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FLAT RANK OF AUTOMORPHISM GROUPS OF BUILDINGS

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Abstract. The flat rank of a totally disconnected locally compact group G, denoted flat-rk(G), is an invariant of the topological group structure of G. It is defined thanks to a natural distance on the space of compact open subgroups of G. For a topological Kac–Moody group G with Weyl group W, we derive the inequalities alg-rk$(W) \leqslant$ flat-rk$(G) \leqslant$ rk$(|W|_0)$. Here, alg-rk(W) is the maximal $\mathbb{Z}$-rank of abelian subgroups of W, and rk$(|W|_0)$ is the maximal dimension of isometrically embedded flats in the $\mathsf{CAT}(0)$-realization $|W|_0$. We can prove these inequalities under weaker assumptions. We also show that for any integer $n \geqslant 1$ there is a simple, compactly generated, locally compact, totally disconnected group G, with flat-rk$(G) = n$ and which is not linear.

Introduction

The general structure theory of locally compact groups is a well-established topic in mathematics. One of its main achievements is the solution to Hilbert's 5th Problem on characterizing Lie groups. The general structure results [MZ66] are still used in recent works. For instance, the No Small Subgroup Theorem [MZ66, 4.2] is used in

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Gromov's characterization of finitely generated groups of polynomial growth [Gro81]. More recently, the theory was used in rigidity problems for discrete groups, in order to attach suitable boundaries to quite general topological groups [BM02].

Simultaneous with these applications, recent years have seen substantial progress in extending known results about connected, locally compact groups to arbitrary locally compact groups. For example, in the theory of random walks on groups, it was shown in [JRW96] that the concentration functions for an irreducible probability measure on a noncompact group converge to 0, while [DSW06] contains the classification of ergodic $\mathbb{Z}^d$-actions on a locally compact group by automorphisms.

This progress has been due to structure theorems for totally disconnected, locally compact groups established in [Wil94], and further advanced in [Wil01] and [Wil04].

The study of particular classes of examples has played an important role in informing further developments of the structure theory of totally disconnected, locally compact groups, beginning with the study of the classes of p-adic Lie groups [Glö98], [GW01] and automorphism groups of graphs [Möl02]. This paper starts the examination of the topological invariants of totally disconnected groups which are closed automorphism groups of buildings with sufficiently transitive actions.

Topological Kac–Moody groups form a subclass of the latter class of groups, which is of particular interest to us. From a combinatorial viewpoint [RR06, 1.C] topological Kac–Moody groups generalize semisimple algebraic groups and therefore should be expected to inherit some of the properties of linear groups. For instance, their Tits system structure and the virtual pro-p-ness of their maximal compact subgroups are used to prove their topological simplicity [Rém04, 2.A.1]; these properties are well known in the algebraic case. On the other hand, it is known that some of these groups are nonlinear [Rém04, 4.C.1] (in fact, it follows from the not yet published work [CR] that *most* of these groups are nonlinear). This may imply that the topological invariants of topological Kac–Moody groups differ substantially from those of algebraic groups over local fields in some important aspects. In this context we mention a challenging question, which also motivates our interest in topological Kac–Moody groups, namely, whether a classification of topologically simple, compactly generated, totally disconnected, locally compact groups is a reasonable goal.

In this paper we focus on the most basic topological invariant of topological Kac–Moody groups G, the *flat rank* of G, denoted flat-rk(G). This rank is defined using the space $\mathcal{B}(G)$ of compact open subgroups of G. This space is endowed with a natural distance: for $V, W \in \mathcal{B}(G)$, the numbers $d(V,W) = \log\big(|V : V \cap W| \cdot |W : V \cap W|\big)$ define a discrete metric, for which conjugations in G are isometries. A subgroup $O \leqslant G$ is called *tidy* for an element $g \in G$ if it minimizes the displacement function of g on $\mathcal{B}(G)$; a subgroup H is called *flat* if all its elements have a common tidy subgroup. A flat subgroup H has a natural abelian quotient, whose rank is called its flat rank. Finally, the flat rank of G is the supremum of the flat ranks of the flat subgroups of G. For details, we refer to Subsection 1.3. The two main results of the paper provide an upper and lower bound for the flat rank of a sufficiently transitive automorphism group G of a building of finite thickness.

A summary of the main results about the upper bound for flat-rk(G) is given by the following statement. The assumptions made in this theorem are satisfied by topological Kac–Moody groups (Subsection 1.2).

Let us recall that the rank $\mathrm{rk}(X)$ of a $\mathsf{CAT}(0)$-space X is the maximal dimension of isometrically embedded flats in X.

Theorem A. *Let $(\mathcal{C}, S)$ be a building of finite thickness with Weyl group W. Denote by δ and by X the W-distance and the $\mathsf{CAT}(0)$-realization of $(\mathcal{C}, S)$, respectively. Let G be a closed subgroup of the group of automorphisms of $(\mathcal{C}, S)$. Assume that the G-action is transitive on ordered pairs of chambers at a given δ-distance. Then the following statements hold:*

(i) *The map $\varphi\colon X \to \mathcal{B}(G)$, mapping a point to its stabilizer, is a quasi-isometric embedding.*
(ii) *For any point $x \in X$, the image of the orbit map $g \mapsto g.x$, restricted to a flat subgroup of flat rank n in G, is an n-dimensional quasi-flat of X.*
(iii) *We have* $\text{flat-rk}(G) \leqslant \mathrm{rk}(X)$.
(iv) *If X contains an n-dimensional flat, so does any of its apartments: in other words,* $\mathrm{rk}(X) = \mathrm{rk}(|\,W\,|_0)$.

As a consequence, we obtain $\text{flat-rk}(G) \leqslant \mathrm{rk}(|W|_0)$, *where* $\mathrm{rk}(|W|_0)$ *is the maximal dimension of flats in the* $\mathsf{CAT}(0)$*-realization* $|W|_0$.

The strategy for the proof of Theorem A is as follows. The inequality $\text{flat-rk}(G) \leqslant \mathrm{rk}(|W|_0)$ is obtained from the statements (i)–(iv), which are proven in the listed order under weaker hypotheses. Statement (i) is part of Theorem 7 — usually called the Comparison Theorem in this paper, (ii) is proved in Proposition 9 under (i) as an assumption. Finally, (iii) is a formal consequence of (ii) using some results of Kleiner, and (iv) is Proposition 14.

The main result about the lower bound for $\text{flat-rk}(G)$ is the second half of Theorem 18, which we reproduce here as Theorem B. The class of groups satisfying the assumptions of Theorem B is contained in the class of groups satisfying the assumptions of Theorem A and contains all topological Kac–Moody groups (Subsection 1.2).

Theorem B. *Let G be a group with a locally finite twin root datum of associated Weyl group W. We denote by $\overline{G}$ the geometric completion of G, i.e. the closure of the G-action in the full automorphism group of the positive building of G. Let A be an abelian subgroup of W. Then A lifts to a flat subgroup $\widetilde{A}$ of $\overline{G}$ such that* $\text{flat-rk}(\widetilde{A}) = \mathrm{rank}_{\mathbb{Z}}(A)$. *As a consequence, we obtain* $\text{alg-rk}(W) \leqslant \text{flat-rk}(G)$, *where* $\text{alg-rk}(W)$ *is the maximal $\mathbb{Z}$-rank of abelian subgroups of W.*

The first half of Theorem 18 (not stated here) asserts that the flat rank of the rational points of a semisimple group over a local field k coincides with the algebraic k-rank of this group. This is enough to exhibit simple, compactly generated, locally compact, totally disconnected groups of arbitrary flat rank $d \geqslant 1$, e.g. by taking the sequence $\big(\mathrm{PSL}_{d+1}(\mathbb{Q}_p)\big)_{d\geqslant 1}$. Theorem C (Theorem 26 in the text) enables us to exhibit a sequence of *nonlinear* groups with the same properties.

Theorem C. *For every integer $n \geqslant 1$ there is a nonlinear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n.*

These examples are provided by Kac–Moody groups. The combinatorial data defining them are obtained by gluing a hyperbolic Coxeter diagram arising from a nonlinear Kac–Moody group, together with an affine diagram which ensures the existence of a sufficiently large abelian group in the resulting Weyl group. The proof relies on the nonlinearity results of [Rém04]. In fact, up to using the not yet published papers [CER] and [CR], we prove in the last section the following stronger statement: *for every integer* $n \geqslant 1$ *there is a nonlinear, abstractly simple, compactly generated, locally compact, totally disconnected group of flat rank* n.

Let us finish this introduction with a conjecture[1]. In Theorems A and B, the upper and lower bound on flat-rk(G) depend only on the associated Weyl group W. We think that the bounds are equal. If this is indeed so, then for every geometric completion G of a finitely generated Kac–Moody group we have flat-rk(G) = alg-rk(W) (where W is the associated Weyl group), hence, thanks of Krammer [Kra94, Theorem 6.8.3], the present paper computes flat-rk(G) (see also the paragraph following Definition 28).

Conjecture. *Let* W *be a finitely generated Coxeter group. Then we have* $\mathrm{rk}(|W|_0) = \text{alg-rk}(W)$*, where* $\mathrm{rk}(|W|_0)$ *is the maximal dimension of flats in the* $\mathsf{CAT}(0)$*-realization of the Coxeter complex of* W *and* alg-rk(W) *is the maximal rank of free abelian subgroups of* W*.*

The following, more general, conjecture seems to be the natural framework for questions of this kind: *if a group* G *acts cocompactly on a proper* $\mathsf{CAT}(0)$*-space* X*, then the rank of* X *equals the maximal rank of an abelian subgroup of* G*.* Some singular cases [BB94], [BB96] of this generalization as well as the smooth analytic case [BBS85], [BS91], have been proved. More recent contributions to this problem are due to Kapovich and Kleiner [KK07] and to Wise [Wis05].

