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Non-distortion of twin building lattices

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Abstract We show that twin building lattices are undistorted in their ambient group; equivalently, the orbit map of the lattice to the product of the associated twin buildings is a quasi-isometric embedding. As a consequence, we provide an estimate of the quasi-flat rank of these lattices, which implies that there are infinitely many quasi-isometry classes of finitely presented simple groups. Finally, we describe how non-distortion of lattices is related to the integrability of the structural cocycle.

Keywords Lattice · Locally compact group · Kac–Moody group · Building · Distortion

Mathematics Subject Classification (2000) 20E · 20E32 · 20F55 · 20F65 · 22E65 · 51E24

1 Introduction

1.1 Distortion

Let G be a locally compact group and $\Gamma < G$ be a finitely generated lattice. Then G is compactly generated [11, Lemma 2.12] and therefore both G and Γ admit word metrics, which are well defined up to quasi-isometry. It is a natural question to understand the relation between the word metric of Γ and the restriction to Γ of the word metric on G.

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In order to address this issue, let us fix some compact generating set $\widehat{\Sigma}$ in G and denote by $\|g\|_{\widehat{\Sigma}}$ the word length of an element $g \in G$ with respect to $\widehat{\Sigma}$; we denote by $d_{\widehat{\Sigma}}$ the associated word metric. Similarly, we fix a finite generating set Σ for Γ and denote by $|\gamma|_{\Sigma}$ the word length of an element $\gamma \in \Gamma$ with respect to Σ, and by d_{Σ} the associated word metric. The lattice Γ is called **undistorted** in G if d_{Σ} is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to Γ. The condition amounts to saying that the inclusion of Γ in G defines a quasi-isometric embedding from the metric space (Γ, d_{Σ}) to the metric space $(G, d_{\widehat{\Sigma}})$.

As is well-known, any cocompact lattice is undistorted: this follows from the Švarc–Milnor Lemma [3, Proposition I.8.19]. The question of distortion thus centres around non-uniform lattices. The main result of Lubotzky et al. [19] is that if G is a product of higher-rank semi-simple algebraic groups over local fields (Archimedean or not), then any lattice of G is undistorted. This relies on the deep arithmeticity theorems due to Margulis in characteristic 0 and Venkataramana in positive characteristic, and on a detailed analysis of the distortion of unipotent subgroups.

Besides the higher-rank lattices in semi-simple groups, a class of non-uniform lattices that has attracted some attention in recent years are the so-called Kac–Moody lattices (see [24] or [9]). A more general class of lattices is that of twin building lattices [12]: a **twin building lattice** is an irreducible lattice $\Gamma < G = G_+ \times G_-$ in a product of two groups G_+ and G_- acting strongly transitively on (locally finite) buildings X_+ and X_- respectively, and such that Γ preserves a twinning between X_+ and X_-. Recall that Γ is then finitely generated and that, in this general context, **irreducible** means that each of the projections of Γ to $G_\pm$ is dense.

Theorem 1.1 *Any twin building lattice $\Gamma < G_+ \times G_-$ is undistorted.*

It should be noted that each individual group G_+ or G_- also possesses non-uniform lattices, obtained for instance by intersecting Γ with a compact open subgroup (*e.g.*, a facet stabilizer) of G_- or G_+, respectively. Other non-uniform lattices have been constructed by Gramlich and Mühlherr [15]. We emphasize that, beyond the affine case (*i.e.* when G_+ is a semi-simple group over a local function field), a non-uniform lattice in a single irreducible factor G_+ (or G_-) should be expected to be automatically *distorted* (see Sect. 3.3 below).

1.2 Quasi-isometry classes

Non-distortion of a lattice Γ in G relates the intrinsic geometry of Γ to the geometry of G. In the case of twin building lattices, the latter geometry is (quasi-isometrically) equivalent to the geometry of the product building $X_+ \times X_-$ on which G acts cocompactly. Non-distortion is especially relevant when studying quasi-isometric rigidity of Γ (which is still an open problem). As a consequence of Theorem 1.1, we can estimate a quasi-isometric invariant of a twin building lattice Γ for $X_+ \times X_-$, namely the maximal dimension of quasi-isometrically embedded flat subspaces into (Γ, d_{Σ}). This rank is bounded from below by the maximal dimension of an isometrically embedded flat in $X_\pm$ and from above by twice the same quantity (Sect. 3.4); furthermore, thanks to Krammer's thesis [18], this metric rank of $X_\pm$ can be computed concretely by means of the Coxeter diagram of the Weyl group of $X_\pm$. This enables us to draw the following group-theoretic consequence.

Corollary 1.2 *There exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.*

This corollary may also be deduced from the work of Dymara and Schick [13], which gives an estimate of another quasi-isometry invariant for twin building lattices, namely the *asymptotic dimension*.

Any finite simple group is of course quasi-isometric to the trivial group. Moreover any finitely presented simple group constructed by Burger and Mozes [4] is quasi-isometric to the product of free groups $F_2 \times F_2$; this is due to Papasoglu [23] and to the fact that the latter groups are constructed as suitable (torsion-free) uniform lattices in products of trees. Furthermore, concerning the finitely presented simple groups constructed by Higman and Thompson [16], as well as their avatars in [6,7,26,27], we are not aware of a classification up to quasi-isometry as of today. However the results of Burillo et al. [2] seem to indicate that a number of finitely presented simple groups in this class may be quasi-isometric to one another.

