic spectrum* of A is the set $\mathscr{M}(A)$ of multiplicative seminorms $A \to \mathbf{R}_{\geqslant 0}$ whose restrictions to A are bounded with respect to $\|\cdot\|_A$; this space is endowed with the coarsest topology making the evaluation maps $x \mapsto x(f)$ continuous ($f \in A$) and we henceforth use the notation $|\,f(x)\,|$ for $x(f)$. At last, to each variety V over k is attached a Berkovich analytic space over k, which is denoted by $V^{\rm an}$. Loosely speaking, the good local connectivity properties of Berkovich analytic spaces come from the fact that the class of maps $x \mapsto f(x)$ is replaced by the wider class of maps $x \mapsto |\,f(x)\,|$. Recall that Spec(A) is in one-to-one correspondence with the set of equivalence classes of ring homomorphisms from A to an arbitrary field, where two maps are identified if they factorize through a common third map, and note that an algebraic map $x \mapsto f(x)$ can be composed with many absolute values coming from huge extensions of k.

If we go back to the compactification problem, we shall merely say that a crucial property is the fact that the attachment $V \mapsto V^{\rm an}$ is functorial and satisfies:

(i) if V is affine with coordinate ring $k[V]$, then $V^{\rm an}$ consists of all the multiplicative seminorms $k[V] \to \mathbf{R}_+$ extending the absolute value of k;

(ii) if V is projective, then $V^{\rm an}$ is compact.

Another key ingredient is a partially functorial behavior of the Bruhat-Tits building with respect to field extensions [66] combined with the possibility to work with any complete extension of k. In some sense, this implies the possibility to see any point (possibly in the relative interior of a cell) in X as a good vertex in the huger Bruhat-Tits building of G over some non-archimedean extension of k. By adapting faithfully flat descent in this context, one obtains the possibility to attach to each point $x \in X$ a Berkovich analytic subgroup G_x (defined over k as an analytic space), and the assignment $x \mapsto G_x$ is injective (in particular it takes distinct values for any two distinct points, even if they lie in the same cell). Finally, the following result [69, 2.1] is the main step to obtain an analytic filling of the group-theoretic compactification $\overline{V}_X^{\rm gp}$ of 2.1.

Theorem 2.4. *Let X be the building associated to a simply connected semisimple algebraic group G over a local field k.*

(i) *For any $x \in X$, there is an analytic subgroup G_x of $G^{\rm an}$ defined over k such that for any non-archimedean extension K/k, we have: $G_x(K) = {\rm Stab}_{G(K)}(x)$.*

(ii) *For any $x \in X$, there is a unique point $\vartheta(x) \in G^{\rm an}$ such that:*
$$G_x = \{g \in G^{\rm an} : |\,f(g)\,| \leqslant |\,f(\vartheta(x))\,| \text{ for any } f \in k[G]\}.$$

(iii) *The resulting map $x \mapsto \vartheta(x)$ is a $G(k)$-equivariant embedding of X into $G^{\rm an}$ with closed image.*

This result gives a map $X \to G^{\rm an}$ and then, in order to obtain equivariant compactifications of X, it suffices to compose it with analytifications of algebraic maps from G to proper varieties (e.g., the maps to flag varieties $G \to G/P$ where P is a parabolic k-subgroup of G). The desired compactifications are the closures of X under these maps. When P varies over all the conjugacy classes of parabolic k-subgroups of G, one obtains all the expected analogues of the Satake-Furstenberg compactifications.

Remark 2.5. Together with the asymptotic cones [16, I.5] of symmetric spaces and Euclidean buildings alluded to in 1.3, the Bruhat-Tits buildings of G over non-archimedean extensions of k with dense valuations are other examples of non-simplicial Euclidean buildings that appear naturally.

The paper [69] also contains a Lie-theoretic description of the boundary structure of these compactifications and some extensions, from Bruhat-Tits' theory, of useful decompositions of the rational points $G(k)$. The paper [70] describes a variant of this compactification procedure which uses highest-weight theory and is closer in spirit to I. Satake's original ideas.

