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# Automorphisms of Drinfeld half-spaces over a finite field

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## ABSTRACT

We show that the automorphism group of Drinfeld’s half-space over a finite field is the projective linear group of the underlying vector space. The proof of this result uses analytic geometry in the sense of Berkovich over the finite field equipped with the trivial valuation. We also take into account extensions of the base field.

## Introduction

In this note we determine the automorphism group of Drinfeld half-spaces over a finite field. Given a finite-dimensional vector space  $V$  over a finite field  $k$ , the Drinfeld half-space  $\Omega(V)$  is defined as the complement of all  $k$ -rational hyperplanes in the projective space  $\mathbf{P}(V)$ ; it is an affine algebraic variety over  $k$ . We show that every  $k$ -automorphism of  $\Omega(V)$  is induced by a  $k$ -automorphism of  $\mathbf{P}(V)$ . Hence the automorphism group of  $\Omega(V)$  is equal to  $\mathrm{PGL}(V)$ .

More generally, for an arbitrary field extension  $K$  of  $k$ , we prove that the natural injection of  $\mathrm{PGL}(V)$  into  $\mathrm{Aut}_K(\Omega(V) \otimes_k K)$  is an isomorphism. Our result answers a question of Dat *et al.* [DOR10, p. 338], which was motivated by the analogous statement for Drinfeld half-spaces over a non-Archimedean local field (with non-trivial absolute value).

Drinfeld defined his  $p$ -adic upper half-spaces in [Dri74]. They are the founding examples of the theory of period domains [RZ96]. Analogs of period domains over finite fields have been studied by Rapoport in [Rap97]; they are open subvarieties of flag varieties characterized by a semi-stability condition. Recently, they have been studied by Rapoport, Orlik and others; see, e.g., [Orl01, OR08]. A good introduction is given in the book [DOR10].

Over local non-Archimedean fields with non-trivial absolute value, Drinfeld half-spaces are no longer algebraic varieties and must be defined in the context of analytic geometry. In this setting, it was shown by Berkovich that every automorphism is induced by a projective linear transformation [Ber95]. This was generalized to products of Drinfeld half-spaces by Alon [Alo06], who also pointed out and corrected a discrepancy in Berkovich’s proof. Berkovich’s strategy was based on the fact that in the case of a local non-Archimedean ground field with non-trivial absolute value, the Bruhat–Tits building of the group  $\mathrm{PGL}(V)$  is contained in  $\Omega(V)$  as the subset of points satisfying a natural maximality condition. This implies that every automorphism of  $\Omega(V)$  induces an automorphism of the Bruhat–Tits building, and with some further work (see [Alo06]) one can prove the claim.

One could in fact use a similar strategy to determine the automorphism group of  $\Omega(V)$  over a finite field. Indeed, if we endow the finite ground field with the trivial absolute value and look at the corresponding Berkovich analytic space  $\Omega(V)^{\mathrm{an}}$ , then by [Ber90] the *vectorial building*

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associated to the group  $\mathrm{PGL}(V)$  is contained in  $\Omega(V)^{\mathrm{an}}$ . We believe that one can then follow Berkovich’s and Alon’s arguments to deduce that this automorphism comes from an element of  $\mathrm{PGL}(V)$ .

However, in this note we adopt a slightly different, and perhaps more natural, viewpoint. Thereby, we wish to highlight that the true content of this theorem is about *extension* of automorphisms and that it has, in fact, very little to do with buildings; see Remark 2.3. Our approach is the following. We consider the space  $X$  obtained by blowing up all  $k$ -rational linear subspaces of the projective space  $\mathbf{P}(V)$ . Irreducible components of the boundary divisor correspond bijectively to linear subspaces of  $\mathbf{P}(V)$ . Moreover, a family of components has non-empty intersection if and only if the corresponding linear subspaces form a flag. We use Berkovich analytic geometry to prove, in Proposition 2.1, that every automorphism of  $\Omega(V)$  preserves the set of discrete valuations on the function field induced by boundary components of  $X$ . Hence, by Proposition 1.4, it extends to an automorphism of  $X$ . In §3, by taking a closer look at the Chow ring of  $X$  we deduce that this automorphism preserves the set of discrete valuations corresponding to hyperplanes, which allows us to conclude that it induces an automorphism of the projective space.