1.3 Integrability of the structural cocycle

Non-distortion of lattices is also relevant, in a more subtle way, to the theory of unitary representations and its applications. More precisely, given a lattice $\Gamma < G$ and a unitary Γ-representation π, one considers the induced G-representation $\mathrm{Ind}_\Gamma^G \pi$. For rigidity questions (at least) and also because the structure of G is richer than that of Γ, it is desirable that the cocycles of Γ with coefficients in π extend to continuous cocycles of G with coefficients in $\mathrm{Ind}_\Gamma^G \pi$. As explained in Shalom [29, Proposition 1.11], a sufficient condition for this to hold is that Γ be **square-integrable**. By definition, for any $p \in [1; \infty)$ it is said that Γ (or more precisely the inclusion $\Gamma < G$) is **p-integrable** if there is a Borel fundamental domain $\Omega \subset G$ for G/Γ such that, for each $g \in G$, we have:

$$\int_\Omega (|\alpha(g,h)|_\Sigma)^p \, \mathrm{d}h < \infty,$$

where $\alpha : G \times \Omega \to \Gamma$ is the induction cocycle defined by $\alpha(g,h) = \gamma \Leftrightarrow gh\gamma \in \Omega$. Mimicking Shalom's arguments in [29, §2], the following statement will be established in the final section of the paper (with the above notation for generating sets).

Theorem 1.3 *Let G be a totally disconnected locally compact group and let $\Gamma < G$ be a finitely generated lattice. Assume there is a Borel fundamental domain $\Omega \subset G$ for G/Γ such that for some $p \in [1; \infty)$ we have:*

$$\int_\Omega \left(\|h\|_{\widehat{\Sigma}}\right)^p \, \mathrm{d}h < \infty.$$

Then, if Γ is non-distorted, it is p-integrable

For S-arithmetic groups, the existence of fundamental domains satisfying the condition of Theorem 1.3 is established in [20, Proposition VIII.1.2] by means of Siegel domains. As we shall see, in the case of twin building lattices the condition is straightforward to check once a fundamental domain provided by the specific combinatorial properties of these lattices is used. In particular, combining Theorem 1.1 with Theorem 1.3, we recover the main result of [25]. We finish by mentioning that square-integrability of lattices is also relevant for lifting Γ-actions to G-actions in geometric situations which are much more general than unitary actions on Hilbert spaces, see [14,22].

This article is written as follows. Section 2 consists of preliminaries. Section 3 provides the aforementioned geometric proof of non-distortion and deals with the various metric notions of ranks that can be better understood thanks to non-distortion; we apply this to quasi-isometry classes of finitely generated simple groups. Section 4 is independent of the previous setting of twin building lattices and establishes a relationship between non-distortion and square-integrability of lattices in general totally disconnected locally compact groups.

2 Lifting galleries from the buildings to the lattice

We refer to [1] for basic definitions and facts on buildings and twinnings, and to [12] for twin building lattices. In this preliminary section, we merely fix the notation and recall one basic fact on twin buildings which plays a key role at different places in this paper.

Let $X = (X_+, X_-)$ be a twin building with Weyl group W associated to a group Γ admitting a root group datum. In particular Γ acts strongly transitively on X. We let d_{X_+} (resp. d_{X_-}) denote the combinatorial distance on the set of chambers of X_+ (resp. X_-). We further denote by S the canonical generating set of W and by $\mathrm{Opp}(X)$ the set of pairs of opposite chambers of X. Throughout the paper, we fix a base pair $(c_+, c_-) \in \mathrm{Opp}(X)$ and call it the **fundamental opposite pair** of chambers. Two opposite pairs (x_+, x_-) and $(y_+, y_-) \in \mathrm{Opp}(X)$ are called **adjacent** if there is some $s \in S$ such that x_+ is s-adjacent to y_+ and x_- is s-adjacent to y_-. Recall that an opposite pair $x \in \mathrm{Opp}(X)$ is contained in unique twin apartment, which we shall denote by $\mathbb{A}(x) = \mathbb{A}(x_+, x_-)$. The positive (resp. negative) half of $\mathbb{A}(x)$ is denoted by $\mathbb{A}(x)_+$ (resp. $\mathbb{A}(x)_-$).

The following key property is well known to the experts, and appear implicitly in the proof of Proposition 5 in [30].

Lemma 2.1 *Let $\varepsilon \in \{+, -\}$. Given any gallery $(x_0, x_1, \ldots, x_n)$ in X_ε and any chamber $y_0 \in X_{-\varepsilon}$ opposite x_0, there exists a gallery $(y_0, y_1, \ldots, y_n)$ in $X_{-\varepsilon}$ such that the following hold for all $i = 1, \ldots, n$:*

(i) $(x_i, y_i) \in \mathrm{Opp}(X)$;
(ii) (x_i, y_i) *is adjacent to* (x_{i-1}, y_{i-1});
(iii) y_i *belongs to the twin apartment* $\mathbb{A}(x_0, y_0)$.