Remark 2.6. When G is split over k, the idea to combine Bruhat-Tits' theory and Berkovich geometry can be found already in [5, §5].

2.3. Cohomological and related questions.

Non-compact Riemannian symmetric spaces and Bruhat-Tits buildings are contractible spaces acted upon properly by the Lie groups they are associated with. These actions are therefore very useful to compute or estimate the cohomology of discrete subgroups of reductive Lie groups. Using this action and suitable compactifications, A. Borel and J.-P. Serre proved, among other things, that arithmetic and even S-arithmetic groups in characteristic 0 are of type F_∞ [9]. Recall that a group Γ is said to be *of type F_m* if it admits a free action on a contractible CW-complex whose m-skeleton has finitely many Γ-orbits; it is said to be *of type F_∞* if it is of type F_n for any n. These conditions are related to other more algebraic finiteness properties stated in terms of resolutions [18, VIII]. The *finiteness length* of Γ is the largest m such that Γ is of type F_m, i.e. admits a classifying space with finite m-skeleton.

In the case when the global ground field leading to the arithmetic groups under consideration is not of characteristic 0, things get much more complicated for cohomology. For instance in characteristic $p > 0$, finite generation is not always true for arithmetic groups, and finitely generated lattices needn't be virtually torsion-free either. Still, combined efforts by K.-U. Bux, R. Köhl, S. Witzel and K. Wortman led to the following result.

Theorem 2.7. *Let K be a global function field, let S be a finite set of places of K and let $\mathscr{O}_S$ be the ring of S-integers in K. Let G be a connected, absolutely almost simple, K-isotropic K-group. For each $v \in S$ let r_v be the rank of G over the completion K_v of K with respect to v. Then the finiteness length of the S-arithmetic group $G(\mathscr{O}_S)$ is equal to $(\sum_{v\in S} r_v) - 1$.*

It was proved in [24] that $(\sum_{v\in S} r_v) - 1$ is an upper bound for the finiteness length of $G(\mathscr{O}_S)$ and equality was proved in [23]. The nice feature of many results in this vein is the mixture of classical techniques such as reduction theory in positive characteristic [45], K. Brown's criterion from algebraic topology [17] and the use of recent tools from geometric group theory such as singular Morse theory [7].

Note that, so far in this report, the fact that for a simple group G, the Bruhat-Tits building is a simplicial complex, has not been exploited yet (in general a Bruhat-Tits building is a polysimplicial complex). Examples of works where this geometric fact is used are given by the papers [59] and [60] which provide a key step towards an almost complete answer to the congruence subgroup problem. We will see in 3.3 that this can also be used to develop a singular version of Hodge theory in order to obtain some vanishing results for the cohomology of automorphism groups of exotic buildings.

Remark 2.8. In this section, most applications of Bruhat-Tits' theory that are presented (except [59] and [60]) mainly deal with the building-theoretic aspect of it and not with the delicate theory of forms of reductive algebraic groups over the valuation ring of the valued ground field. The volume formula proved by G. Prasad [58], which eventually leads to the classification of fake projective planes [61], is an example of a result that needs, among other things, Bruhat-Tits' theory at the latter level of subtlety.

3. Kac-Moody theory and exotic buildings

In this section, we are interested in families of non-classical buildings admitting sufficiently large groups of automorphisms, and being therefore good candidates for the comparison with symmetric spaces and Bruhat-Tits buildings associated to semisimple Lie groups. The main source of such buildings comes from an algebraic machinery which was not *a priori* designed for these purposes, namely Kac-Moody theory. We explain here why the analogy is indeed fruitful. In fact, Kac-Moody groups provide a good balance between persistence of classical results from the theory of arithmetic groups and appearance of new phenomena. This is true in the framework of discrete groups, as well as in that of non-discrete locally compact groups. Moreover it is likely that this theory is also the source of many interesting profinite groups. As mentioned before, the three main questions organizing the study of Kac-Moody groups are those about linearity, rigidity and simplicity (but they are not the only ones).