Organization of the paper. Section 1 is devoted to recalling basic facts on buildings (chamber systems), on the combinatorial approach to Kac–Moody groups (twin root data), and on the structure theory of totally disconnected locally compact groups (flat rank). We prove the upper bound inequality in Section 2, and the lower bound inequality in Section 3. In Section 4 we exhibit a family of nonlinear groups of any desired positive flat rank.

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1. Buildings and totally disconnected groups

1.1. Buildings and their automorphism groups

Buildings as chamber systems. In this paper a building is a chamber system, denoted $\mathcal{C}$ throughout, together with a distance function with values in a Coxeter group. A chamber system is a set $\mathcal{C}$, called *the set of chambers*, together with a family S of partitions of $\mathcal{C}$, called *the adjacency relations.* Each element s of S defines an equivalence relation which we will not distinguish from s; members of the same s-equivalence class will be called *s-adjacent.* For each s in S the equivalence classes of s should be thought of as the set of chambers sharing a fixed 'face' of 'colour' s.

[1] Which has been verified by Caprace and Haglund in [CH] while this paper was being refereed.

A finite sequence of chambers such that consecutive members are adjacent (that is, contained in some adjacency relation) is called a *gallery*. A gallery is said to *join* its first and last terms. A gallery is called *nonstuttering* if consecutive members are different. For every nonstuttering gallery $(c_0, \ldots, c_n)$, any word $s_1 \cdots s_n$ in the free monoid S^* on S such that c_{j-1} and c_j are s_j-adjacent for all $1 \leqslant j \leqslant n$, is called a *type* of the gallery $(c_0, \ldots, c_n)$ (a gallery does not necessarily have a unique type). A gallery having a type contained in the submonoid of S^* generated by a subset T of S is called a T*-gallery*. A maximal subset of chambers which can be joined by a T-gallery is called a T*-residue*. We say that a chamber system $(\mathcal{C}, S)$ has *finite thickness* if S and all $\{s\}$-residues for $s \in S$ are finite.

A *morphism* between two chamber systems over the same index set, S say, is a map between the underlying sets of chambers that preserves s-adjacency for all s in S. A permutation of the underlying set $\mathcal{C}$ of a chamber system $(\mathcal{C}, S)$ is said to be an *automorphism* of $(\mathcal{C}, S)$ if it induces a permutation of S. The group of all automorphisms of $(\mathcal{C}, S)$ will be denoted $\mathsf{Aut}(\mathcal{C}, S)$. An automorphism of $(\mathcal{C}, S)$ is said to be *type-preserving* if it induces the identity permutation of S. The group of all type-preserving automorphisms of $(\mathcal{C}, S)$ will be denoted $\mathsf{Aut}_0(\mathcal{C}, S)$.

Each Coxeter system (W, S) gives rise to a chamber system: its set of chambers is W, and for $s \in S$, we say that w and w' are s-adjacent if and only if $w' \in \{w, ws\}$. A word f in the free monoid on S is called *reduced* if it has minimal length among all such words representing their product s_f as an element of W. If (W, S) is a Coxeter system and T is a subset of S, then the subgroup of W generated by T is called a *special subgroup* and is denoted W_T. A subset T of S is called *spherical* if W_T is finite.

Let (W, S) be a Coxeter system. A *building* of type (W, S) is a chamber system $\mathcal{C}$ with adjacency relations indexed by the elements of S, each consisting of sets containing at least two elements. We also require the existence of a W*-distance* $\delta\colon \mathcal{C} \times \mathcal{C} \to W$ such that whenever f is a reduced word on S, then for chambers x and y we have $\delta(x, y) = s_f$ if and only if there is a (nonstuttering) gallery of type f joining x to y. In a building, a nonstuttering gallery has a unique type. Other basic properties of W-distances can be found in [Ron89, 3.1]. Every Coxeter system (W, S) is a building with W-distance δ defined by $\delta(x, y) := x^{-1}y$. The image of a map which preserves the corresponding W-distances form the Coxeter system (W, S) into a building of type (W, S) is called an *apartment* of the building.

Nonpositively curved realization of a building. A chamber system can be realized as a topological space so that each chamber is homeomorphic to a model space X, and adjacency of chambers is represented by them sharing a preassigned subspace of X as a common 'face'. We now explain a very flexible way to do this for chamber systems which are buildings. The general method is due to Vinberg [Vin71]. It was later applied by Davis to associate to each Coxeter group a locally finite metric realization of the Coxeter system and Moussong then checked that the piecewise Euclidean metric on it is $\mathsf{CAT}(0)$. We follow Davis' exposition [Dav98].

Let $(\mathcal{C}, S)$ be a building of type (W, S). We start out with a topological space X, which will be our model for a chamber, and a family of closed subspaces $(X_s)_{s \in S}$, which will be our supply of 'faces'. The pair $(X, (X_s)_{s \in S})$ will be called a *model space*. For each point x in X we define a subset $S(x)$ of S by setting $S(x) := \{s \in S \colon x \in X_s\}$. Furthermore, we define an equivalence relation $\sim$ on the set $\mathcal{C} \times X$ by $(c, x) \sim (c', x')$ if and only if $x = x'$ and $\delta(c, c') \in W_{S(x)}$. The **$X$-realization of** $(\mathcal{C}, S)$, written $X(\mathcal{C})$, is the quotient space $(\mathcal{C} \times X)/\!\sim$, where $\mathcal{C}$ carries the discrete topology.

If $(\mathcal{C}, S)$ is a building, and $(X, (X_s)_{s\in S})$ is a model space, then any type-preserving automorphism of $(\mathcal{C}, S)$ induces a homeomorphism of $X(\mathcal{C})$ via the induced permutation action on $\mathcal{C} \times X$. Furthermore, this assignment defines a homomorphism from $\mathsf{Aut}_0(\mathcal{C}, S)$ into the group of homeomorphisms of $X(\mathcal{C})$. This homomorphism is injective if $X \backslash \bigcup_{s\in S} X_s$ is nonempty. Automorphisms of $(\mathcal{C}, S)$ which are not type-preserving will not induce homeomorphisms of $X(\mathcal{C})$ unless the model space $(X, (X_s)_{s\in S})$ admits symmetries realizing the possible type permutations. We will spell out appropriate conditions below for specific choices of model spaces.

We now introduce the model spaces $(X, (X_s)_{s\in S})$ which define the Davis realization of a building $(\mathcal{C}, S)$ leading to a $\mathsf{CAT}(0)$-structure on $X(\mathcal{C})$. A variant of it, available for a subclass of buildings, is used by Moussong to define a $\mathsf{CAT}(-1)$-structure on $X(\mathcal{C})$ [Mou88]. *We assume from now on that $(\mathcal{C}, S)$ is a building of finite rank, i.e. S is finite.* The model spaces X in both cases are metric simplicial complexes (with a family of subcomplexes $(X_s)_{s\in S}$) with the same underlying abstract simplicial complex (and the same subcomplexes X_s), namely, the flag complex of the poset of spherical subsets of S ordered by inclusion. For $s \in S$, the subcomplex X_s is the union of all chains starting with the set $\{s\}$ (and all their subchains). This model space always supports a natural piecewise Euclidean structure [Dav98] as well as a piecewise hyperbolic structure [Mou88, Section 13].

Our assumption that S is finite implies that X is a finite complex. In particular, $X(\mathcal{C})$ has only finitely many cells up to isometry, so Bridson's theorem [BH99, I.7.50] implies that $X(\mathcal{C})$ with the path metric is a complete geodesic space. Moreover, X has finite diameter since only compact simplices are used for the hyperbolic structure. Suppose that all s-residues of $(\mathcal{C}, S)$ are finite for $s \in S$. Then, because of the way we defined the family of subspaces $(X_s)_{s\in S}$ encoding the faces, $X(\mathcal{C})$ is locally finite. The geometric realization of a Coxeter complex based on model spaces for the Davis realization is $\mathsf{CAT}(0)$ (and the Moussong realization of a Coxeter complex is $\mathsf{CAT}(-1)$ if and only if the Coxeter group is Gromov-hyperbolic) [Mou88]. Using retractions onto apartments one shows that analogous results hold for buildings whose Coxeter group is of the appropriate type [Dav98, Section 11].

For both the Davis and Moussong realizations the map which assigns to a type-preserving automorphism of the building $(\mathcal{C}, S)$ the self-map of $X(\mathcal{C})$ induced by the permutation of $\mathcal{C}$ defines a homomorphism from $\mathsf{Aut}_0(\mathcal{C}, S)$ into the group of simplicial isometries of $X(\mathcal{C})$, which we will denote by $\mathsf{Isom}(X(\mathcal{C}))$. The metric structures on the corresponding model spaces are in addition invariant under all diagram automorphisms of the Coxeter diagram of the building. Hence automorphisms of $(\mathcal{C}, S)$ also induce simplicial isometries of $X(\mathcal{C})$ in both cases. Since the vertex $\varnothing$ of X is not contained in any of the subcomplexes X_s for $s \in S$, these homomorphisms are injective. We denote the Davis and Moussong realizations of a building $(\mathcal{C}, S)$ by $|\mathcal{C}|_0$ and $|\mathcal{C}|_{-1}$, respectively.

Whenever we make a claim involving the Moussong realization $|\mathcal{C}|_{-1}$ of a building, we implicitly assume that the Coxeter group associated to the building is Gromov-hyperbolic. In fact, we do not use the Moussong realization for any of our results on topological automorphism groups of buildings, so that the reader may ignore all occurrences of $|\mathcal{C}|_{-1}$.

The natural topology on the group of automorphisms of a chamber system is the permutation topology.