### 1. Automorphisms of Drinfeld half-spaces

Let  $k$  be a finite field and let  $V$  be a  $k$ -vector space. We denote by  $\mathbf{P}(V)$  the projective scheme  $\mathrm{Proj}(\mathrm{Sym}^\bullet V)$  and define the  $k$ -scheme  $\Omega(V)$  to be the complement of all (rational) hyperplanes in  $\mathbf{P}(V)$ :

$$\Omega(V) = \mathbf{P}(V) - \bigcup_{\substack{W \subset V \\ \dim W = 1}} \mathbf{P}(V/W).$$

For every field extension  $K/k$ , we denote by  $V_K = V \otimes_k K$  the induced vector space over  $K$ . Then the base change  $\Omega(V)_K = \Omega(V) \otimes_k K$  is the complement of all  $k$ -rational hyperplanes in  $\mathbf{P}(V_K) = \mathbf{P}(V) \otimes_k K$ .

The main result of this note is the following.

**THEOREM 1.1.** *Let  $V$  be a vector space of finite dimension over a finite field  $k$ .*

(i) *The restriction map*

$$\mathrm{PGL}(V) = \mathrm{Aut}_k(\mathbf{P}(V)) \rightarrow \mathrm{Aut}_k(\Omega(V)), \quad \varphi \mapsto \varphi|_{\Omega(V)}$$

*is an isomorphism. Equivalently, every  $k$ -automorphism of  $\Omega(V)$  extends to a  $k$ -automorphism of  $\mathbf{P}(V)$ .*

(ii) *For every field extension  $K/k$ , the natural map*

$$\mathrm{PGL}(V) \longrightarrow \mathrm{Aut}_K(\Omega(V)_K)$$

*is an isomorphism. Equivalently, every  $K$ -automorphism of  $\Omega(V)_K$  comes, by base change, from a  $k$ -automorphism of  $\mathbf{P}(V)$ .*

This result holds trivially if  $\dim V \leq 1$ , because then  $\Omega(V) = \mathbf{P}(V)$ . From now on, we assume that  $V$  has dimension at least 2 and we set  $n = \dim V - 1$ .

The proof combines analytic geometry in the sense of Berkovich with algebraic arguments. As a first step, we show that every  $k$ -automorphism of  $\Omega(V)$  can be extended to an automorphism of the  $k$ -scheme  $X$  which we get by blowing up all linear subspaces of  $\mathbf{P}(V)$ . For this step, we use Berkovich analytic geometry over the field  $k$  endowed with the trivial absolute value. The second step is of an algebraic nature and consists of checking that this automorphism of  $X$  is

induced by a  $k$ -automorphism of  $\mathbf{P}(V)$ . Here we analyze the geometry of the boundary divisor more closely and use an induction argument.

Given a proper subvector space  $W$  of  $V$ , applying  $\text{Proj}$  to the natural map  $\text{Sym}^\bullet(V) \rightarrow \text{Sym}^\bullet(V/W)$  leads to a closed immersion  $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$  whose image  $L$  is called a *linear subspace* of  $\mathbf{P}(V)$ . Such a subscheme is said to be trivial if  $L = \emptyset$  or  $L = \mathbf{P}(V)$ ; it is called a *hyperplane* if it is of codimension 1. We denote by  $\mathcal{L}^i(V)$  the set of linear subspaces of dimension  $i$  in  $\mathbf{P}(V)$ , and by  $\mathcal{L}(V) = \bigcup_{0 \leq i \leq n-1} \mathcal{L}^i(V)$  the set of non-trivial linear subspaces.