Proof The desired gallery is constructed inductively as follows. Let $i > 0$. If y_{i-1} is opposite x_i, then set $y_i = y_{i-1}$. Otherwise the codistance $\delta^*(x_i, y_{i-1})$ is an element $s \in S$ and there is a unique chamber in the twin apartment $\mathbb{A}(x_0, y_0)$ which is s-adjacent to y_{i-1}. Define y_i to be that chamber. It follows from the axioms of a twinning that y_i is opposite x_i. The gallery $(y_0, y_1, \ldots, y_n)$ constructed in this way satisfies all the desired properties. □

3 Non-distortion of twin building lattices

In this section, we show that a twin building lattice is non-distorted for its natural diagonal action on its pair of twinned buildings. The arguments are elementary and use the basic combinatorial geometry of buildings.

3.1 An adapted generating system

Let Σ denote the subset of Γ consisting of those elements γ such that $(\gamma.c_+, \gamma.c_-)$ is adjacent to (c_+, c_-), where $(c_+, c_-) \in \mathrm{Opp}(X)$ denotes the fundamental opposite pair. Notice that

$$\max\{d_{X_+}(c_+, \gamma.c_+); d_{X_-}(c_-, \gamma.c_-)\} \leqslant 1$$

for all $\gamma \in S$.

The graph structure on $\mathrm{Opp}(X)$ induced by the aforementioned adjacency relation is isomorphic to the Cayley graph associated to the pair (Γ, Σ). Lemma 2.1 readily implies that this graph is connected. Thus Σ is a generating set for Γ.

Lemma 3.1 *Let $z = (z_+, z_-)$ be a pair of opposite chambers such that*

$$\max\{d_{X_+}(c_+, z_+); d_{X_-}(c_-, z_-)\} \leqslant 1.$$

Then there exists $\sigma \in \Sigma$ such that $\sigma.z = c$.

Proof It is enough to deal with the case when $\max\{d_{X_+}(c_+, z_+); d_{X_-}(c_-, z_-)\} = 1$.

If both z_- and z_+ belong to the twin apartment $\mathbb{A} = \mathbb{A}_- \sqcup \mathbb{A}_+$, we can write $z_+ = w_+.c_+$ and $z_- = w_-.c_-$ for $w_\pm \in W$ uniquely defined by $z_\pm$. Since z_- and z_+ are assumed to be opposite, the codistance $\delta^*(z_-, z_+)$ is by definition equal to 1_W. Since the diagonal Γ-action on $X_- \times X_+$ preserves codistances, we deduce that $w_+ = w_-$. At last since $\max\{d_{X_+}(c_+, z_+); d_{X_-}(c_-, z_-)\} = 1$, we deduce that there exists a canonical reflection $s \in S$ such that $w_\pm = s$ and this reflection is represented by an element $n_s \in \mathrm{Stab}_\Gamma(\mathbb{A})$; we clearly have $n_s \in \Sigma$.

We henceforth deal with the case when at least one of the elements $z_\pm$ does not lie in $\mathbb{A}$. Up to switching signs, we may—and shall—assume that $z_- \notin \mathbb{A}_-$. Let s be the canonical reflection such that z_- is s-adjacent to c_-. By the Moufang property, the group $U_{-\alpha_s}$ acts simply transitively on the chambers $\neq c_-$ which are s-adjacent to c_-. By conjugating by an element n_s as above and since $z_- \neq s.c_-$ (because $z_- \notin \mathbb{A}_-$), we conclude that there exists $u_+ \in U_{\alpha_s} \setminus \{1\}$ such that $u_+.z_- = c_-$. Moreover u_+ stabilizes c_+ so the chamber $u_+.z_+$ is adjacent to c_+.

If $u_+.z_+ \in \mathbb{A}_+$, then since the Γ-action preserves the codistance, the chamber $u_+.z_+ \in \mathbb{A}_+$ is the unique chamber in $\mathbb{A}$ which is opposite $c_- = u_+.z_-$, namely c_+; we are thus done in this case because we clearly have $u_+ \in \Sigma$.

We finish by considering the case when $u_+.z_+ \notin \mathbb{A}_+$. Then there exists some canonical reflection $t \in S$ such that $u_+.z_+$ is t-adjacent to c_+ and we can find similarly an element $u_- \in U_{-\alpha_t} \setminus \{1\}$ such that $u_-.(u_+.z_+) = c_+$. Setting $\sigma = u_-u_+$, we obtain an element of Γ sending $z_\pm$ to $c_\pm$. Since the Γ-action preserves each adjacency relation, hence the combinatorial distances, we have $\sigma \in \Sigma$ because $d_{X_-}(c_-, \sigma.c_-) = d_{X_-}(u_-^{-1}.c_-, u_+.c_-) = d_{X_-}(c_-, u_+.c_-) = 1$ and $d_{X_+}(c_+, \sigma.c_+) = d_{X_+}(c_+, u_-.c_+) = 1$. ◻

3.2 Proof of non-distortion

We define the combinatorial distance d_X of the chamber set of X by

$$d_X\left((x_+, x_-), (y_+, y_-)\right) = d_{X_+}(x_+, y_+) + d_{X_-}(x_-, y_-).$$

Since the G-action on X is cocompact, it follows from the Švarc–Milnor lemma [3, Proposition I.8.19] that G is quasi-isometric to X. Hence Theorem 1.1 is an immediate consequence of the following.