Definition 1. Let a group G act on a set M. For any subset F of M we denote by $\mathrm{Fix}_G(F)$ the pointwise stabilizer $\{g \in G : g.x = x \text{ for every } x \in F\}$. The *permutation topology* on G is the topology with the family $\{\mathrm{Fix}_G(F) \mid F \text{ finite subset of } M\}$ as a neighbourhood base of the identity in G.

The automorphism group of a chamber system is a subgroup of the permutation group on the set of chambers. For $\epsilon \in \{0, -1\}$, the permutation topology maps to another natural topology under the monomorphism $\mathsf{Aut}(\mathcal{C}, S) \to \mathsf{Isom}(|\mathcal{C}|_\epsilon)$.

Lemma 2. *Let $(\mathcal{C}, S)$ be a building of finite thickness. Then the permutation topology on $\mathsf{Aut}(\mathcal{C}, S)$ maps to the compact-open topology under the map $\mathsf{Aut}(\mathcal{C}, S) \to \mathsf{Isom}(|\mathcal{C}|_\epsilon)$ for $\epsilon \in \{0, -1\}$.* $\square$

1.2. Topological automorphism groups of buildings

The examples of topological automorphism groups of buildings we are most interested in are Kac–Moody groups over finite fields. We will not define them and refer the reader instead to [Tit87, Subsection 3.6] and [Rém02b, Section 9] for details. A Kac–Moody group over a finite field is an example of a group G with twin root datum $\big((U_a)_{a\in\Phi}, H\big)$ (of type (W, S); compare [Rém02b, 1.5.1] for the definition) such that all the root groups are finite. We will call a group which admits a twin root datum consisting of finite groups *a group with a locally finite twin root datum.*

Any group G with a locally finite twin root datum of type (W, S) admits an action on a twin building (compare [Rém02b, 2.5.1] for the definition) of type (W, S) having finite s-residues for all $s \in S$. Let $(\mathcal{C}, S)$ be the positive twin (which is isomorphic to the negative twin). Its geometric realizations $|\mathcal{C}|_0$ and $|\mathcal{C}|_{-1}$ (if defined) are locally finite. The group H is the fixator of an apartment $\mathcal{A}$ of $(\mathcal{C}, S)$ with respect to this action of G. Hence we have a short exact sequence $1 \to H \to N \to W \to 1$, where N is the stabilizer of $\mathcal{A}$ (note that both BN-pairs are saturated).

In fact, the action of G on $(\mathcal{C}, S)$ is *strongly transitive* in the following sense [Ron89, p. 56].

Definition 3. Suppose a group G acts on a building $(\mathcal{C}, S)$ with Weyl group W and W-distance function δ:

(i) The action of G on $(\mathcal{C}, S)$ is said to be *δ-2-transitive* if whenever (c_1, c_2), and (c_1', c_2') are ordered pairs of chambers of $(\mathcal{C}, S)$ with $\delta(c_1, c_2) = \delta(c_1', c_2')$, then the diagonal action of some element of G maps (c_1, c_2) to (c_1', c_2').

(ii) If, moreover, the stabilizer N of some apartment $\mathcal{A}$ is transitive on the chambers of $\mathcal{A}$, we say that the action is *strongly transitive.*

Some of our result are valid for δ-2-transitive group actions which are not necessarily strongly transitive. Note that if the action of G on a building is δ-2-transitive, then so is its restriction to the finite index subgroup of type-preserving automorphisms in G. Upon taking $c_1 = c_2$ and $c_1' = c_2'$ in the definition of a δ-2-transitive action, it is seen that a δ-2-transitive action is transitive on the set of chambers. Hence, in a building $(\mathcal{C}, S)$ that admits a δ-2-transitive action, all residues of the same type have the same cardinality. We have learned that our notion of δ-2-transitivity has been called Weyl-transitivity by Abramenko and Brown in their preprint [AB], where they construct examples of δ-2-transitive actions which are not strongly transitive following a suggestion by Tits.

Finally, if G is a group with a locally finite twin root datum $\big((U_a)_{a\in\Phi}, H\big)$, then the G-actions on each building have a common kernel K. Moreover, the root groups embed in G/K and $\big((U_a)_{a\in\Phi}, H/K\big)$ is a twin root datum for G/K with the same associated Coxeter system and twin buildings [RR06, Lemma 1]. When G is a Kac–Moody group over a finite field, we have $K = Z(G)$. The *topological group associated to* G, denoted $\overline{G}$, is the closure of G/K with respect to the topology on $\mathsf{Aut}(\mathcal{C}, S)$ defined in the previous subsection.

1.3. Structure of totally disconnected, locally compact groups

The structure theory of totally disconnected, locally compact groups is based on the notions of a tidy subgroup for an automorphism and the scale function. These notions were defined in [Wil94] in terms of the topological dynamics of automorphisms and the definitions were reformulated in [Wil01]. We take the geometric approach to the theory as outlined in [BW06] and further elaborated on in [BMW04]. An overview of the geometric approach can be found in [Bau07].

Let G be a totally disconnected, locally compact group and let $\mathsf{Aut}(G)$ be the group of bicontinuous automorphisms of G. We want to analyze the action of subgroups of $\mathsf{Aut}(G)$ on G; we will be interested primarily in groups of inner automorphisms of G. To that end we consider the induced action of $\mathsf{Aut}(G)$ on the set

$$\mathcal{B}(G) := \{V \mid V \text{ is a compact, open subgroup of } G\}.$$

The function

$$d(V, W) := \log(|V : V \cap W| \cdot |W : W \cap V|)$$

defines a metric on $\mathcal{B}(G)$ and $\mathsf{Aut}(G)$ acts by isometries on the discrete metric space $(\mathcal{B}(G), d)$. Let α be an automorphism of G. An element O of $\mathcal{B}(G)$ is called *tidy for* α if the *displacement function* of α, denoted by $d_\alpha \colon \mathcal{B}(G) \to \mathbb{R}$ and defined by $d_\alpha(V) = d(\alpha(V), V)$, attains its minimum at O. Since the set of values of the metric d on $\mathcal{B}(G)$ is a well-ordered discrete subset of $\mathbb{R}$, every $\alpha \in \mathsf{Aut}(G)$ has a subgroup tidy for α. Suppose that O is tidy for α. The integer

$$s_G(\alpha) := |\alpha(O) : \alpha(O) \cap O|,$$

which is also equal to $\min\big\{|\alpha(V) : \alpha(V) \cap V| \mid V \in \mathcal{B}(G)\big\}$, is called *the scale of* α. An element of $\mathcal{B}(G)$ is called *tidy for a subset* $\mathcal{M}$ of $\mathsf{Aut}(G)$ if and only if it is tidy for every element of $\mathcal{M}$. A subgroup $\mathcal{H}$ of $\mathsf{Aut}(G)$ is called *flat* if and only if there is an element of $\mathcal{B}(G)$ which is tidy for $\mathcal{H}$. We will call a subgroup H of G flat if and only if the group of inner automorphisms induced by H is flat.

Later we will uncover implications of the flatness condition for groups acting in a nice way on $\mathsf{CAT}(0)$-spaces. They are based on the following properties of flat groups which hold for automorphism groups of general totally disconnected, locally compact groups. Suppose that $\mathcal{H}$ is a flat group of automorphisms. The set

$$\mathcal{H}(1) := \{\alpha \in \mathcal{H} \mid s_G(\alpha) = 1 = s_G(\alpha^{-1})\}$$

is a normal subgroup of $\mathcal{H}$ and $\mathcal{H}/\mathcal{H}(1)$ is free abelian. The *flat rank* of $\mathcal{H}$, denoted flat-rk($\mathcal{H}$), is the $\mathbb{Z}$-rank of $\mathcal{H}/\mathcal{H}(1)$. If $\mathcal{A}$ is a group of automorphisms of the totally disconnected, locally compact group G, then its flat rank is defined to be the supremum of the flat ranks of all flat subgroups of $\mathcal{A}$. The flat rank of the group G itself is the flat rank of the group of inner automorphisms of G.

If $\mathcal{H}$ is a flat group of automorphisms with O tidy for $\mathcal{H}$, then by setting $\|\alpha\mathcal{H}(1)\|_{\mathcal{H}} := d(\alpha(O), O)$ one defines a norm on $\mathcal{H}/\mathcal{H}(1)$. That is, $\|\cdot\|_{\mathcal{H}}$ satisfies the axioms of a norm on a vector space with the exception that we restrict scalar multiplication to integers. This norm can be given explicitly in terms of a set of epimorphisms $\Phi(\mathcal{H}, G) \subseteq \mathrm{Hom}(\mathcal{H}, \mathbb{Z})$ of *root functions* and a set of *scaling factors* s_ρ; $\rho \in \Phi(\mathcal{H}, G)$ associated to $\mathcal{H}$. In terms of these invariants of $\mathcal{H}$ the norm may be expressed as $\|\alpha\mathcal{H}(1)\|_{\mathcal{H}} = \sum_{\rho\in\Phi} \log(s_\rho)|\rho(\alpha)|$ [Bau07, Proposition 6]. In particular, the function $\|\cdot\|_{\mathcal{H}}$ extends to a norm in the ordinary sense on the vector space $\mathbb{R} \otimes \mathcal{H}/\mathcal{H}(1)$. For further information on flat groups, see [Wil04].

2. Geometric rank as an upper bound

In this section we show that the flat rank of a locally compact, strongly transitive group G of automorphisms of a building of finite thickness is bounded above by the geometric rank of the Weyl group. This inequality actually holds under the weaker assumption that the G-action is δ-2-transitive (a notion which we introduced in Definition 3 of Subsection 1.2).

2.1. Consequences of δ-2-transitivity

In this subsection our main result is Theorem 7, which compares the Davis-realization of a building of finite thickness with the space of compact open subgroups of a closed subgroup of its automorphism group acting δ-2-transitively.