3.1. Kac-Moody theory. Roughly speaking, Kac-Moody Lie algebras are infinite-dimensional generalizations of complex semisimple Lie algebras [47] and Kac-Moody groups integrate these Lie algebras over $\mathbf{Z}$, thus providing infinite-dimensional generalizations of Chevalley schemes [36]. Our goal in this section is to introduce the two versions of Kac-Moody groups, namely the minimal (possibly twisted) Kac-Moody groups and the complete ones; they are both presented and compared in J. Tits' Bourbaki talk [79].

Combinatorial Kac-Moody objects. The starting point to define all these objects is a *generalized Cartan matrix*; i.e. an integral matrix $A = [A_{s,t}]_{s,t\in S}$ satisfying: $A_{s,s} = 2$, $A_{s,t} \leqslant 0$ when $s \neq t$ and $A_{s,t} = 0 \Leftrightarrow A_{t,s} = 0$. It is more accurate to start with a *Kac-Moody root datum*, namely a 5-tuple $\mathcal{D} = \big(S, A, \Lambda, (c_s)_{s\in S}, (h_s)_{s\in S}\big)$, where A is a generalized Cartan matrix indexed by a finite set S and where Λ is a free $\mathbf{Z}$-module (with $\mathbf{Z}$-dual $\Lambda^\vee$); the elements c_s of Λ and h_s of $\Lambda^\vee$ are requested to satisfy $c_s(h_t) = A_{ts}$ for all $s, t \in S$. One defines then a complex Lie algebra $\mathfrak{g}_{\mathcal{D}}$ by a presentation generalizing Serre's presentation of finite-dimensional semisimple Lie algebras, involving $(h_s)_{s\in S}$ and the usual generators $(e_s)_{s\in S}$, $(f_s)_{s\in S}$ so that in particular $\mathbf{C}e_s \oplus \mathbf{C}h_s \oplus \mathbf{C}f_s \simeq \mathfrak{sl}_2(\mathbf{C})$.

Using the free abelian group $Q = \bigoplus_{s\in S} \mathbf{Z}\alpha_s$ on the symbols α_s, one defines a Q-gradation on $\mathfrak{g}_{\mathcal{D}}$ in which the degrees with non-trivial corresponding spaces belong to $Q^+ \cup Q^-$, where $Q^+ = \sum_{s\in S} \mathbf{N}\alpha_s$ and $Q^- = -Q^+$. The latter non-zero degrees are called *roots* and if $(c_s)_{s\in S}$ is free over $\mathbf{Z}$, they have the usual interpretation in terms of weight spaces. The *height* of a root $\alpha = \sum_{s\in S} n_s\alpha_s$ is the integer $\mathrm{ht}(\alpha) = \sum_{s\in S} n_s$. There is a natural action on the lattice Q by a Coxeter group W generated by involutions denoted again by $s \in S$; it is defined by setting $s.a_t = a_t - A_{st}a_s$. A root is called *real* if it is in the W-orbit of a *simple* root, i.e. some α_s; otherwise, it is said to be *imaginary*. The set of roots (resp. real roots, imaginary roots) is denoted by Δ (resp. Δ_{re}, Δ_{im}).