Proposition 3.2 *Let $\Gamma < G = G_+ \times G_-$ be a twin building lattice associated with the twin building $X = X_+ \times X_-$ and let $c = (c_+, c_-) \in X$ be a pair of opposite chambers. Then for each $\gamma \in \Gamma$, we have:*

$$\frac{1}{2} d_X(c, \gamma.c) \leqslant |\gamma|_\Sigma \leqslant 2 d_X(c, \gamma.c).$$

Proof of Proposition 3.2 Writing $\gamma \in \Gamma$ as a product of $|\gamma|_\Sigma$ elements of the generating set Σ and using triangle inequalities, we obtain

$$d_X(c, \gamma.c) \leqslant 2|\gamma|_\Sigma$$

by the definition of d_X and of Σ.

It remains to prove the other inequality, which says that Γ-orbits spread enough in X. We set $x = (x_+, x_-) = \gamma^{-1}.c$. Let us pick a minimal gallery in X_-, from x_- to c_-. Using auxiliary positive chambers, one opposite for each chamber of the latter gallery, a repeated use of Lemma 3.1 shows that there exists $\gamma_- \in \Gamma$ such that $\gamma_-.x_- = c_-$ and

$$(*) \quad |\gamma_-|_\Sigma \leqslant d_{X_-}(c_-, x_-).$$

Moreover as in the first paragraph, we have:

$$(**) \quad d_{X_+}(c_+, \gamma_-.c_+) \leqslant |\gamma_-|_\Sigma,$$

by the definition of Σ. We deduce:

$$\begin{aligned} d_{X_+}(c_+, \gamma_-.x_+) &\leqslant d_{X_+}(c_+, \gamma_-.c_+) + d_{X_+}(\gamma_-.c_+, \gamma_-.x_+) \\ &\leqslant |\gamma_-|_\Sigma + d_{X_+}(c_+, x_+) \\ &\leqslant d_{X_-}(c_-, x_-) + d_{X_+}(c_+, x_+), \end{aligned}$$

successively by the triangle inequality, by $(**)$ and the fact the Γ-action is isometric for the combinatorial distances on chambers, and by $(*)$. Therefore, by definition of d_X, we already have:

$$(***) \quad d_{X_+}(c_+, \gamma_-.x_+) \leqslant d_X(c, x).$$

We now construct a suitable element $\gamma_+ \in \Gamma$ such that $\gamma_+.x_+ = c_+$ and $\gamma_+.c_- = c_-$. Let $\gamma_-.x_+ = z_0, z_1, \ldots, z_k = c_+$ be a minimal gallery in X_+ from $\gamma_-.x_+$ to c_+. Let $\mathbb{A} = \mathbb{A}_+ \sqcup \mathbb{A}_-$ be the twin apartment defined by the opposite pair $c = (c_+, c_-)$. Let $c_0 = c_-, c_1, \ldots, c_k$ be the gallery contained in $\mathbb{A}_+$ and associated to $z_0, z_1, \ldots, z_k = c_+$ as in Lemma 2.1. Notice that, since c_k is opposite $z_k = c_+$ and since c_- is the *unique* chamber of $\mathbb{A}_-$ opposite c_+, we have $c_k = c_-$.

By Lemma 3.1, there exists $\sigma_1 \in \Sigma$ such that $\sigma_1.z_{k-1} = z_k$ and $\sigma_1.c_{k-1} = c_k$. Moreover a straightforward inductive argument yields for each $i \in \{1, \ldots, k\}$ an element $\sigma_i \in \Sigma$ such that $\sigma_i\sigma_{i-1}\cdots\sigma_1.z_{k-i} = z_k$ and $\sigma_i\sigma_{i-1}\cdots\sigma_1.c_{k-i} = c_k$. Let now $\gamma_+ = \sigma_k \cdots \sigma_1$, so that $|\gamma_+|_\Sigma \leqslant k = d_{X_+}(c_+, \gamma_-.x_+)$. By construction, we have $\gamma_+.(\gamma_-.x_+) = c_+$ and $\gamma_+.c_- = c_-$, that is $(\gamma_+\gamma_-).x = c$. Therefore $(\gamma_+\gamma_-\gamma^{-1}).c = c$ and hence there is $\sigma \in \Sigma$ such that $\gamma = \sigma\gamma_+\gamma_-$. In fact, since σ fixes c, it follows that $\sigma\sigma' \in \Sigma$ for each $\sigma' \in \Sigma$. Upon replacing σ_k by $\sigma\sigma_k$, we may—and shall—assume that $\gamma = \gamma_+\gamma_-$. Therefore we have:

$$\begin{aligned} |\gamma|_\Sigma &\leqslant |\gamma_+|_\Sigma + |\gamma_-|_\Sigma \\ &\leqslant d_{X_+}(c_+, \gamma_-.x_+) + d_{X_-}(c_-, x_-), \end{aligned}$$

the last inequality coming from $|\gamma_+|_\Sigma \leqslant k = d_{X_+}(c_+, \gamma_-.x_+)$ and $(*)$ above. By $(***)$ and the definition of d_X, this finally provides $|\gamma|_\Sigma \leqslant 2 \cdot d_X(c, \gamma.c)$, which finishes the proof. □

3.3 A remark on distortion of lattices in rank one groups

Let $G = G_+ \times G_-$ be product of two totally disconnected locally compact groups, let $\pi_\pm : G \to G_\pm$ denote the canonical projections and let $\Gamma < G$ be a finitely generated lattice. Assume that $\overline{\pi_-(\Gamma)}$ is cocompact in G_- (this is automatic for example if Γ is irreducible). Let also $U_- < G_-$ be a compact open subgroup and set $\Gamma_- = \Gamma \cap (G_+ \times U_-)$. Then the projection of Γ_- to G_+ is a lattice, and it is straightforward to verify that, *if* Γ_- *is finitely generated and undistorted in* G_-*, then* Γ *is undistorted in* G.