The following proposition allows us to compute the distance between stabilizers of chambers for such groups.

Proposition 4. *Suppose that the action of a group G on a building of finite thickness $(\mathcal{C}, S)$ with W-distance δ is δ-2-transitive and type-preserving. For each type s let $q_s + 1$ be the common cardinality of s-residues in $(\mathcal{C}, S)$. Let (c, c') be a pair of chambers of $(\mathcal{C}, S)$ and let $s_1 \cdots s_l$ be the type of some minimal gallery connecting c to c'. Then $|G_c \colon G_c \cap G_{c'}| = \prod_{j=1}^{l} q_{s_j}$.*

Proof. Set $w = \delta(c, c')$. The index $|G_c \colon G_c \cap G_{c'}|$ is the cardinality of the orbit of the chamber c' under the action of G_c. Since the action of G is type-preserving, this orbit is contained in the set $c_w = \{d \mid \delta(c, d) = \delta(c, c')\}$. Since the action of G is δ-2-transitive as well, the orbit is equal to this set. It remains to show that the cardinality of c_w is equal to $\prod_{j=1}^{l} q_{s_j}$.

Pick $d \in c_w$. By definition of buildings in terms of W-distance and by [Ron89, (3.1)v], the chamber c is connected to d by a unique minimal gallery of type $s_1 \cdots s_l$. On the other hand, any endpoint of a gallery of type $s_1 \cdots s_l$ starting at c will belong to c_w. Therefore the cardinality of c_w, hence the index $|G_c : G_c \cap G_{c'}|$, equals the number of galleries of type $s_1 \cdots s_l$ starting at c. That latter number is equal to $\prod_{j=1}^{l} q_{s_j}$, establishing the claim. $\square$

Let α be a (bicontinuous) automorphism of a locally compact group G. Recall that the *module* of α is the positive real number $\Delta_G(\alpha)$ such that $\alpha_* \mathrm{d}g = \Delta_G(\alpha) \cdot \mathrm{d}g$, where $\mathrm{d}g$ is any left Haar measure on G and $\alpha_* \mathrm{d}g$ is its push-forward by α. The number $\Delta_G(\alpha)$ doesn't depend on the choice of $\mathrm{d}g$ and we say that G is *unimodular* if the module of any inner automorphism is equal to 1 [Bou63, Chapitre 7]. Before we continue with the preparation of Theorem 7, we derive the following corollary (which is used only in the proof of Remark 8).

Corollary 5. *A closed subgroup G of the automorphism group of a building of finite thickness, satisfying the conditions of Proposition* 4*, is unimodular. In particular, the scale function of G assumes the same value at a group element and its inverse.*

Proof. Let α be a bicontinuous automorphism of a totally disconnected, locally compact group G. Then the module of α equals $|\alpha(V) : \alpha(V) \cap V| \cdot |V : \alpha(V) \cap V|^{-1}$, where V is an arbitrary compact, open subgroup of G.

If we apply this observation in the situation of Proposition 4 with $V = G_c$ and α being inner conjugation by $x \in G$, we conclude that the modular function of G takes the value $|G_{x.c} : G_{x.c} \cap G_c| \cdot |G_c : G_{x.c} \cap G_c|^{-1}$ at x. Since traversing a minimal gallery joining $x.c$ to c in the opposite order gives a minimal gallery joining c to $x.c$, the formula for the index derived in Proposition 4 shows that x has module 1. Since x was arbitrary, G is unimodular.

Our second claim follows from the first one, because the value of the modular function of G at an automorphism α is $s_G(\alpha)/s_G(\alpha^{-1})$. We recall the proof. Choose V to be tidy for the automorphism α of G. Then $|\alpha(V) : \alpha(V) \cap V|$ equals the scale of α while $|V : \alpha(V) \cap V|$ equals the scale of α^{-1}. By the definition of the modular function Δ_G of G, we have $\Delta_G(\alpha) = s_G(\alpha)/s_G(\alpha^{-1})$ as claimed. □

The following concept will be used to make the comparison in Theorem 7.

Definition 6 (Adjacency Graph of a Chamber System)**.** Let $(\mathcal{C}, S)$ be a chamber system and let $d : s \mapsto d_s$ be a map from S to the positive real numbers. The *adjacency graph of* $(\mathcal{C}, S)$ *with respect to* $(d_s)_{s\in S}$ is defined as follows. It is the labelled graph $\Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ having $\mathcal{C}$ as a set of vertices; two vertices $c, c' \in \mathcal{C}$ are connected by an edge of label s if and only if c and c' are s-adjacent for some $s \in S$; each edge of label s is defined to have length d_s.

We are ready to state and prove our comparison theorem.

Theorem 7 (Comparison Theorem)**.** *Suppose that $(\mathcal{C}, S)$ is a building of finite thickness with W-distance function δ and let $\epsilon \in \{0, -1\}$.*

(1) *Suppose that $(d_s)_{s\in S}$ is a set of positive real numbers indexed by S. Then any map $\psi\colon |\mathcal{C}|_\epsilon \to \Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ which sends a point x to some chamber $c \in \mathcal{C}$ such that $x \in |c|$ is a quasi-isometry.*
(2) *Let G be a closed subgroup of the group of automorphisms of $(\mathcal{C}, S)$ such that the action of G is δ-2-transitive. Assume that all the cardinalities of the s-residues for s in S are bigger than* 2*. Then the map $\varphi\colon |\mathcal{C}|_\epsilon \to \mathcal{B}(G)$ mapping a point to its stabilizer is a quasi-isometric embedding.*

Remark 8. The range of the map φ considered in part (2) above cannot be quasi-dense unless the Weyl group of $(\mathcal{C}, S)$ (and hence the building itself) is finite.

Proof of Remark 8. As will be seen in the proof of Theorem 7 below, we may assume that the G-action is type-preserving, so that we may apply Proposition 4.

As argued at the end of the forthcoming proof of Theorem 7, there is a constant M such that the stabilizer of any point is at a distance of at most M to the stabilizer of the centre of a chamber in $|\mathcal{C}|_\epsilon$.

We will show however, under the condition that the Weyl group of $(\mathcal{C}, S)$ is infinite, that for each positive K there is a compact, open subgroup V_K of G whose distance to each chamber stabilizer exceeds K, which proves Remark 8.

Since the group G is unimodular by Corollary 5, me may take for V_K any compact open subgroup of G such that the Haar measure of some (hence, every) chamber stabilizer is $\exp(K)$ times the Haar measure of V_K.

In order to construct V_K, fix a chamber c in $(\mathcal{C}, S)$ and choose a chamber c' whose gallery-distance to c is at least $\log_2(\exp(K))$, where $\log_2$ denotes the logarithm to base 2.

Proposition 4 shows that the index of the group $V_K := G_c \cap G_{c'}$ in G_c is at least $\exp(K)$, which implies that its Haar measure is smaller than the Haar measure of G_c (hence that of every other chamber stabilizer) by a factor of at least K, which finishes the proof. $\square$

Proof of Theorem 7. We begin by proving the first claim.

Both $m := \min\{d_s \mid s \in S\}$ and $M := \max\{d_s \mid s \in S\}$ are finite and positive, because S is a finite set and $\{d_s \mid s \in S\}$ is a set of positive numbers. The image of ψ is obviously $M/2$-quasi-dense in $\Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ and we need to prove that ψ is a quasi-isometric embedding as well.

To see this, we compare distances between points in the space $|\mathcal{C}|_\epsilon$ and the graph $\Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ to the gallery-distance between chambers corresponding to these points. To that end, denote by $d_\mathcal{C}$, d_Γ and d_ϵ the gallery-distance on the set of chambers, the distance in the graph $\Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ and the distance on $|\mathcal{C}|_\epsilon$, respectively. Factor ψ as the composite of a map $\psi' \colon (|\mathcal{C}|_\epsilon, d_\epsilon) \to (\mathcal{C}, d_\mathcal{C})$, sending each point to some chamber containing it and the map $\iota \colon \big(\mathcal{C}, d_\mathcal{C}\big) \to \big(\Gamma(\mathcal{C}, S, (d_s)_{s\in S}), d_\Gamma\big)$ induced by the identity on $\mathcal{C}$.

For any pair of chambers c and c' we have

$$m\, d_\mathcal{C}(c, c') \leqslant d_\Gamma(c, c') \leqslant M d_\mathcal{C}(c, c'),$$

which shows that ι is a quasi-isometric embedding.

Furthermore, given two points, x and y say, in $|\mathcal{C}|_\epsilon$, there is a minimal gallery such that the geometric realizations of the chambers in that gallery do cover the geodesic joining x to y. (This can be seen as follows. By convexity of apartments, it suffices to prove this statement in the case where $(\mathcal{C}, S)$ is a Coxeter complex. The geodesic segment $[x, y]$ intersects precisely those walls, for which there are some chambers, c_x say, containing x and, c_y say, containing y which are separated by those same walls. The existence of a minimal gallery covering the segment $[x, y]$ is then proved by induction on the number of these walls.)

Denoting by D the diameter of the geometric realization $|c|$ of a chamber in $|\mathcal{C}|_\epsilon$, and by r the maximal gallery-distance between chambers in the same spherical residue we conclude, using the description of minimal galleries in terms of reduced words in the Coxeter group, that

$$d_\epsilon(x, y) - D \leqslant D\, d_\mathcal{C}(\psi'(x), \psi'(y)) \leqslant r\, d_\epsilon(x, y) + 2rD.$$

It follows that ψ' is a quasi-isometric embedding, and so ψ is as well, finishing the proof of the first claim.