Minimal Kac-Moody groups. Using the divided powers $\frac{1}{n!}e_s^n$ and $\frac{1}{n!}f_s^n$ of the canonical generators e_s and f_s and of their Weyl group conjugates, J. Tits defined a certain $\mathbf{Z}$-form $\mathcal{U}_{\mathcal{D}}$ of the universal enveloping algebra $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$. The ring $\mathcal{U}_{\mathcal{D}}$ has a filtration indexed by Q; some subrings as well as their completions with respect to some subsemigroups of Q are used to construct Kac-Moody groups. For the adjoint action on $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$, the real root spaces have a locally nilpotent action which can be exponentiated to produce 1-parameter unipotent subgroups in the automorphism group of the $\mathbf{Z}$-form $\mathcal{U}_{\mathcal{D}}$ for suitable restrictions of parameters and elements in $\mathfrak{g}_{\mathcal{D}}$. By and large, the *minimal Kac-Moody group functor* $\mathfrak{G}_{\mathcal{D}}$ is an amalgamation of a split torus with character group Λ and of a quotient of the subgroup generated by these 1-parameter subgroups [78]. To each real root $\gamma \in \Delta_{\mathrm{re}}$ is attached a subgroup functor $\mathfrak{U}_\gamma$, which is isomorphic to the 1-dimensional additive group functor, but there is no subgroup associated to imaginary roots in minimal Kac-Moody groups. Non-split versions of minimal Kac-Moody groups can also constructed [63].

Example 3.1. The functor which sends a field k to the group $\mathrm{SL}_{n+1}(k[t,t^{-1}])$ is a minimal Kac-Moody group functor of affine type $\widetilde{A}_n$.

Complete Kac-Moody groups. More generally, minimal Kac-Moody groups generalize groups of the form $\mathbf{G}(k[t,t^{-1}])$ where k is a field and $\mathbf{G}$ is a k-isotropic semisimple group. Accordingly, complete Kac-Moody groups generalize groups like $\mathbf{G}\big(k(\!(t)\!)\big)$. We present here a construction due to G. Rousseau [67] which provides group functors defined over $\mathbf{Z}$; the functors have a structure of ind-scheme generalizing constructions due to O. Mathieu [54] or Sh. Kumar [49] over the complex numbers.

For suitable affine group schemes over fields in characteristic 0, the algebra of invariant distributions [37, II §4 n°6] can be identified with the universal enveloping algebra of the Lie algebra of the group. Moreover $\mathbf{Z}$-forms of this algebra can be used to define, by duality, $\mathbf{Z}$-forms of the rings of regular functions: this eventually leads to group schemes over $\mathbf{Z}$ extending the initial groups over $\mathbf{C}$. For a Kac-Moody root datum $\mathcal{D}$, G. Rousseau associates to any closed set of roots Ψ, a pro-unipotent group scheme $\mathfrak{U}^{\mathrm{ma}}_\Psi$ defined over $\mathbf{Z}$ [67, 3.1]. With this approach, imaginary roots do lead to root groups (which seems to be a promising property for this version of Kac-Moody groups) and all the groups $\mathfrak{U}^{\mathrm{ma}}_\Psi$ have a filtration described thanks to the root system. At last, if k is a finite field of characteristic p, the groups $\mathfrak{U}^{\mathrm{ma}}_\Psi(k)$ are pro-p.

Remark 3.2. L. Carbone and H. Garland also defined a representation theoretic completion $\mathfrak{G}^{\mathrm{cg}\lambda}_{\mathcal{D}}(k)$ of $\mathfrak{G}_{\mathcal{D}}(k)$ for each dominant weight λ [33].

Connection with building theory. We can now go back to the main subject matter of this report, i.e. building theory. One crucial fact about minimal Kac-Moody groups is that any such group $\mathfrak{G}_{\mathcal{D}}(k)$ over some field k enjoys a combinatorial structure refining that of a Tits system [13, IV §2], and called a *twin* BN-*pair*. As a consequence, there is a pair $X_\pm$ of twin buildings as mentioned in 1.3 such that $\mathfrak{G}_{\mathcal{D}}(k)$ acts strongly transitively on each of them. The apartments are explicitly described thanks to the Weyl group W of $\mathcal{D}$ and the buildings $X_\pm$ are locally finite if and only if the ground field k is finite (if so, the full isometry groups $\mathrm{Iso}(X_\pm)$ are then locally compact for the compact open topology). Similarly, the complete group $\mathfrak{G}^{\mathrm{ma}+}_{\mathcal{D}}(k)$ has a natural strongly transitive action on a single building which is closely related to the twin buildings $X_\pm$ [67, Corollaire 3.18]. In the latter case, a chamber stabilizer is isomorphic to the semi-direct product of a finite-dimensional split torus and of the pro-unipotent group scheme $\mathfrak{U}^{\mathrm{ma}}_\Psi$ associated to $\Psi = \Delta^+$ (where Δ^+ is the set of all positive roots).