We emphasize however that, in the case of twin building lattices, the lattice Γ_- should not be expected to be undistorted in G_- beyond the affine case (which corresponds to the classical case of arithmetic lattices in semi-simple groups over local function fields). Indeed, a typical non-affine case is when G_+ and G_- are Gromov hyperbolic (equivalently, the Weyl group is Gromov hyperbolic or, still equivalently, each of the buildings X_+ and X_- are Gromov hyperbolic). Then a non-uniform lattice in G_+ is always distorted, as follows from the following.

Lemma 3.3 *Let G be a compactly generated Gromov hyperbolic totally disconnected locally compact group and $\Gamma < G$ be a finitely generated lattice. Then the following assertions are equivalent.*

(i) Γ *is a uniform lattice.*
(ii) Γ *is undistorted in G.*
(iii) Γ *is a Gromov hyperbolic group.*

Proof (i) $\Rightarrow$ (ii) Follows from the Švarc–Milnor Lemma.
(ii) $\Rightarrow$ (iii) Follows from the well-known fact that a quasi-isometrically embedded subgroup of a Gromov hyperbolic group is quasi-convex.
(iii) $\Rightarrow$ (i) By Serre's covolume formula (see [28]) a non-uniform lattice in a totally disconnected locally compact group possesses finite subgroups of arbitrary large order, and therefore cannot be Gromov hyperbolic. □

3.4 Various notions of rank

As a consequence of Theorem 1.1, we obtain the following estimate for one of the most basic quasi-isometric invariants attached to a finitely generated group.

Corollary 3.4 *Let $\Gamma < G = G_+ \times G_-$ be a twin building lattice with finite symmetric generating subset Σ. Let r denote the quasi-flat rank of (Γ, d_Σ) and let R denote the flat rank of the building $X_\pm$. Then we have: $R \leqslant r \leqslant 2R$.*

Recall that by definition, the **flat rank** (resp. **quasi-flat rank**) of a metric space is the maximal rank of a flat (resp. quasi-flat), *i.e.* an isometrically embedded (resp. quasi-isometrically embedded) copy of $\mathbf{R}^n$. By [10] the flat rank of a building coincides with the maximal rank of a free Abelian subgroup of its Weyl group W, and this quantity may be computed explicitly in terms of the Coxeter diagram of W, see [18, Theorem 6.8.3].

Proof of Corollary 3.4 Let us first prove $r \leqslant 2R$. Let $\varphi : (\mathbf{R}^r, d_{\mathrm{eucl}}) \to (\Gamma, d_\Sigma)$ denote a quasi-isometric embedding of a Euclidean space in the Cayley graph of Γ. With the notation of Proposition 3.2, we know that the orbit map $\omega_c : \Gamma \to X_+ \times X_-$ defined by $\gamma \mapsto \gamma.c$ is a quasi-isometric embedding. Therefore the composed map $\omega_c \circ \varphi : (\mathbf{R}^r, d_{\mathrm{eucl}}) \to X_+ \times X_-$ is a quasi-isometric embedding. By [17, Theorem C], this implies the existence of flats of dimension r in the product of two spaces of flat rank R; hence $r \leqslant 2R$.

We now turn to the inequality $R \leqslant r$. As mentioned above, it is shown in [10] that the flat rank of a building coincides with the flat rank of any of its apartment. Since the standard twin apartment is contained in the image of Γ under the orbit map $\Gamma \to X_+ \times X_-$, the desired inequality follows directly from the non-distortion of Γ established in Proposition 3.2. □

Note that another notion of rank, relevant to Willis' general theory of totally disconnected locally compact groups, is discussed for the full automorphism groups $G_\pm = \mathrm{Aut}(X_\pm)$ in [8], and turns out to coincide with the above notions of rank.

Finally, we now provide the proof of existence of infinitely many quasi-isometry classes of finitely presented simple groups.

Proof of Corollary 1.2 Since there exist twin buildings of arbitrary flat rank (choose for instance Dynkin diagrams such that the associated Coxeter diagram contains more and more commuting $\widetilde{A}_2$-diagrams), we deduce that twin building lattices fall into infinitely many quasi-isometry classes. This observation may be combined with the simplicity theorem from [12] to yield the desired result. □

4 Integrability of undistorted lattices

In this section, we give up the specific setting of twin building lattices and provide a simple condition ensuring that non-distorted finitely generated lattices in totally disconnected groups are square-integrable.