The second claim will be derived from the first and Proposition 4, once we have shown that we can reduce to the case where G acts by type-preserving automorphisms. Let G° be the subgroup of type-preserving automorphisms in G. It is a closed subgroup of finite index, say n, in G. Therefore it is open in G, and $d(O, G^\circ \cap O) \leqslant \log(|G : G^\circ|) = \log n$ for each open subgroup O of G. It follows that the map $\mathcal{B}(G^\circ) \to \mathcal{B}(G)$ induced by the inclusion $G^\circ \hookrightarrow G$ is a quasi-isometry. Since G° is δ-2-transitive as well, we may — and shall — assume that $G = G^\circ$.

Choose some map ψ satisfying the conditions on the map with the same name in part (1). As before denote the flag consisting of the single vertex $\varnothing$ in X by $(\varnothing)$. It defines a vertex of the simplicial complex underlying the model space X of the Davis and Moussong realizations. Denote by $[c, x]$ the equivalence class with respect to the relation $\sim$ containing the pair $(c, x) \in \mathcal{C} \times X$. To avoid heavy notation, we set $\hat{c} := [c, |\varnothing|]$ for each $c \in \mathcal{C}$. We also set $\mathcal{C}_\varnothing := \{\hat{c} \mid c \in \mathcal{C}\}$, a set of vertices of the simplicial complex underlying $|\mathcal{C}|_\epsilon$. Let q_s be the common cardinality of s-residues in $(\mathcal{C}, S)$. Proposition 4 implies that for the choice $d_s = 2\log q_s$ for each $s \in S$ the map $\nu\colon \operatorname{im}(\varphi|_{\mathcal{C}_\emptyset}) \to \Gamma(\mathcal{C}, S, (d_s)_{s\in S})$ defined by $\nu(G_{\hat{c}}) := \psi(\hat{c})\,(= c)$ is an isometric embedding (note that by our assumption on the cardinalities of s-residues for s in S, the stabilizers in G of different chambers are at a positive distance by Proposition 4, hence ν is well defined). Since the composite of the restriction of φ to $\mathcal{C}_\varnothing$ with ν equals the restriction of ψ to $\mathcal{C}_\varnothing$ it follows that the restriction of φ to $\mathcal{C}_\varnothing$ is a quasi-isometric embedding, because ψ is by part (1), which we already proved.

It follows that φ is a quasi-isometric embedding as well, because $\mathcal{C}_\varnothing$ is quasi-dense in $|\mathcal{C}|_\epsilon$ and the distance between the stabilizer of a point in the geometric realization $|c|$ of a chamber c and the stabilizer of the point $\hat{c} \in \mathcal{C}_\varnothing \cap |c|$ is bounded above by a constant M independent of c since $|c|$ is a finite complex and G acts transitively on chambers. □

2.2. Consequences of the Comparison Theorem

The most important consequence of the Comparison Theorem (Theorem 7) is the inequality between the flat rank of a δ-2-transitive automorphism group and the rank of the building the group acts on. This result follows from the following proposition.

Proposition 9. *Let G be a totally disconnected, locally compact group. Suppose that G acts on a metric space X in such a way that the G-stabilizers of points are compact, open subgroups of G. Assume that the map $X \to \mathcal{B}(G)$, which assigns a point its stabilizer, is a quasi-isometric embedding. Let H be a flat subgroup of G of finite flat rank n. Then, for any point x in X, the inclusion of the H-orbit of x defines an n-quasi-flat in X.*

Proof. Let x be any point in X. Since the map $x \mapsto G_x$ is an equivariant quasi-isometric embedding, the orbit of x under H is quasi-isometric to the orbit of its stabilizer G_x under H acting by conjugation. The latter orbit is quasi-isometric to the H-orbit of a tidy subgroup, say O, for H. But $H.O$ is isometric to the subset $\mathbb{Z}^n$ of $\mathbb{R}^n$ with the norm $\|\cdot\|_H$ introduced in Subsection 1.3. The subset $\mathbb{Z}^n$ is quasi-dense in $\mathbb{R}^n$ equipped with that norm and, therefore, we obtain a quasi-isometric embedding of $\mathbb{R}^n$ equipped with $\|\cdot\|_H$ into X. But the identity map between $\mathbb{R}^n$ equipped with $\|\cdot\|_H$ and $\mathbb{R}^n$ with the

Euclidean norm is bi-Lipschitz. Composing with this map, we obtain an n-quasi-flat in X as claimed. $\square$

Before deriving the rank inequality, we prove the following characterization for the existence of fixed points.

Corollary 10. *Let G be a totally disconnected, locally compact group. Suppose that G acts isometrically on a complete $\mathsf{CAT}(0)$-space X with compact, open point stabilizers. Assume that the map $X \to \mathcal{B}(G)$, which assigns a point its stabilizer, is a quasi-isometric embedding. Then an element g of G has a fixed point in X if, and only if, $s_G(g) = 1 = s_G(g^{-1})$.*

Proof. Let g be an element of G and let x be a point of X. The subgroup $\langle g \rangle$ is a flat subgroup of G of flat rank 0 or 1 and we have flat-rk$(\langle g \rangle) = 0$ if and only if $s_G(g) = 1 = s_G(g^{-1})$. Hence to prove our claim, we need to show that g has a fixed point in X if and only if flat-rk$(\langle g \rangle) = 0$.

Proposition 9 is applicable with H equal to $\langle g \rangle$. Therefore the set $\langle g \rangle.x$ is quasi-isometric to a point or the real line in the cases flat-rk$(\langle g \rangle) = 0$ and flat-rk$(\langle g \rangle) = 1$, respectively. We conclude that the set $\langle g \rangle.x$ is bounded if and only if flat-rk$(\langle g \rangle) = 0$.

Since $\langle g \rangle$ acts by isometries on the complete $\mathsf{CAT}(0)$-space X, if it has a bounded orbit, it admits a fixed point by [BH99, II.2, Corollary 2.8(1)]. The converse of the latter statement is trivial. We conclude that g has a fixed point in X if and only if flat-rk$(\langle g \rangle) = 0$ as had to be shown. $\square$

We adopt the following definition for the rank of a complete $\mathsf{CAT}(0)$-space. For alternative definitions see [Gro93, pp. 127–133].

Definition 11. The *rank* of a complete $\mathsf{CAT}(0)$-space X, denoted rk(X), is the maximal dimension of an isometrically embedded Euclidean space in X.

Recall that a metric space X is called *cocompact* if and only if the isometry group of X acts cocompactly on X [BH99, p. 202].

Theorem 12. *Let G be a totally disconnected, locally compact group. Suppose that G acts isometrically on a complete, locally compact, cocompact $\mathsf{CAT}(0)$-space X with compact, open point stabilizers. Assume that the map $X \to \mathcal{B}(G)$, which assigns to a point its stabilizer, is a quasi-isometric embedding. Then* flat-rk$(G) \leqslant$ rk(X)*; in particular, the flat rank of G is finite. We have* flat-rk$(G) = 0$ *if, and only if, every element of G fixes a point in X.*

Proof. The hypotheses on X guarantee that the rank of X is finite and equals the maximal dimension of quasi-flats in X by [Kle99, Theorem C]. The hypotheses on the action of G on X enable us to apply Proposition 9, which, together with the first observation of this proof, implies that flat-rk$(G) \leqslant$ rk$(X) < +\infty$. The last statement follows from Corollary 10. $\square$

The rank of a Gromov-hyperbolic $\mathsf{CAT}(0)$-space is 1. This leads to the following special case of Theorem 12.

Corollary 13. *Let G, X and $X \to \mathcal{B}(G)$ be as above. Assume further that X is Gromov-hyperbolic. Then* flat-rk$(G) = 1$*, unless every element of G has a fixed point in X, in which case* flat-rk$(G) = 0$. $\square$

2.3. Equality of the rank of a building and the rank of an apartment

The purpose of this subsection is to prove that the Davis-realization of a building has the same rank as any of its apartments. We do not claim that any flat of the building is contained in an apartment, though this is probably true as well[2]. This stronger statement is known to be true for Euclidean buildings by Theorem 1 in [Bro89, Chapter VI, Section 7]. We believe we can show this to be true also if the building is Moufang and of finite thickness.

Proposition 14. *Let $(\mathcal{C}, S)$ be a building with S finite and Weyl group W. If $|\mathcal{C}|_0$ contains a d-flat, then so does $|W|_0$. Hence* $\mathrm{rk}(|\mathcal{C}|_0) = \mathrm{rk}(|W|_0)$.

Proof. Let F be a d-flat in $|\mathcal{C}|_0$. Applying Lemma 9.34 in Chapter II of [BH99] with Y equal to $\mathbb{R}^d$ and X equal to $|W|_0$, and using the isomorphism of the geometric realization of any apartment with $|W|_0$, we see that it suffices to show that for each n in $\mathbb{N}$ there is an apartment A_n of $(\mathcal{C}, S)$ which contains an isometric copy of the ball of radius n around $\mathbf{0}$ in $\mathbb{R}^d$.

This isometric copy, say B_n, of the ball of radius n around $\mathbf{0}$ in $\mathbb{R}^d$ will be taken to lie inside F. Let n be a natural number, o some point in F and B_n the ball of radius n around o in F. To show that there is an apartment A_n containing B_n, we will prove that there are two chambers c_n and c'_n such that the minimal galleries connecting c_n and c'_n cover B_n. Then, by combinatorial convexity, any apartment A_n containing c_n and c'_n contains B_n and by our introductory remark the proposition follows, because n was arbitrary.

To determine how we should choose the chambers c_n and c'_n, we first take a look at the way walls in $|\mathcal{C}|_0$ intersect the flat F. Since any geodesic joining two points of a wall lies entirely inside that wall, the intersection of a wall with F is an affine subspace of F. (It can be shown that the affine subspaces of F arising in this way are either empty or of codimension at most 1 in F, but we will not make use of this additional information.) Note further that the family of affine subspaces arising as intersections of walls in $|\mathcal{C}|_0$ with F is locally finite.