Remark 3.3. When k is finite, there is another more elementary completion $\mathfrak{G}^{\mathrm{geom}}_{\mathcal{D}}(k)$ obtained by taking the closure of the image of $\mathfrak{G}_{\mathcal{D}}(k)$ in the isometry group $\mathrm{Iso}(X_\pm)$ [68, 1.B].

3.2. Non-linearity, simplicity and rigidity.

This subsection presents joint work with P.-E. Caprace.

We can now consider our three main questions: non-linearity, simplicity and rigidity, when dealing with minimal Kac-Moody groups over finite fields. The point is that these groups are, by definition, finitely generated groups which generalize arithmetic groups in positive characteristic like $\mathrm{SL}_{n+1}(\mathbf{F}_q[t,t^{-1}])$. Therefore the general theme is to try to answer the following question.

(∗) *To what extent is a finitely generated Kac-Moody group close to a discrete subgroup in a non-archimedean semisimple Lie group?*

In what follows, Λ denotes a minimal Kac-Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbf{F}_q)$ over some finite field of characteristic p.

Lattice property. The first result supporting the analogy of (∗) was proved independently in [32] and [62].

Theorem 3.4. *Assume that the Weyl group W of Λ is infinite and denote by $W(t) = \sum_{w\in W} t^{\ell(w)}$ its growth series. If $W(\frac{1}{q}) < \infty$, then the group Λ is a lattice of $X_+ \times X_-$; it is never cocompact.*

The statement means that the homogeneous space $\big(\mathrm{Isom}(X_+) \times \mathrm{Isom}(X_-)\big)/\Lambda$ carries an invariant measure of finite total volume. The proof relies on a simple measure-theoretic formula and an explicit description of a fundamental domain for the diagonal Λ-action of $X_+ \times X_-$. The fundamental domain is given by the product of a chamber and of a suitably chosen apartment of opposite sign. This can be seen by combinatorial arguments relevant to Tits systems (it is an analogue of the geometric interpretation of the Cartan decomposition mentioned in 2.1).

Normal subgroup property. The previous theorem suggests to try to prove the main results of the theory of discrete subgroups of Lie groups in the case of finitely generated Kac-Moody groups. A particularly well adapted part of this theory is G. Margulis' work on lattices in Lie groups [51] because many proofs there rely on measure-theoretic techniques (which can be more easily adapted to non-linear groups than arguments from pure algebraic group theory). One striking result in this field is a strong dichotomy called the normal subgroup property for higher-rank lattices. More precisely, one says that a group Γ has the *normal subgroup property* if for any $N \lhd \Gamma$ either N is finite and central in Γ, or N has finite index in Γ. Here is the result for Kac-Moody groups.

Theorem 3.5. *If the finitely generated Kac-Moody group Λ is a lattice of $X_+ \times X_-$, then it has the normal subgroup property.*