4.1 Schreier graphs and lattice actions

Let us a consider a totally disconnected, locally compact group G. As before we assume that G contains a finitely generated lattice, say Γ, which implies that G is compactly generated [11, Lemma 2.12]. By [5, III.4.6, Corollaire 1], we know that G contains a compact open subgroup, say U. Let C be a compact generating subset of G which, upon replacing C by $C \cup C^{-1}$, we may—and shall— assume to be symmetric: $C = C^{-1}$. We set $\widehat{\Sigma} = UCU$, which is still a symmetric generating set for G.

We now introduce the **Schreier graph** $\mathfrak{g}_{U,\widehat{\Sigma}}$, or simply $\mathfrak{g}$, associated to the above choices. It is the graph whose set of vertices is the discrete set G/U, which is countable whenever G is σ-compact. Two distinct vertices gU and hU are connected by an edge if, and only if, we have $g^{-1}h \in \widehat{\Sigma}$ [21, §11.3]. The natural G-action on $\mathfrak{g}$ by left translation is proper, and it is isometric whenever we endow $\mathfrak{g}$ with the metric $d_\mathfrak{g}$ for which all edges have length 1. We view the identity class $1_G U$ as a base vertex of the graph $\mathfrak{g}$, which we denote by v_0.

Denoting by $\|\cdot\|_{\widehat{\Sigma}}$ the word metric on G attached to $\widehat{\Sigma}$, we have: $\|g\|_{\widehat{\Sigma}} = d_{\widehat{\Sigma}}(1_G, g)$ for any $g \in G$. Notice that the generating set $\widehat{\Sigma}$ of G consists by definition of those elements $g \in G$ such that $d_\mathfrak{g}(v_0, g.v_0) \leqslant 1$. In particular, for all $g, h \in G$, we have:

$$d_\mathfrak{g}(g.v_0, h.v_0) \leqslant d_{\widehat{\Sigma}}(g, h) \leqslant d_\mathfrak{g}(g.v_0, h.v_0) + 1.$$

Moreover $d_\mathfrak{g}(g.v_0, h.v_0) = d_{\widehat{\Sigma}}(g, h)$ whenever $g.v_0 \neq h.v_0$.

In the present setting, using again [5, III.4.6, Corollaire 1] and the discreteness of the Γ-action, we may—and shall—work with a Schreier graph $\mathfrak{g}$ defined by a compact open subgroup U small enough to satisfy $\Gamma \cap U = \{1_G\}$. Thus we have:

$$\mathrm{Stab}_\Gamma(v_0) = \Gamma \cap U = \{1_G\}.$$

Let $\mathscr{V} = \{v_0, v_1, \ldots\}$ be a set of representatives for the Γ-orbits of vertices. The element v_0 is the previous one, and for each $i > 0$, we choose v_i in such a way that $d_\mathfrak{g}(v_i, v_0) \leqslant d_\mathfrak{g}(v_i, \gamma.v_0)$ for all $\gamma \in \Gamma$; this is possible because the distance $d_\mathfrak{g}$ takes integral values. We set $g_0 = 1$; for each $i > 0$, since the G-action on the vertices of $\mathfrak{g}$ is transitive, there exists $g_i \in G$ such that $g_i.v_i = v_0$. Thus for any $g \in G$ there exists $j \geqslant 0$ such that $g.v_0 \in \Gamma.v_j$, which provides the partition:

$$G = \bigsqcup_{j \geqslant 0} \Gamma g_j^{-1} U.$$

Furthermore, for each $i \geqslant 0$, we choose a Borel subset $V_i \subset U$ which is a section of the right U-orbit map $U \to \Gamma \setminus (\Gamma g_i^{-1} U)$ defined by $u \mapsto \Gamma g_i^{-1} u$. Setting $F_i = g_i^{-1} V_i g_i$, we obtain a subset F_i of $\mathrm{Stab}_G(v_i)$ such that

$$\mathscr{F} = \bigsqcup_{i \geqslant 0} F_{v_i} g_i^{-1}$$

is a Borel fundamental domain for Γ in G. We normalize the Haar measure on G so that $\mathscr{F}$ has volume 1.

4.2 Non-distortion implies square-integrability

We can now turn to the proof of Theorem 1.3 from the introduction. Let us first recall its precise statement.

Theorem 1.3 *Let G be a totally disconnected locally compact group and let $\Gamma < G$ be a finitely generated lattice. Assume there is a Borel fundamental domain $\Omega \subset G$ for G/Γ such that for some $p \in [1; \infty)$ we have:*

$$\int_{\Omega} \left(\|h\|_{\widehat{\Sigma}}\right)^p \mathrm{d}h < \infty.$$

Then, if Γ is non-distorted, it is p-integrable.

Proof Let $g \in G$ and $h \in \mathscr{F}$.