If M is a wall in $|\mathcal{C}|_0$ and two points p and p' in F are separated by $M \cap F$ in F, then p and p' are separated by M in $|\mathcal{C}|_0$. Therefore, if we demonstrate that it is possible to choose chambers c_n and c'_n to contain points p_n and p'_n in F, such that no intersection of a wall with F separates any point in B_n from both p_n and p'_n, then the minimal galleries connecting c_n and c'_n cover B_n. The following lemma demonstrates that such a choice of p_n and p'_n is always possible. This concludes the proof modulo Lemma 15. $\square$

The following lemma completes the proof of Proposition 14. We follow the suggestion of the referee, who provided a simpler proof for (a slightly more general version of) this result.

Lemma 15. *Let B be a nonempty, bounded subset of $\mathbb{R}^d$ and let $\mathcal{M}$ be a locally finite collection of hyperplanes in $\mathbb{R}^d$. Then there exist two points p and p' in the complement of $\bigcup \mathcal{M}$ such that no element of $\mathcal{M}$ separates any point in B from both p and p'.*

[2] This has been confirmed by Caprace and Haglund in [CH] while the present paper was being refereed.

Proof. Let $\mathcal{M}_\cap$ be the subfamily of $\mathcal{M}$ consisting of those members of $\mathcal{M}$ which intersect B. Since $\mathcal{M}$ is locally finite and B is bounded, $\mathcal{M}_\cap$ is finite.

We will choose the points p and p' from the complement of $\bigcup \mathcal{M}$ to have the following two properties:

(1) The line segment $[p, p']$ intersects B.
(2) Each element of $\mathcal{M}_\cap$ separates p from p'.

The first property ensures that for every element M of $\mathcal{M} \setminus \mathcal{M}_\cap$ at least one of the points p and p' lies on the same side of M as B, while the second property ensures that for every element M of $\mathcal{M}_\cap$ any point of B lies in the same closed half-space with respect to M as either p or p'. Therefore the two conditions ensure that no element of $\mathcal{M}$ separates any point in B from both p and p', and it will suffice to conform with conditions (1) and (2) to conclude the proof.

In what follows we may assume that d is at least 1, because the statement to be proved is obvious if $d = 0$.

In order to find p and p', choose a point, b say, in the nonempty set B and a line, say l, through b that is transverse to all the finitely many hyperplanes in $\mathcal{M}_\cap$.

We will choose our points p and p' to lie on l with b on the segment $[p, p']$. Any such choice of p and p' will satisfy condition (1) above so that, in what follows, we only have to worry about condition (2).

The set $l \cap \bigcup \mathcal{M}_\cap$ is a finite subset of l while the set $l \cap \bigcup \mathcal{M}$ is a locally finite subset of l.

We may therefore choose points p and p' on l which do not belong to $\bigcup \mathcal{M}$ and such that all points of the set $l \cap \bigcup \mathcal{M}_\cap$ lie on the segment $[p, p']$.

By construction, the pair (p, p') satisfies property (2) and we have already noted that it satisfies property (1) as well. As seen above, this proves that p and p' have the property sought in the statement of the lemma. $\square$

3. Algebraic rank as a lower bound

In this section we show that when G is either a semisimple algebraic group over a local field or a topological Kac–Moody group over a finite field, the algebraic rank of the Weyl group W (Definition 17 below) is a lower bound for the flat rank of G.

One strategy to get a lower bound of flat-rk(G) of geometric nature, i.e. coming from the building X, is to use the stabilizer map $\varphi : x \mapsto G_x$. Indeed, according to Theorem 7, φ maps an n-quasi-flat in $|\mathcal{C}|_\epsilon$ to an n-quasi-flat in $\mathcal{B}(G)$, which can even be assumed to consist of vertex stabilizers. Still, in order to make this strategy work, one needs to establish a connection between flats in the space $\mathcal{B}(G)$ and flat subgroups in G itself. A connection which should be useful for the task described above is stated by the following conjecture.

Conjecture 16. *Let $\mathcal{H}$ be a group of automorphisms of a totally disconnected, locally compact group G. Suppose that some (equivalently, every) orbit of $\mathcal{H}$ in $\mathcal{B}(G)$ is quasi-isometric to $\mathbb{R}^n$. Then $\mathcal{H}$ is flat, of flat rank n.*

This conjecture has been verified for $n = 0$ [BW06, Proposition 5]. Going back to the announced algebraic lower bound, we now introduce the algebraic rank of a group.

Definition 17. Let H be a group. The algebraic rank of H, denoted alg-rk(H), is the supremum of the ranks of the free abelian subgroups of H.

In order to achieve the aim of this section, we will show that given a free abelian subgroup A of the Weyl group W of the building X, we can lift A to a flat subgroup of the isometry group G. This holds in both classes of the groups considered.

Theorem 18.

(1) *Let* $\mathbf{G}$ *be an algebraic semisimple group* $\mathbf{G}$ *over a local field* k*, and let* $G = \mathbf{G}(k)$ *be its rational points. Let* $\mathbf{S}$ *be a maximal* k*-split torus in* $\mathbf{G}$*, and let* $W = W_{\text{aff}}(\mathbf{S}, \mathbf{G})$ *be the affine Weyl group of* $\mathbf{G}$ *with respect to* $\mathbf{S}$*. Then* $\mathbf{S}(k)$ *is a flat subgroup of* G*, of flat rank* alg-rk$(W) = k$-rk$(\mathbf{G})$*. In particular, we have* alg-rk$(W) \leqslant$ flat-rk(G)*.*

(2) *Let* G *be a group with a locally finite twin root datum of associated Weyl group* W*. Let* A *be an abelian subgroup of* W *and let* $\widetilde{A}$ *be the inverse image of* A *under the natural map* $N \to W$*. Then* $\widetilde{A}$ *is a flat subgroup of* $\overline{G}$ *and* flat-rk$(\widetilde{A}) = \text{rank}_{\mathbb{Z}}(A)$*. In particular, we have* alg-rk$(W) \leqslant$ flat-rk(G)*.*

Before beginning the proof of the above theorem, we note the following corollary, which was obtained by different means in unpublished work of the first and third authors.

Corollary 19. *Let* $\mathbf{G}$ *be an algebraic semisimple group over a local field* k*, with affine Weyl group* W*. Then* flat-rk$(\mathbf{G}(k)) = k$-rk$(\mathbf{G}) =$ alg-rk(W)*.*

This result and its proof are valid for any closed subgroup lying between $\mathbf{G}(k)$ and its closed subgroup $\mathbf{G}(k)^+$ generated by unipotent radicals of parabolic k-subgroups — see, e.g., [Mar89, I.2.3] for a summary about $\mathbf{G}(k)^+$. It suffices to replace $\mathbf{S}(k)$ by $\mathbf{S} \cap \mathbf{G}(k)^+$.

Proof of Corollary 19. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ and let W be the affine Weyl group of $\mathbf{G}$ with respect to $\mathbf{S}$. Part (1) of Theorem 18 shows that k-rk$(\mathbf{G}) =$ alg-rk$(W) \leqslant$ flat-rk$(\mathbf{G}(k))$.

To show the inequality alg-rk$(W) \geqslant$ flat-rk$(\mathbf{G}(k))$, note that the action of $\mathbf{G}(k)$ on its affine building X is given by a BN-pair in $\mathbf{G}(k)$, which implies that this action is strongly transitive. In particular, the action of $\mathbf{G}(k)$ on X is δ-2-transitive for the canonical W-metric δ.

Part (2) of Theorem 7 shows that the map assigning a point of X its stabilizer in $\mathbf{G}(k)$ is a quasi-isometric embedding. Hence Theorem 12 is applicable and yields flat-rk$(\mathbf{G}(k)) \leqslant$ rk(X). In the case at hand, we have rk$(X) =$ alg-rk(W) because maximal flats in X are apartments in X [Bro89, VI.7], and W is virtually free abelian.

We finally conclude that k-rk$(\mathbf{G}) =$ alg-rk$(W) =$ flat-rk$(\mathbf{G}(k))$. □

Proof of Theorem 18. We treat cases (1) and (2) separately.

In case (1), the group $\mathbf{S}(k)$ is flat by Theorem 5.9 of [Wil04], being topologically isomorphic to a power of the multiplicative group of k, which itself is generated by the group of units and a uniformizer of k.

It remains to show that flat-rk$(\mathbf{S}(k)) = \text{rank}_{\mathbb{Z}} (\mathbf{S}(k)/(\mathbf{S}(k))(1))$ equals alg-rk$(W) = k$-rk$(\mathbf{G})$.

As seen in the proof of Corollary 19, the map assigning to a point of the affine building X its stabilizer in $\mathbf{G}(k)$ is a quasi-isometric embedding. Hence Corollary 10 can be applied to the action of $G = \mathbf{G}(k)$ on X. Therefore $(\mathbf{S}(k))(1)$ is the subgroup of

$\mathbf{S}(k)$ of elements admitting fixed points in X. Denote by $A_{\mathbf{S}}$ the affine apartment of X associated to $\mathbf{S}$. The group $\mathbf{S}(k)$ leaves $A_{\mathbf{S}}$ invariant. Since $A_{\mathbf{S}}$ is a convex subspace of the $\mathsf{CAT}(0)$-space X, every element of $\mathbf{S}(k)$ admitting a fixed point in X fixes a point of $A_{\mathbf{S}}$ (the projection of the fixed point onto $A_{\mathbf{S}}$). Therefore $\mathbf{S}(k)/(\mathbf{S}(k))(1)$ identifies with the group of translations of $A_{\mathbf{S}}$ induced by $\mathbf{S}(k)$, which is a subgroup of finite index in the translation lattice of W, which is itself an abelian subgroup of W of maximal rank. Therefore flat-rk$(\mathbf{S}(k))$ = alg-rk(W), which coincides with k-rk$(\mathbf{G})$. This settles case (1).