The proof is mainly a consequence of deep results due to Y. Shalom [73] and to U. Bader and Y. Shalom [3]. The idea is to follow Margulis' strategy: to sum up, we can assume that we are in a situation where $N \lhd \Lambda < \mathrm{Isom}(X_-) \times \mathrm{Isom}(X_+)$ for a center-free Λ; hence we have to prove that Λ/N is finite, i.e. is compact for the discrete topology! This apparently naive remark is a crucial trick because being compact here is equivalent to being amenable and having Kazhdan's property (T). Then the idea is to use a criterion due to Y. Shalom (resp. U. Bader and Y. Shalom) which says that in order to prove property (T) (resp. amenability) for the discrete quotient group Λ/N, it is enough to check it on the topological quotients $\overline{\mathrm{pr}_\pm(\Lambda)}/\overline{\mathrm{pr}_\pm(N)}$, where $\mathrm{pr}_\pm$ is the natural projection from $\mathrm{Isom}(X_-) \times \mathrm{Isom}(X_+)$ to $\mathrm{Isom}(X_\pm)$. Checking the latter points is easier because the involved topological groups have more structure: indeed, $\overline{\mathrm{pr}_\pm(\Lambda)}$ acts strongly transitively on the building $X_\pm$ since so does Λ. In fact, using Tits system arguments, one can see that the each topological quotient $\overline{\mathrm{pr}_\pm(\Lambda)}/\overline{\mathrm{pr}_\pm(N)}$ is compact. The paper [73] considers *cocompact* irreducible lattices in direct products, but the cocompactness assumption can relaxed to a weaker integrability condition involving an induction cocycle, which is checked in [65] thanks to combinatorial arguments.

Simplicity. The general strategy to prove simplicity of suitable (i.e. non-affine, irreducible) Kac-Moody lattices owes a lot to M. Burger and Sh. Mozes' seminal works [21] and [22]. Among other things, these papers prove the existence of finitely presented torsion free simple groups; these groups are constructed as lattices acting on products of two trees with a compact fundamental domain (in fact, the groups can be chosen to act transitively on the vertices of the square complex). The general idea is first to see the discrete groups under consideration as analogues of lattices in Lie groups in order to rule out infinite quotients, and then to exploit decisive differences with linear groups in order to rule out finite quotients too.

The first step, exploiting the analogy with lattices in Lie groups, is of course what was mentioned before in the Kac-Moody case. The point is to obtain the normal subgroup property without relying on any algebraic group structure on the ambient topological group. This structure is replaced by the fact that the latter topological group is the direct product of isometry groups of trees or buildings. The second step, where one has to stand by non-linear phenomena, is so far specific to each of the two situations: uniform lattices in products of trees in [22], or non-uniform lattices for products of (usually higher-dimensional) buildings in the Kac-Moody case. In the case of products of trees, this step relies on the possibility of obtaining some non-residual finiteness criteria involving transitivity conditions on the local actions (around each vertex) for the projection of the lattice on each of the two trees; this part was eventually improved by the possibility to embed explicitly well-known non-residually finite groups into suitable cocompact lattices of products of trees. In the Kac-Moody case, the arguments are relevant to Coxeter groups. This is where non-affineness of the Weyl group has to be exploited crucially: a strengthening of Tits' alternative for Coxeter groups implies that Coxeter complexes Σ of non-affine Coxeter groups contain lots of hyperbolic triples of roots (seen as half-spaces of Σ), i.e. with empty pairwise intersections. Combining this with a trick on infinite root systems and some defining relations for Kac-Moody groups leads to the following wide source of infinite finitely generated simple groups [29, §4].

Theorem 3.6. *Let Λ be a Kac-Moody group defined over the finite field $\mathbf{F}_q$. Assume that the Weyl group W is infinite and irreducible, and that $W(\frac{1}{q}) < \infty$. Then Λ is simple (modulo its finite center) whenever the buildings $X_\pm$ are not Euclidean and Λ is generated by its root subgroups.*

Remark 3.7. The assumption on generation by root groups is mild since the initial group Λ can be replaced by its finite index subgroup generated by the root groups, but the assumption excluding affine Weyl groups is crucial: indeed, groups of the form $\mathbf{G}(\mathbf{F}_q[t, t^{-1}])$, where $\mathbf{G}$ is a semisimple group over $\mathbf{F}_q$, are affine Kac-Moody groups and admit lots of (congruence) quotients.

Remark 3.8. Of course, the question of abstract simplicity for complete Kac-Moody groups makes sense too. Topological simplicity can be proved easily in this case by using Tits system arguments and the fact complete Kac-Moody groups over $\mathbf{F}_q$ are locally pro-p [64]. Using a beautiful mixture of dynamical and Lie-theoretic arguments, T. Marquis proved the (much better) abstract simplicity of the same groups [52].