On the one hand, by definition of the induction cocycle $\alpha : G \times \mathscr{F} \to \Gamma$, the element $\alpha(g, h) = \gamma \in \Gamma$ is defined by $\gamma hg \in \mathscr{F}$. Therefore, by construction of the fundamental domain $\mathscr{F}$, there exist $i \geqslant 0$ and $u \in F_i$ such that $\gamma hg = u g_i^{-1}$. Let us apply the latter element to the origin v_0 of $\mathfrak{g}$. We obtain $\gamma hg.v_0 = u g_i^{-1} v_0 = u.v_i$, and since $u \in F_i$ and $F_i \subset \mathrm{Stab}_G(v_i)$, this finally provides $\gamma hg.v_0 = v_i$. By this and the choice of v_i in its Γ-orbit, we have:

$$(\star) \quad d_{\mathfrak{g}}(v_0, v_i) \leqslant d_{\mathfrak{g}}(v_0, \gamma^{-1}.v_i) = d_{\mathfrak{g}}(v_0, hg.v_0).$$

On the other hand, let Σ be a finite symmetric generating set for Γ and let d_Σ be the associated word metric; we set $|\gamma|_\Sigma = d_\Sigma(1_G, \gamma)$ for $\gamma \in \Gamma$. Since the metric spaces $(G, d_{\widehat{\Sigma}})$ and $(\mathfrak{g}, d_{\mathfrak{g}})$ are quasi-isometric (Sect. 4.1), the assumption that Γ is undistorted is equivalent to the fact that the Γ-orbit map $\Gamma \to \mathfrak{g}$ of v_0 defined by $\gamma \mapsto \gamma.v_0$ is a quasi-isometric embedding. In particular, there exist constants $L \geqslant 1$ and $C \geqslant 0$ such that

$$|\gamma|_\Sigma \leqslant L \cdot d_{\mathfrak{g}}(v_0, \gamma.v_0) + C$$

for all $\gamma \in \Gamma$.

Moreover $d_{\mathfrak{g}}$ takes integer values and $\mathrm{Stab}_\Gamma(v_0) = \{1_G\}$, so for all non-trivial $\gamma \in \Gamma$ we have: $L.d_{\mathfrak{g}}(v_0, \gamma.v_0) + C \leqslant (L + C).d_{\mathfrak{g}}(v_0, \gamma.v_0)$. Therefore, upon replacing L by a larger constant we may—and shall—assume that $C = 0$.

Our aim is to evaluate $|\gamma|_\Sigma = |\alpha(g, h)|_\Sigma$ in terms of $\|g\|_{\widehat{\Sigma}}$ and $\|h\|_{\widehat{\Sigma}}$. Note that $|\gamma|_\Sigma = |\gamma^{-1}|_\Sigma$ since Σ is symmetric.

First, we deduce successively from non-distortion, from the triangle inequality inserting $\gamma^{-1}.v_i$, and from the fact that the Γ-action on $\mathfrak{g}$ is isometric, that:

$$\begin{aligned}|\gamma^{-1}|_\Sigma &\leqslant L\cdot d_{\mathfrak{g}}(v_0,\gamma^{-1}.v_0)\\ &\leqslant L\cdot\big(d_{\mathfrak{g}}(v_0,\gamma^{-1}.v_j)+d_{\mathfrak{g}}(\gamma^{-1}.v_0,\gamma^{-1}.v_j)\big)\\ &\leqslant L\cdot\big(d_{\mathfrak{g}}(v_0,\gamma^{-1}.v_j)+d_{\mathfrak{g}}(v_0,v_j)\big).\end{aligned}$$

Then, we deduce successively from ($\star$), from the triangle inequality inserting $h.v_0$, and from the fact that the G-action on $\mathfrak{g}$ is isometric, that:

$$\begin{aligned}|\gamma^{-1}|_\Sigma &\leqslant 2L\cdot d_{\mathfrak{g}}(v_0,hg.v_0)\\ &\leqslant 2L\cdot\big(d_{\mathfrak{g}}(v_0,h.v_0)+d_{\mathfrak{g}}(h.v_0,hg.v_0)\big)\\ &\leqslant 2L\cdot\big(d_{\mathfrak{g}}(v_0,h.v_0)+d_{\mathfrak{g}}(v_0,g.v_0)\big)\,.\end{aligned}$$

Finally, by definition of the Schreier graph we deduce that $|\gamma^{-1}|_\Sigma\leqslant 2L\cdot\big(\|g\|_{\widehat{\Sigma}}+\|h\|_{\widehat{\Sigma}}\big)$. Recall that we want to prove that the function $h\mapsto|\alpha(g,h)|_\Sigma$ belongs to $\mathrm{L}^p(\mathscr{F},\mathrm{d}h)$. Since $\mathrm{Vol}(\mathscr{F},\mathrm{d}h)=1$, so does the constant function $h\mapsto\|g\|_{\widehat{\Sigma}}$, therefore it remains to prove the lemma below. □

Lemma 4.1 *The function $h\mapsto\|h\|_{\widehat{\Sigma}}$ belongs to $\mathrm{L}^p(\mathscr{F},\mathrm{d}h)$.*

Proof Let $h\in\mathscr{F}$.

By construction of the fundamental domain $\mathscr{F}$, there exist $i\geqslant 0$ and u_i in F_i, hence in $\mathrm{Stab}_G(v_i)$, such that $h=u_ig_i^{-1}$. This implies $h.v_0=u_i.(g_i^{-1}.v_0)=u_i.v_i=v_i$, and also $(\gamma h).v_0=\gamma.v_i$ for each $\gamma\in\Gamma$. Now the explicit form of the quasi-isometry equivalence (Sect. 4.1) between $(\mathfrak{g},d_{\mathfrak{g}})$ and $(G,d_{\widehat{\Sigma}})$ implies:

$$d_{\mathfrak{g}}(v_0,h.v_0)\leqslant\|h\|_{\widehat{\Sigma}}\leqslant d_{\mathfrak{g}}(v_0,h.v_0)+1,$$

and

$$d_{\mathfrak{g}}(v_0,(\gamma h).v_0)\leqslant\|\gamma h\|_{\widehat{\Sigma}}\leqslant d_{\mathfrak{g}}(v_0,(\gamma h).v_0)+1.$$