DEFINITION 1.2. We denote by  $\pi : X \rightarrow \mathbf{P}(V)$  the blow-up of  $\mathbf{P}(V)$  along the full hyperplane arrangement. To be precise,  $X$  is defined as

$$X = X_{n-1} \xrightarrow{\pi_{n-1}} X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 \xrightarrow{\pi_0} X_{-1} = \mathbf{P}(V)$$

with

$$\pi = \pi_0 \circ \pi_1 \circ \cdots \circ \pi_{n-1},$$

where  $\pi_i$  denotes the blow-up of  $X_{i-1}$  along the strict transforms of linear subspaces of  $\mathbf{P}(V)$  of dimension  $i$ .

The scheme  $X$  is projective and smooth over  $k$ . It contains  $\Omega(V)$  as an open dense subscheme, since each  $\pi_i$  induces an isomorphism over  $\Omega(V)$ . We write  $D = X - \Omega(V)$  for the complement.

Note that  $\pi_{n-1}$  is an isomorphism and that the strict transforms of two distinct linear subspaces  $L, L' \subset \mathbf{P}(V)$  of dimension  $i$  in  $X_{i-1}$  are disjoint since (the strict transform of)  $L \cap L'$  has previously been blown up.

Each non-trivial linear subspace  $L \subset \mathbf{P}(V)$  defines a smooth and irreducible hypersurface  $E_L$  in  $X$  as follows. If  $L$  has dimension  $i$ , its strict transform by  $\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1}$  in  $X_{i-1}$  (which is, by convention,  $L$  itself if it is a point) is blown up under the map  $\pi_i : X_i \rightarrow X_{i-1}$  to give rise to a hypersurface  $E_L^{(i)}$  in  $X_i$ . The (codimension-1) subscheme  $E_L$  of  $X$  is then the strict transform of  $E_L^{(i)}$  by  $\pi_{i+1} \circ \cdots \circ \pi_{n-1}$ . The induced map  $E_L \rightarrow E_L^{(i)}$  coincides with the blow-up of  $E_L^{(i)}$  along the hypersurface arrangement induced by hyperplanes of  $\mathbf{P}(V)$  containing  $L$ . We have an alternative description of  $E_L$  as the closure

$$\overline{\pi^{-1}\left(L - \bigcup_{\substack{L' \in \mathcal{L}(V) \\ L' \subsetneq L}} L'\right)}$$

taken in  $X$ .

It follows from the construction of  $X$  that the boundary divisor  $D$  is the union of all hypersurfaces  $E_L$ , i.e. we have

$$D = \pi^{-1}\left(\bigcup_{\substack{W \subset V \\ \dim W=1}} \mathbf{P}(V/W)\right) = \bigcup_L E_L.$$

Two components  $E_L$  and  $E_{L'}$  have non-empty intersection if and only if  $L \subset L'$  or  $L' \subset L$ . Indeed, if none of the inclusions holds, then  $L$  and  $L'$  intersect along a smaller linear subspace, say of dimension  $i$ , and the strict transforms of  $L$  and  $L'$  in  $X_i$  are disjoint. It follows that a family of components has non-empty intersection if and only if it is indexed by linear subspaces lying in a flag. We define the stratum  $Z_{\mathcal{F}}$  corresponding to a flag  $\mathcal{F}$  by

$$Z_{\mathcal{F}} = \bigcup_{L \in \mathcal{F}} E_L - \bigcup_{L' \notin \mathcal{F}} E_{L'}.$$

LEMMA 1.3. *The divisor  $D$  has simple normal crossings. Moreover, if  $Z = Z_{\mathcal{F}}$  is the stratum corresponding to the flag  $\mathcal{F}$ , then*

$$U_Z = X - \bigcup_{L \notin \mathcal{F}} E_L$$

is an affine open subset of  $X$  containing  $Z$  as a closed subset.

*Proof.* We start by considering a complete flag  $\mathcal{F} = (L_0, \dots, L_n)$ . In order to get an explicit description of  $U_Z$  in this case, we first compare  $X$  to the blow-up  $Y$  of  $\mathbf{P}(V)$  along  $\mathcal{F}$ . To be precise, we define

$$p : Y = Y_{n-1} \xrightarrow{p_{n-1}} Y_{n-2} \longrightarrow \dots \longrightarrow Y_1 \xrightarrow{p_1} Y_0 \xrightarrow{p_0} Y_{-1} = \mathbf{P}(V),$$

where  $p_i$  denotes the blow-up of  $Y_{i-1}$  along the strict transform of  $L_i$ . By the universal property of blow-up, there exists a (unique) morphism of towers  $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ .

Now, we want to show that  $f$  identifies  $U_Z$  with the complement  $W_Z$  in  $Y$  of the strict transforms of all linear subspaces not contained in  $\mathcal{F}$ . Note that  $W_Z$  is also the complement of the strict transforms of all hyperplanes distinct from  $L_{n-1}$ . We argue by induction along the towers of blow-ups. For every  $i \in \{-1, \dots, n-1\}$ , we define two open subsets  $U_i \subset X_i$  and  $W_i \subset Y_i$  as follows:

- $U_{-1} = W_{-1}$  is the complement in  $\mathbf{P}(V)$  of all 0-dimensional linear subspaces distinct from  $L_0$ ;
- if  $0 \leq i \leq n-2$ , then  $U_i$  (respectively,  $W_i$ ) is the complement in  $\pi_i^{-1}(U_{i-1})$  (respectively, in  $p_i^{-1}(W_{i-1})$ ) of the strict transforms of all  $(i+1)$ -dimensional linear subspaces  $L \subset \mathbf{P}(V)$  that are not in  $\mathcal{F}$ ;
- $U_{n-1} = \pi_{n-1}^{-1}(U_{n-2})$  and  $W_{n-1} = p_{n-1}^{-1}(W_{n-2})$ .

Arguing by induction on  $i$ , we see that  $U_i = f_i^{-1}(W_i)$  and that  $f_i$  induces an isomorphism between  $U_i$  and  $W_i$  which respects the restrictions of exceptional divisors. It is clear that

$$U_{n-1} = U_Z = X - \bigcup_{L \notin \mathcal{F}} E_L.$$

On the other hand, we claim that  $W_{n-1}$  coincides with  $W_Z$ . The inclusion  $W_{n-1} \subset W_Z$  is obvious. For every point  $y \in Y - W_{n-1}$  there exists an index  $i \in \{-1, \dots, n-2\}$  such that the image  $y_i$  of  $y$  in  $Y_i$  lies in the strict transform of a  $(i+1)$ -dimensional linear subspace  $L \subset \mathbf{P}(V)$  distinct from  $L_{i+1}$ . Let us consider a hyperplane  $H$  which contains  $L$ . By construction,  $y_i$  is contained in the strict transform of  $H$  in  $Y_i$ . Since  $L_j \not\subset H$  for  $j \in \{i, \dots, n-2\}$ , the subspaces  $L_j$  and  $H$  are transverse. Blowing up along some smooth subschemes can only decrease the order of contact; hence the strict transform  $\tilde{H}$  of  $H$  in  $Y_j$  is transverse to the center of  $p_{j+1}$ . This implies that the strict transform of  $H$  in  $Y_{j+1}$  coincides with the inverse image of  $\tilde{H}$  in  $Y_{j+1}$ . It follows that  $y$  belongs to the strict transform of  $H$  in  $Y$ , and thus  $y \in Y - W_Z$ . This proves the converse inclusion  $W_Z \subset W_{n-1}$ .

Given a basis  $(e_0, e_1, \dots, e_n)$  of  $V$  such that  $L_i = Z(e_{i+1}, \dots, e_n)$  for every  $i \in \{0, \dots, n-1\}$ , we have a commutative diagram

$$\begin{CD} \mathrm{Spec}(k[t_1, \dots, t_n]) @<j<< Y \\ @VqVV @VVpV \\ \mathrm{Spec}(k[x_1, \dots, x_n]) @<<< \mathbf{P}(V) \end{CD}$$

where the horizontal arrows are open immersions identifying  $t_1, \dots, t_n$  (respectively,  $x_1, \dots, x_n$ ) with the rational functions  $e_1/e_0, \dots, e_n/e_{n-1}$  (respectively,  $e_1/e_0, \dots, e_n/e_0$ ) and  $q$  is the morphism defined by  $q^*(x_i) = \prod_{j \leq i} t_j$ .