We now reduce case (2) to Proposition 20, which we will prove later. Let $(H, (U_a)_{a\in\Phi})$ be the locally finite root datum in G, $(\mathcal{C}, S)$ the positive building of $(H, (U_a)_{a\in\Phi})$ and $\mathcal{A}$ the standard apartment in $(\mathcal{C}, S)$. Let δ be the W-distance on $(\mathcal{C}, S)$ and let $\epsilon \in \{0, -1\}$.

Since the group H is finite, $\widetilde{A}$ is finitely generated. Since A is abelian, the commutator of two elements of $\widetilde{A}$ is contained in H, hence is of finite order. It follows that $\widetilde{A}$ is a flat subgroup of $\overline{G}$ by Proposition 20.

The G-action on $(\mathcal{C}, S)$ is δ-2-transitive, hence so is the $\overline{G}$-action. Hence the map sending a point in $|\mathcal{C}|_\epsilon$ to its stabilizer in $\overline{G}$ is a quasi-isometric embedding by Theorem 7. Corollary 10 then implies that the subgroup $\widetilde{A}(1)$ is the set of elements in $\widetilde{A}$ which fix some point in $|\mathcal{C}|_\epsilon$.

The group N leaves $\mathcal{A}$ invariant and the induced action of N on $|\mathcal{A}|_\epsilon$ is equivariant to the action of W when $\mathcal{A}$ is identified with the Coxeter complex of W. Since the kernel of the action of N on $|\mathcal{A}|_\epsilon$ is the finite group H, the action of N on $|\mathcal{A}|_\epsilon$ is proper. Hence an element of infinite order in N does not have a fixed point in $|\mathcal{A}|_\epsilon$ and therefore has no fixed point in $|\mathcal{C}|_\epsilon$ either, because $|\mathcal{A}|_\epsilon$ is an N-invariant convex subspace of the $\mathsf{CAT}(0)$-space $|\mathcal{C}|_\epsilon$. Therefore $\widetilde{A}(1)$ is the set of elements of $\widetilde{A}$ of finite order. This implies that $\mathrm{rank}_{\mathbb{Z}}(\widetilde{A}/\widetilde{A}(1)) = \mathrm{rank}_{\mathbb{Z}}(A)$, as claimed. □

The remainder of this section is devoted to the proof of the following.

Proposition 20. *Let G be a totally disconnected locally compact group and let A be a subgroup of G which is a finite extension of a finitely generated abelian group. Then A is a flat subgroup of G.*

We split the proof into several subclaims.

Lemma 21. *If C is a compact subgroup of G, then there is a base of neighbourhoods of e consisting of C-invariant, compact, open subgroups. In particular, any compact subgroup is flat.*

Proof. If V is a compact, open subgroup of G, then $O = \bigcap_{c\in C} cVc^{-1}$ is a compact, open subgroup of G which is contained in V and is normalized by C. Since G has a base of neighbourhoods consisting of compact, open subgroups, this proves the first claim. The second claim follows from it. □

We remind the reader on the tidying procedure defined in [Wil04] which for any automorphism α of G outputs a subgroup tidy for α. It will be used in the proofs of Lemmas 23 to 25.

Algorithm 22 (α-Tidying Procedure)**.**

[0] *Choose a compact open subgroup* $O \leqslant G$.

[1] *Let* ${}^kO := \bigcap_{i=0}^k \alpha^i(O)$. *For some* $n \in \mathbb{N}$ (*hence for all* $n' \geqslant n$) *we have*

$$ {}^nO = \Big(\bigcap_{i\geqslant 0} \alpha^i({}^nO)\Big) \cdot \Big(\bigcap_{i\geqslant 0} \alpha^{-i}({}^nO)\Big). $$

Set $O' = {}^nO$.

[2] *For each compact, open subgroup* V, *let*

$$ \mathcal{K}_{\alpha,V} := \big\{k \in G \mid \{\alpha^i(k)\}_{i\in\mathbb{Z}} \text{ is bounded and } \alpha^n(k) \in V \text{ for } n \gg 0\big\}. $$

Put $K_{\alpha,V} = \overline{\mathcal{K}}_{\alpha,V}$ *and* $K_\alpha = \bigcap\{K_{\alpha,V} \mid V \text{is a compact, open subgroup}\}$.

[3] *Form* $O^* := \{x \in O' \mid kxk^{-1} \in O'K_\alpha \ \forall k \in K_\alpha\}$ *and define* $O'' := O^*K_\alpha$. *The group* O'' *is tidy for* α *and we output* O''.

Lemma 23. *Let* α *be an automorphism of* G *and* C *an* α*-invariant compact subgroup of* G. *Choose a* C*-invariant compact, open subgroup* O *of* G *as in Lemma* 21. *Then* $\alpha(O)$ *and* K_α *are* C*-invariant. Hence the output derived from applying the* α*-tidying procedure in Algorithm* 22 *to* O *is* C*-invariant.*

Proof. To show that $\alpha(O)$ is C-invariant, let c be an element of C. Then

$$ c\alpha(O)c^{-1} = \alpha\big(\alpha^{-1}(c)O\alpha^{-1}(c)^{-1}\big) = \alpha(O), $$

which shows that $\alpha(O)$ is C-invariant.

Towards proving that K_α is C-invariant, we first prove that if C is α-invariant and V is a C-invariant compact, open subgroup of G, then $\overline{\mathcal{K}}_{\alpha,V}$ is C-invariant. Let c be an element of C and k an element of $\mathcal{K}_{\alpha,V}$. Then, for all $n \in \mathbb{Z}$,

$$ \alpha^n(ckc^{-1}) = \alpha^n(c)\alpha^n(k)\alpha^n(c^{-1}) \subseteq C\{\alpha^n(k) \mid n \in \mathbb{Z}\}C, $$

and hence the set $\{\alpha^n(ckc^{-1}) \mid n \in \mathbb{Z}\}$ is bounded. Furthermore, for all n such that $\alpha^n(k) \in V$, we have

$$ \alpha^n(ckc^{-1}) \in \alpha^n(c)V\alpha^n(c)^{-1} = V, $$

proving that indeed $\overline{\mathcal{K}}_{\alpha,V}$ is C-invariant if C is α-invariant and V is C-invariant.

To derive that K_α is C-invariant, note that whenever $V' \leqslant V$ are compact, open subgroups of G, then $\overline{\mathcal{K}}_{\alpha,V'} \leqslant \overline{\mathcal{K}}_{\alpha,V}$. Hence, if $\mathcal{O}$ is a base of neighbourhoods of e, consisting of compact, open subgroups of G, then $K_\alpha = \bigcap\{\overline{\mathcal{K}}_{\alpha,V} \mid V \in \mathcal{O}\}$. Since C is compact, there is a base of neighbourhoods $\mathcal{O}$ of e consisting of C-invariant, compact, open subgroups by Lemma 21. We conclude that K_α is the intersection of a family of C-invariant sets, hence is itself C-invariant.

Finally, we establish that the output of Algorithm 22 is C-invariant. The group O' is C-invariant since it is the intersection of C-invariant subgroups. Therefore O^* is C-invariant as well. Since K_α is C-invariant, it follows that the output O'' is also C-invariant as claimed. □

Lemma 24. *Let* $\mathcal{A}$ *be a set of automorphisms of* G, C *a compact subgroup of* G *invariant under each element of* $\mathcal{A}$ *and let* γ *be an automorphism of* G *stabilizing* C *such that* $[\gamma, \alpha]$ *is an inner automorphism in* C *for each* α *in* $\mathcal{A}$. *Suppose there is a common tidy subgroup for* $\mathcal{A}$ *which is* C*-invariant. Then there is a common tidy subgroup for* $\mathcal{A} \cup \{\gamma\}$ *which is* C*-invariant.*

Proof. Let O be a common tidy subgroup for $\mathcal{A}$ which is C-invariant. We will show that the output O'' of the γ-tidying procedure, Algorithm 22, on the input O produces a common tidy subgroup for $\mathcal{A} \cup \{\gamma\}$ which is C-invariant. Lemma 23 shows that O'' is C-invariant and we have to prove it is tidy for each element α in $\mathcal{A}$.

Since $[\gamma, \alpha] \in C$, for all C-invariant, compact, open subgroups V, we have

$$|\alpha\gamma(V) \colon \alpha\gamma(V) \cap \gamma(V)| = |\gamma(\alpha(V)) \colon \gamma(\alpha(V) \cap V)| = |\alpha(V) \colon \alpha(V) \cap V|.$$

Hence, if V is α-tidy and C-invariant, then $\gamma(V)$ is α-tidy and it is C-invariant by Lemma 23 as well. Using this observation, induction on i shows that $\gamma^i(O)$ is α-tidy for each $i \in \mathbb{N}$. Since any finite intersection of α-tidy subgroups is α-tidy by Lemma 10 in [Wil94], the output O' of step [1] of Algorithm 22 will be α-tidy.

We will show next that $\alpha(K_\gamma) = K_\gamma$. Theorem 3.3 in [Wil04] then implies that O'' is tidy for α, finishing the proof.