Infinitely many quasi-isometry classes of simple groups. The wide choice of buildings admitting simple lattices is a very useful fact in geometric group theory. Recall that, after M. Gromov, it is natural to attach to each group Γ with finite symmetric generating set $S = S^{-1}$ its *Cayley graph*, i.e. the graph in which the vertices are the elements of Γ, which are declared to be adjacent if and only if they differ from the right by an element of S. This is the starting point to see these groups as metric spaces. One important notion in this context is that of quasi-isometry between metric spaces, that is almost bi-Lipschitz equivalence except that additive constant are allowed. More precisely, two metric spaces (X, d_X) and (Y, d_Y) are said to be *quasi-isometric* to one another if there is a map $f : X \to Y$ such that there exist $C \geqslant 1$ and $D \geqslant 0$ satisfying for each $x, x' \in X$:

$$\frac{1}{C} \cdot d_X(x, x') - D \leqslant d_Y(f(x), f(x')) \leqslant C \cdot d_X(x, x') + D$$

and such that for any $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leqslant D$. The first condition says that f is a *quasi-isometric embedding* and the second condition is a coarse metric surjectivity assumption.

Now let G be a locally compact group admitting a finitely generated lattice Γ; then G admits a compact generating subset, say $\widehat{\Sigma}$. We denote by $d_{\widehat{\Sigma}}$ the word metric associated with $\widehat{\Sigma}$ and we fix a finite generating set Σ for Γ, leading to an associated word metric d_Σ. The lattice Γ is called *undistorted* in G if d_Σ is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to Γ. This amounts to saying that the inclusion of Γ in G is a quasi-isometric embedding from the metric space (Γ, d_Σ) to the metric space $(G, d_{\widehat{\Sigma}})$.

It is proved in [30] that any Kac-Moody lattice $\Lambda < \mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$ is undistorted, and the most important consequence of this statement in geometric group theory is the following.

Theorem 3.9. *There exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.*

Note that since any two trees are bi-Lipschitz equivalent, all uniform lattices of products of trees lie in the same quasi-isometry class.

3.3. Cohomological and related questions.

It was mentioned in 2.3 that buildings, being simplicial complexes, are particularly well adapted to cohomology computation. Techniques from Hodge theory can be pushed quite far in this singular context [56]. Another approach, introduced by J. Dymara and T. Januszkiewicz, uses representation theoretic techniques as stated in [11] in the classical case, and leads to important results. The result below is a special case of [38, Theorem E].

Theorem 3.10. *Let Λ be a minimal Kac-Moody group over $\mathbf{F}_q$, defined by a generalized Cartan matrix A of size $n \times n$. Let $m < n$ be an integer such that all the principal submatrices of size $m \times m$ of A are Cartan matrices (i.e. are of finite type). Then for $1 \leqslant k \leqslant m-1$ and $q \gg 1$, the continuous cohomology groups $\mathrm{H}^k_{\mathrm{ct}}\big(\mathrm{Aut}(X_\pm), \rho\big)$ vanish for any unitary representation ρ.*

Degree 1 is of particular interest since $\mathrm{H}^1_{\mathrm{ct}}(G, \rho) = \{0\}$ for any unitary representation ρ is equivalent to Kazhdan's property (T) for G [46, Chap. 4]. When Λ is 2-spherical (i.e. when we have $m \geqslant 2$ above) Theorem 3.10 implies property (T) for the full automorphism groups $\mathrm{Aut}(X_\pm)$ with $q \gg 1$, hence for their product, and finally for any lattice in this product [51, III]. As a consequence, many Kac-Moody lattices have property (T) and this can be used to prove a super-rigidity result for isometric actions of higher-rank Kac-Moody groups on negatively curved metric spaces [29, §7].