Moreover by the choice of v_i in its Γ-orbit, we have $d_{\mathfrak{g}}(v_0,h.v_0)\leqslant d_{\mathfrak{g}}(v_0,(\gamma h).v_0)$ for any $\gamma\in\Gamma$. This allows us to put together the above two double inequalities, and to obtain (after forgetting the extreme upper and lower bounds):

(†) $\|h\|_{\widehat{\Sigma}}\leqslant\|\gamma h\|_{\widehat{\Sigma}}+1.$

for any $h\in\mathscr{F}$ and $\gamma\in\Gamma$.

Recall that $p\in[1;+\infty)$ is an integer such that we have a Borel fundamental domain Ω for which $\int_\Omega\big(\|h\|_{\widehat{\Sigma}}\big)^p\,\mathrm{d}h<\infty$. Since $G=\bigsqcup_{\gamma\in\Gamma}\gamma^{-1}\Omega$ we can write:

$$\int_{\mathscr{F}}\big(\|h\|_{\widehat{\Sigma}}\big)^p\,\mathrm{d}h=\sum_{\gamma\in\Gamma}\int_{\mathscr{F}\cap\gamma^{-1}\Omega}\big(\|h\|_{\widehat{\Sigma}}\big)^p\,\mathrm{d}h.$$

But in view of (†) and of the unimodularity of G (which contains a lattice), we have:

$$\int_{\mathscr{F}\cap\gamma^{-1}\Omega}\big(\|h\|_{\widehat{\Sigma}}\big)^p\,\mathrm{d}h\leqslant\int_{\mathscr{F}\cap\gamma^{-1}\Omega}\big(\|\gamma h\|_{\widehat{\Sigma}}+1\big)^p\,\mathrm{d}h=\int_{\gamma\mathscr{F}\cap\Omega}\big(\|h\|_{\widehat{\Sigma}}+1\big)^p\,\mathrm{d}h,$$

which finally provides

$$\int_{\mathscr{F}} \left(\|h\|_{\widehat{\Sigma}}\right)^p \mathrm{d}h \leqslant \sum_{\gamma\in\Gamma} \int_{\gamma\mathscr{F}\cap\Omega} \left(\|h\|_{\widehat{\Sigma}}+1\right)^p \mathrm{d}h = \int_{\Omega} \left(\|h\|_{\widehat{\Sigma}}+1\right)^p \mathrm{d}h.$$

The conclusion follows because $\mathscr{F}$ has Haar volume equal to 1 and because by assumption $h \mapsto \|h\|_{\widehat{\Sigma}}$ belongs to $\mathrm{L}^p(\Omega, \mathrm{d}h)$. □

4.3 p-integrability of twin building lattices

Let us finish by mentioning the following fact which, using Theorem 1.3, allows us to prove the main result of [25] in a more conceptual way.

Lemma 4.2 *Let Γ be a twin building lattice and let G be the product of the automorphism groups of the associated buildings $X_\pm$. Let W be the Weyl group and $\sum_{n\geqslant 0} c_n t^n$ be the growth series of W with respect to its canonical set of generators S, i.e., $c_n = \#\{w \in W : \ell_S(w) = n\}$. Let $q_{\min}$ denote the minimal order of root groups and assume that $\sum_{n\geqslant 0} c_n q_{\min}^{-n} < \infty$. Then Γ admits a fundamental domain $\mathscr{F}$ in G, with associated induction cocycle $\alpha_{\mathscr{F}}$, such that $h \mapsto \alpha_{\mathscr{F}}(g,h)$ belongs to $\mathrm{L}^p(\mathscr{F}, \mathrm{d}h)$ for any $g \in G$ and any $p \in [1; +\infty)$.*

Proof We freely use the notation of Sect. 3.1 and [25]. We denote by $\mathscr{B}_\pm$ the stabilizer of the standard chamber $c_\pm$ in the closure $\overline{\Gamma}^{\mathrm{Aut}(X_\pm)}$. By [loc. cit.] there is a fundamental domain $\mathscr{F} = D = \bigsqcup_{w\in W} D_w$ such that $\mathrm{Vol}(D_w, \mathrm{d}h) \leqslant q_{\min}^{-\ell_S(w)}$. If we choose the compact generating set $\widehat{\Sigma} = \bigsqcup_{(s_-,s_+)\in S\times S} \mathscr{B}_- s_- \mathscr{B}_- \times \mathscr{B}_+ s_+ \mathscr{B}_+$, we see that by definition of D_w, which his contained in $\mathscr{B}_- \times \mathscr{B}_+ w$, we have $\|h\|_{\widehat{\Sigma}} \leqslant \ell_S(w)$ for any $w \in W\backslash\{1\}$ and any $h \in D_w$. Therefore for any $p \in [1; +\infty)$ we have: $\int_{\mathscr{F}} \left(\|h\|_{\widehat{\Sigma}}\right)^p \mathrm{d}h \leqslant \sum_{n\geqslant 0} n^p c_n q_{\min}^{-n}$, from which the conclusion follows. □

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