Towards proving our remaining claim $\alpha(K_\gamma) = K_\gamma$, we now show that if V is a C-invariant, compact, open subgroup of G, then $\alpha(\mathcal{K}_{\gamma,V}) = \mathcal{K}_{\gamma,\alpha(V)}$. Using our assumptions $[\gamma, \alpha] \subseteq C$ and $\gamma(C) = C$, the equation

$$[\gamma^n, \alpha] = [\gamma, \alpha]^{\gamma^{n-1}} \cdot [\gamma, \alpha]^{\gamma^{n-2}} \cdots [\gamma, \alpha]^{\gamma^{n-1}} \cdot [\gamma, \alpha]$$

shows that for all n in $\mathbb{Z}$ there are c_n in C such that $\gamma^n\alpha = \kappa(c_n)\,\alpha\gamma^n$, where $\kappa(g)$ denotes conjugation by g. Therefore, if $k \in \mathcal{K}_{\gamma,V}$, then

$$\gamma^n(\alpha(k)) = c_n(\alpha\gamma^n(k))c_n^{-1} \subseteq C\alpha(\gamma^n(k))C$$

is bounded. Since $\alpha(C) = C$ we have $\gamma^n\alpha = \kappa(c_n)\,\alpha\gamma^n = \alpha\kappa(\alpha^{-1}(c_n))\,\gamma^n$ for all n. Put $c'_n = \alpha^{-1}(c_n)$ for $n \in \mathbb{Z}$. Since V is C-invariant, for sufficiently large n in $\mathbb{N}$ we have

$$\gamma^n(\alpha(k)) = \alpha(c'_n\gamma^n(k)c'^{-1}_n) \in \alpha(c'_n V c'^{-1}_n) = \alpha(V).$$

This shows that $\alpha(\mathcal{K}_{\gamma,V}) = \mathcal{K}_{\gamma,\alpha(V)}$, hence $\alpha(\overline{\mathcal{K}}_{\gamma,V}) = \overline{\mathcal{K}}_{\gamma,\alpha(V)}$ for all C-invariant, compact, open subgroups V of G.

Now, if V runs through a neighbourhood base of e consisting of C-invariant compact, open subgroups (which exists by Lemma 21), then $\alpha(V)$ does as well. Since K_γ can be defined as the intersection of the family of all $\overline{\mathcal{K}}_{\gamma,W}$, where W runs through a neighbourhood base of e consisting of compact, open subgroups as already observed in the proof of Lemma 23, we obtain $\alpha(K_\gamma) = K_\gamma$. $\square$

As a corollary of Lemma 24 we obtain the following result.

Lemma 25. *Let $\mathcal{H}$ be a group of automorphisms of G, and C a compact subgroup of G such that $[\mathcal{H}, \mathcal{H}]$ consists of inner automorphisms in C. Then $\mathcal{H}$ has local tidy subgroups, that is, for every finite subset* $\mathbf{f}$ *of $\mathcal{H}$ there is a compact, open subgroup O of G such that for any $\gamma \in \langle \mathbf{f} \rangle$ the group $\gamma(O)$ is tidy for each $\alpha \in \mathbf{f}$. Moreover, O can be chosen C-invariant.*

Proof. First we use induction on the cardinality $f \geqslant 0$ of the finite set $\mathbf{f}$ to derive the existence of a common tidy subgroup for $\mathbf{f}$ which is C-invariant. Lemma 21 proves the induction hypothesis in the case $f = 0$ and provides a basis for the induction. Assume the induction hypothesis is already proved for sets of cardinality $f - 1 \geqslant 0$, and assume that $\mathbf{f}$ is a finite set of cardinality f. Choose an element γ in $\mathbf{f}$ and put $\mathcal{A} = \mathbf{f}\backslash\{\gamma\}$. Then the induction hypothesis implies that there is a common C-invariant tidy subgroup for $\mathcal{A}$ and Lemma 24 implies that the same holds for $\mathbf{f}$.

If O is a common C-invariant tidy subgroup for $\mathbf{f}$, then the first step in the proof of Lemma 24 shows that $\gamma(O)$ is α-tidy for each α in $\mathbf{f}$ since $\gamma \in \mathcal{H}$. □

Proof of Proposition 20. Finally, an application of Theorem 5.5 in [Wil04] enables us to derive Proposition 20 from Lemma 25. □

4. Simple nonlinear groups of arbitrary flat rank

We think that the algebraic and geometric rank of a Coxeter group of finite rank are equal. More generally, Corollary 7 in [GO], in conjunction with [Kle99, Theorem C] leads us to believe that the rank of a proper $\mathsf{CAT}(0)$-space, on which a group G acts properly discontinuously and cocompactly, will turn out to be a quasi-isometry invariant and will be equal to the algebraic rank of G.

Looking for examples with alg-rk$(W) = \mathrm{rk}(|W|_0)$ led us to the existence of nonlinear, topologically simple groups of arbitrary flat rank.

Theorem 26. *For every natural number $n \geqslant 1$ there is a nonlinear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n.*

Proof. The idea of the proof is to show the existence of a sequence of connected Coxeter diagrams $(D_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, we have:

(1) if W_n denotes the Coxeter group with diagram D_n, then alg-rk$(W_n) = \mathrm{rk}(|W_n|_0)$ $= \dim(|W_n|_0)$; and
(2) there is a Kac–Moody root datum of associated Coxeter diagram D_n, and a finite field k such that the corresponding Kac–Moody group G_n over k is centre-free and not linear over any field.

The required nonlinear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n may then be taken to be the completion $\overline{G}_n$.

We now specify a sequence of Coxeter diagrams satisfying the two conditions above. Fix a field k of cardinality at least 5. Let D_1 be a cycle of length $r = 5$ all of whose edges are labelled ∞. Then W_1 is Gromov-hyperbolic and therefore satisfies the first condition above. There is a Kac–Moody root datum whose Coxeter diagram is D_1 and whose group of k-points satisfies the conditions of [Rém04, Theorem 4.C.1] (take the remark following that theorem into account). This shows that the second condition above is satisfied as well. This settles the case $n = 1$.

If $n > 1$, let D_n be the diagram obtained by joining every vertex of D_1 to every vertex of a diagram of type $\widetilde{\mathsf{A}}_n$ by an edge labelled ∞. The translation subgroup of the special subgroup corresponding to the $\widetilde{\mathsf{A}}_n$-subdiagram is an abelian subgroup of rank n in W_n. Furthermore, the dimension of the $\mathsf{CAT}(0)$-realization of W_n can be seen to be n as well, since the size of the maximal spherical subsets of S does not grow. As the dimension of $|W_n|_0$ is an upper bound for $\mathrm{rk}(|W_n|_0)$, the first condition above is satisfied for D_n for any $n > 1$. For every $n > 1$, the Kac–Moody data chosen in the case $n = 1$ can be extended to a Kac–Moody datum such that the associated Coxeter group has diagram D_n.

(This amounts to extending the combinatorial data — that is, the finite index set of cardinality n, the generalized $n \times n$ Cartan matrix, the choice of n of vectors each in a free $\mathbb{Z}$-module of rank n and its dual, such that the matrix of the pairing between these two sets of vectors is the given generalized Cartan matrix — defining the Kac–Moody group functor.

The most restrictive of these tasks is to arrange for the coefficients of a general Cartan matrix to yield the given Weyl group. This is possible as soon as the edge labels of the Coxeter diagram of the Weyl group belong to the set $\{2, 3, 4, 6, \infty\}$.)

By our choice, then G_n contains G_1, and the second condition is also satisfied for any $n > 1$. The group G_n is centre-free whenever the above described Kac–Moody root datum is chosen to be the adjoint datum for the given generalized Cartan matrix. □

The above result is Theorem C of the Introduction. We mention there a stronger version, in which we exhibit groups with exactly the same properties except that topological simplicity is replaced by abstract simplicity. This can be proved by using recent (not yet published) results from [CER] and [CR]. The main idea of the proof is the same (i.e. gluing hyperbolic and affine Dynkin diagrams) but the details are as follows. Glue a (nonaffine but 2-spherical) triangle Dynkin diagram with edges all labelled by 2's to an affine diagram of type, say, $\widetilde{\mathsf{A}}_{n-1}$ along a vertex. Use [CR] applied to the hyperbolic triangle Kac–Moody group to disprove the linearity of the corresponding (hence of the ambient) Kac–Moody group. Use [CER, Theorem 1.2] with a sufficiently large finite ground field to obtain abstract simplicity for the complete Kac–Moody group corresponding to the full diagram.

Remark 27. The referee suggested the Haglund–Paulin simplicity criterion for isometry groups of spaces with walls [HP] as a potentially rich source of abstractly simple groups with the above properties. We agree that this criterion might lead to further examples. The difficulty in this approach is to construct a space with walls which not only has large rank, but whose automorphism group is rich enough to be seen to have the same rank (or, lacking that, large rank) also.

We finally observe that thanks to a theorem of Krammer, the algebraic rank of a Coxeter group of finite rank can be computed from its Coxeter diagram. In order to state that theorem, we first introduce the notion of a standard abelian subgroup of a Coxeter group.

Definition 28. Let (W, S) be a Coxeter system. Let $I_1, \dots, I_n \subseteq S$ be irreducible, nonspherical and pairwise perpendicular (i.e., the order of the product of two elements taken from different subsets is 2). For any i, let H_i be a subgroup of W_{I_i} defined as follows. If I_i is affine, H_i is the translation subgroup; otherwise, H_i is any infinite cyclic subgroup of W_{I_i}. The group $\prod_i H_i$ is called a *standard free abelian subgroup.*

The algebraic rank of a standard abelian subgroup $\prod_i H_i$ as defined above equals $\sum_{I_i \text{ affine}}(\#I_i - 1) + \sum_{I_i \text{ not affine}} 1$. Moreover, since all possible choices of the subsets $I_1, \dots, I_n \subseteq S$ can be enumerated, the maximal algebraic rank of a Coxeter group is achieved by some standard free abelian subgroup because of the following theorem [Kra94, 6.8.3].

Theorem 29. *Let W be an arbitrary Coxeter group of finite rank. Then any free abelian subgroup of W has a finite index subgroup which is conjugate to a subgroup of some standard free abelian subgroup.* □

In particular, the algebraic rank of a Coxeter group of finite rank can be computed from its Coxeter diagram.

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