Finite generation of maximal pro-p subgroups. Let us finish by mentioning another potential source of original results in a new group-theoretic framework. More precisely, if we now consider complete Kac-Moody groups as in 3.1 over finite fields, then we obtain locally pro-p groups (whatever the completion procedure, in fact). Moreover it follows from the Bruhat-Tits fixed point lemma that maximal pro-p subgroups in a given complete Kac-Moody group over $\mathbf{F}_q$ are all conjugate to one another [64]; they are finite index subgroups of chamber stabilizers for their natural action on the associated building.

Let $\mathfrak{G}_{\mathcal{D}}^{\mathrm{ma}+}(\mathbf{F}_q)$ be the algebraic completion of the minimal Kac-Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbf{F}_q)$. Let A be the generalized Cartan matrix of the Kac-Moody root datum $\mathcal{D}$ defining the group functor $\mathfrak{G}_{\mathcal{D}}$. Let $U^{\mathrm{ma}+}$ be a pro-p Sylow subgroup of $\mathfrak{G}_{\mathcal{D}}^{\mathrm{ma}+}(\mathbf{F}_q)$. The following theorem [28, Theorem 2.2] shows that the maximal pro-p subgroups in complete Kac-Moody groups over finite fields of characteristic p have an interesting behavior, which still deserves deeper investigation.

Theorem 3.11. *Assume that the characteristic p of $\mathbf{F}_q$ is greater than the absolute value of any off-diagonal coefficient of the generalized Cartan matrix A. Then $U^{\mathrm{ma}+}$ is finitely generated as a pro-p group.*

Remark 3.12. An argument initially due to L. Carbone, M. Ershov and G. Ritter [31], combining a Frattini sugbroup argument and a Tits system argument, implies the projective simplicity of complete Kac-Moody groups over finite fields for many types A. It can be generalized to all types but still leads to a weaker result than Marquis's theorem [52] because of the assumption on the size of p with respect to the coefficients of A.

The connection with cohomology is as follows: under the assumptions of the theorem, it can be proved that the following more precise statements hold.

(i) The Frattini subgroup $\Phi(U^{\mathrm{ma}+})$ of $U^{\mathrm{ma}+}$ is equal to the abstract derived group $[U^{\mathrm{ma}+}, U^{\mathrm{ma}+}]$.

(ii) We have: $\Phi(U^{\mathrm{ma}+}) = \overline{\langle U_\gamma : \gamma \text{ non-simple positive real root}\rangle}$.

(iii) We have also: $\mathrm{H}_1(U^{\mathrm{ma}+}, \mathbf{Z}/p\mathbf{Z}) \simeq (\mathbf{Z}/p\mathbf{Z})^{\mathrm{size}(A)\cdot[\mathbf{F}_q:\mathbf{Z}/p\mathbf{Z}]}$.

The connection between the Frattini subgroup and homology is that we have $\mathrm{H}_1(V, \mathbf{Z}/p\mathbf{Z}) \cong V/\Phi(V)$ for any pro-p group V [71, Lemma 6.8.6]; moreover $\dim_{\mathbf{Z}/p\mathbf{Z}} \mathrm{H}_1(V, \mathbf{Z}/p\mathbf{Z})$ is the minimal size of a topologically generating set for V. The latter point suggests to compute higher homology groups for pro-p Sylow subgroups of complete locally compact Kac-Moody groups. The next interesting result would be to be able to decide under which conditions these groups are finitely presentable as pro-p groups. This is related to $\mathrm{H}_2(U^{\mathrm{ma}+}, \mathbf{Z}/p\mathbf{Z})$.

Of course the question of simplicity doesn't make sense for pro-p groups, but discussing linearity of these pro-p Sylow subgroups definitely makes sense. One hope would be to disprove linearity for as many examples as possible. There are only partial results in this direction so far [28, §4].

Remark 3.13. By studying full pro-p completions of suitably chosen subgroups of minimal Kac-Moody group over finite fields, M. Ershov could exhibit some examples of Golod-Shafarevich groups with property (T), which leads to the existence of infinite torsion residually finite non-amenable groups [39].

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