Via  $j$ , the open subscheme  $W_Z$  of  $Y$  is isomorphic to the principal open subset  $D(f)$  of  $\text{Spec}(k[t_1, \dots, t_n])$ , where

$$f = \prod_{i=1}^n \prod_{(a_i, \dots, a_n) \in k^{n-i+1}} (1 + a_i t_i + a_{i+1} t_i t_{i+1} + \dots + a_n t_i \dots t_n).$$

In particular,  $W_Z$  is affine. Moreover, the intersection of the exceptional divisor of  $p$  with the open affine set  $W_Z$  coincides with  $\text{div}(t_1 \dots t_n)$  and hence has simple normal crossings. Using the isomorphism between  $U_Z$  and  $W_Z$  induced by  $f$ , we deduce that  $U_Z$  is affine and that  $D \cap U_Z$  has simple normal crossings. Since the sets  $U_Z$  for all choices of complete flags form an open affine covering of  $X$ , the divisor  $D$  has simple normal crossings on  $X$ .

We now claim that the intersection  $\Sigma$  of any family of  $d$  irreducible components of  $D$  is either empty or irreducible. Indeed, assume that  $\Sigma$  is non-empty and reducible. Non-emptiness amounts to saying that these components correspond to linear subspaces in some flag  $\mathcal{F}$ . Pick a complete flag  $\mathcal{F}'$  containing  $\mathcal{F}$ . In the corresponding affine chart  $U_Z$ , the intersection of the  $d$  components which we consider is irreducible; hence there must be a component  $\Sigma_0$  of  $\Sigma$  which lies in  $X - U_Z$ . Since, by construction,  $X - U_Z$  is the union of some irreducible components of  $D$ , we see that  $\Sigma_0$  must be contained in a  $(d + 1)$ th irreducible component of  $D$ . But this contradicts the normal crossing property of  $D$ . In view of the discussion before Lemma 1.3, this shows that the strata of  $D$  are in one-to-one correspondence with flags of linear subspaces.

If we start with a stratum  $Z$  corresponding to a partial flag  $\mathcal{F}$ , the set  $U_Z = X - \bigcup_{L \notin \mathcal{F}} E_L$  is the intersection of all  $U_{Z'}$  for strata  $Z'$  that correspond to complete flags containing  $\mathcal{F}$ . Hence it is open affine as a finite intersection of open affines in a separated  $k$ -scheme.  $\square$

In order to extend an automorphism of  $\Omega(V)$  first to  $X$  and then to  $\mathbf{P}(V)$ , we look at its action on the discrete valuations associated to the components of  $D$ . For each  $L \in \mathcal{L}(V)$ , the local ring at the generic point of the hypersurface  $E_L$  is a discrete valuation ring in the function field  $\kappa(V)$  of  $X$ . We denote by  $\text{ord}_L$  the corresponding discrete valuation on  $\kappa(V)$ , and we write

$$\Gamma(V) = \{\text{ord}_L : L \in \mathcal{L}(V)\}$$

for the set of all these valuations. Note that  $\kappa(V)$  is the function field of both  $\mathbf{P}(V)$  and  $\Omega(V)$ . If  $L$  is a *hyperplane* in  $\mathbf{P}(V)$ , then the valuation  $\text{ord}_L$  is the one given by the local ring of  $\mathbf{P}(V)$  at the generic point of  $L$ .

The sets  $\mathcal{L}(V)$  and  $\Gamma(V)$  come with a natural simplicial structure, for which the  $q$ -simplices correspond to flags of linear subspaces of length  $q - 1$ .

**PROPOSITION 1.4.** *Let  $\varphi$  be a  $k$ -automorphism of  $\Omega(V)$  and let  $\varphi^*$  be the induced automorphism of the set of valuations on the function field  $\kappa(V)$ .*

(i) *The birational map  $\varphi$  extends to a  $k$ -automorphism of  $X$  if and only if  $\varphi^*$  preserves the set  $\Gamma(V)$  and its simplicial structure.*

(ii) *The birational map extends to a  $k$ -automorphism of  $\mathbf{P}(V)$  if and only if  $\varphi^*$  preserves the subset of  $\Gamma(V)$  defined by hyperplanes.*

*Proof.* (i) The condition is necessary because the simplicial set  $\Gamma(V)$  describes the incidence relations between irreducible components of  $D$  (Lemma 1.3). To see that it is sufficient, we use

the covering of  $X$  by the open affine subsets

$$U_Z = X - \bigcup_{L \notin \mathcal{F}} E_L,$$

where  $Z$  denotes a stratum of  $D$  and  $\mathcal{F}$  is the corresponding flag of linear subspaces of  $\mathbf{P}(V)$ . If  $\varphi$  preserves  $\Gamma(V)$  with its simplicial structure, then for every stratum  $Z$  there exists another stratum  $Z'$  such that the rational map

$$U_{Z'} \dashrightarrow U_Z$$

induced by  $\varphi$  is defined at each point of height 1.

Since  $U_Z$  is affine and  $U_{Z'}$  is noetherian and normal, this rational map is everywhere defined on  $U_{Z'}$  (see [EGA4, 20.4.12]), and therefore  $\varphi$  extends to an automorphism from  $X$  to  $X$  (apply this argument to  $\varphi^{-1}$ ).

(ii) If the morphism  $\varphi : \Omega(V) \rightarrow \Omega(V)$  preserves all valuations  $\text{ord}_L$  coming from hyperplanes, then for every hyperplane  $L$  in  $\mathbf{P}(V)$  there exists a hyperplane  $L'$  such that the rational map

$$\mathbf{P}(V) - L' \dashrightarrow \mathbf{P}(V) - L$$

induced by  $\varphi$  is defined at every point of height 1, and the conclusion follows as for (i). □

### 2. Step 1: valuations and analytic geometry

This section is devoted to the first step toward the theorem, namely the fact that every  $k$ -automorphism of  $\Omega(V)$  extends to a  $k$ -automorphism of  $X$ .

**PROPOSITION 2.1.** *Let  $\text{Aut}_k(X, D)$  denote the group of  $k$ -automorphisms of  $X$  which preserve  $D$ . The canonical map*

$$\text{Aut}_k(X, D) \rightarrow \text{Aut}_k(\Omega(V)), \quad \varphi \mapsto \varphi|_{\Omega(V)}$$

*is an isomorphism. Equivalently, every  $k$ -automorphism of  $\Omega(V)$  extends to a  $k$ -automorphism of  $X$ .*

We can study this problem from a nice geometric viewpoint in the framework of Berkovich spaces.

Endowed with the trivial absolute value,  $k$  becomes a complete non-Archimedean field. There is a well-defined category of  $k$ -analytic spaces, together with an analytification functor  $Z \rightsquigarrow Z^{\text{an}}$  from the category of  $k$ -schemes locally of finite type. If  $Z$  is affine, then the topological space underlying  $Z^{\text{an}}$  is the set of multiplicative  $k$ -seminorms on  $\mathcal{O}(Z)$  with the topology generated by evaluation maps  $x \mapsto |f(x)| := x(f)$ , where  $f \in \mathcal{O}(Z)$ . Imposing the additional condition that all seminorms are bounded by 1 on the algebra  $\mathcal{O}(Z)$ , we obtain a compact domain  $Z^{\square}$  in  $Z^{\text{an}}$  equipped with a *specialization* map  $\text{sp} : Z^{\square} \rightarrow Z$  (denoted by  $r$  in [Thu07]) which sends a multiplicative seminorm  $x$  to the prime ideal  $\{f \in \mathcal{O}(Z) : |f(x)| < 1\}$ . The reader is referred to [Ber90, § 3.5] and [Thu07, § 1] for a detailed account.

Working in the analytic category over  $k$  allows us to realize  $\Gamma(V)$  as a set of rays in  $\Omega(V)^{\text{an}}$ : for each  $L \in \mathcal{L}(V)$ , the map

$$\varepsilon_L : (0, 1] \rightarrow \Omega(V)^{\text{an}}, \quad r \mapsto r^{\text{ord}_L(\cdot)}$$

is an embedding and  $\varepsilon_L(1)$  is the canonical point of  $\Omega(V)^{\text{an}}$ , namely the point corresponding to the trivial absolute value on  $\kappa(V)$ . Now, the proposition will follow from the fact that this

collection of rays is the 1-skeleton of a *conical complex*  $\mathfrak{S}(V)$  in  $\Omega(V)^{\text{an}}$  which is preserved by every  $k$ -automorphism of  $\Omega(V)$ .

This conical complex  $\mathfrak{S}(V)$  is the *fan*  $\mathfrak{S}_0(X, D)$  of the toroidal embedding  $\Omega(V) \hookrightarrow X$  introduced in [Thu07, §3.1 and Proposition 4.7], following [Ber99]. Let us describe this construction in the particular case we consider here.

(a) The canonical map

$$r : \mathbf{A}_k^{n, \rhd} \rightarrow [0, 1]^n, \quad x \mapsto (|t_1(x)|, \dots, |t_n(x)|)$$

has a continuous section  $j$  defined by mapping a tuple  $r \in [0, 1]^n$  to the following diagonalizable multiplicative seminorm on  $k[t_1, \dots, t_n]$ :

$$\sum_{\nu \in \mathbb{N}^n} a_\nu t^\nu \mapsto \max_\nu |a_\nu| r_1^{\nu_1} \cdots r_n^{\nu_n}.$$

(b) Let  $D(t_1, \dots, t_n)$  denote the invertibility locus of  $t_1, \dots, t_n$ . Intersecting the image of  $j$  with the open domain  $D(t_1, \dots, t_n)^\rhd$ , we obtain a closed subset  $C_n \subset D(t_1, \dots, t_n)^\rhd$  homeomorphic to the cone  $(0, 1]^n$ . The map  $\tau = r \circ j$  is a retraction of  $D(t_1, \dots, t_n)^\rhd$  onto  $C_n$ . Its fiber over a point  $x \in C_n$  is a  $k$ -affinoid domain whose Shilov boundary is reduced to  $\{x\}$ .

(c) We identify  $C_n$  and  $(0, 1]^n$  via  $r$ . For  $I \subset \{1, \dots, n\}$ , let  $C_n^I$  denote the face of  $C_n$  defined by  $r_i = 0$  for every  $i \in I$ . The specialization map  $\text{sp} : D(t_1, \dots, t_n)^\rhd \rightarrow \mathbf{A}_k^n$  sends the interior of  $C_n^I$  to the generic point of the locally closed subscheme  $Z_I = V(t_i, i \in I) \cap D(t_j, j \notin I)$ . This implies that  $C_n^I$  is contained in  $U^\rhd = \text{sp}^{-1}(U)$  for any open neighborhood  $U$  of the generic point of  $Z_I$ .

(d) We can also recover the monoid  $r_1^{\mathbb{N}} \cdots r_n^{\mathbb{N}}$  defining the integral affine structure on  $(0, 1]^n$  from the analytic structure of  $\mathbf{A}_k^n$ . Indeed, this is precisely the monoid of functions  $|f| : C_n \rightarrow (0, 1]$  induced by germs  $f \in \mathcal{O}_{\mathbf{A}_k^n, 0}$  invertible on  $D(t_1, \dots, t_n)$ . Similarly, the submonoid corresponding to the face  $C_n^I$  comes from germs of  $\mathcal{O}_{\mathbf{A}_k^n}$  at the generic points of  $Z_I$  which are invertible over  $D(t_1, \dots, t_n)$ .

(e) We now return to the scheme  $X$  with its simple normal crossing divisor  $D$ . Fix a stratum  $Z$  with generic point  $\eta_Z$  and let  $\Lambda_Z^+$  denote the monoid of germs in  $\mathcal{O}_{X, \eta_Z}$  whose restriction to  $\Omega(V)$  is invertible. As in the proof of Lemma 1.3, there is an open immersion  $(t_1, \dots, t_n) : U_Z \rightarrow \mathbf{A}_k^n$  identifying  $Z$  with a non-empty open subset of  $Z_I$  for a suitable subset  $I$  of  $\{1, \dots, n\}$ . By transport of structure, we obtain a closed subset  $C_Z$  of  $U_Z^\rhd - Z^\rhd$  such that the natural map

$$C_Z \rightarrow \text{Hom}_{\text{Mon}}(\Lambda_Z^+ / k^\times, (0, 1]), \quad x \mapsto (f \mapsto |f(x)|)$$

is a homeomorphism. Covering  $X$  by the open subschemes  $U_Z$ , we can glue the cones  $C_Z$  along common faces in  $\Omega(V)^{\text{an}}$  to define a cone complex  $\mathfrak{S}(V)$ . This gluing is compatible with local retractions, so we get a retraction of  $\Omega(V)^{\text{an}}$  onto  $\mathfrak{S}(V)$ .

The following property of the conical complex  $\mathfrak{S}(V)$  is specific to our situation and is the key point in the proof of Proposition 2.1. It may be interesting to look for other ‘natural’ toroidal compactifications that satisfy this condition.

LEMMA 2.2. *The map*

$$\iota : \mathfrak{S}(V) \rightarrow \text{Hom}_{\mathbf{Ab}}(\mathcal{O}(\Omega(V))^\times / k^\times, \mathbb{R}_{>0}), \quad x \mapsto (f \mapsto |f(x)|)$$

*is a closed embedding inducing the integral affine structure on each cone. Moreover, (the images of) distinct cones span distinct linear spaces.*

*Proof.* Roughly speaking, this statement means that there are enough invertible functions on  $\Omega(V)$ . Consider a stratum  $Z$  of  $D$  corresponding to a flag  $\mathcal{F}$  of non-trivial linear subspaces of  $\mathbf{P}(V)$ , and pick a basis  $(e_0, \dots, e_n)$  of  $V$  such that  $\mathcal{F}$  is a subflag of

$$Z(e_1, \dots, e_n) \subset Z(e_2, \dots, e_n) \subset \dots \subset Z(e_n).$$

The explicit description of  $X$  given at the end of the proof of Lemma 1.3 shows that the tuple  $(e_1/e_0, e_2/e_1, \dots, e_n/e_{n-1})$  of elements in  $\mathcal{O}_{X, \eta_Z}$  contains a regular system of parameters defining  $D$  at  $\eta_Z$ . Therefore, the map  $\iota$  induces an integral affine embedding of the cone  $C_Z$ .

Furthermore, we claim that the following is true: *given two distinct cones  $C$  and  $C'$ , there exists  $f \in \mathcal{O}(\Omega(V))^\times$  such that  $|f| = 1$  on one of the cones and  $|f| < 1$  on the interior of the other.* Injectivity of the map  $\iota$  and the last statement of the lemma follow immediately.

We finish the proof by establishing the claim. Given two non-zero vectors  $v, v' \in V$  and a non-trivial linear subspace  $L \subset \mathbf{P}(V)$ , the function  $v/v'$  is either a unit, a uniformizer, or the inverse of a uniformizer at the generic point of  $E_L$ , according to the positions of  $Z(v)$  and  $Z(v')$  with respect to  $L$ . It follows that:

- (a)  $|v/v'| < 1$  on  $\varepsilon_L(0, 1)$ , if  $L \subset Z(v)$  and  $L \not\subset Z(v')$ ;
- (b)  $|v/v'| > 1$  on  $\varepsilon_L(0, 1)$ , if  $L \subset Z(v')$  and  $L \not\subset Z(v)$ ;
- (c)  $|v/v'| = 1$  on  $\varepsilon_L(0, 1]$ , if the hyperplanes  $Z(v)$  and  $Z(v')$  are in the same position with respect to  $L$ .

Consider two distinct strata  $Z$  and  $Z'$  of  $D$ , corresponding to distinct flags  $\mathcal{F}$  and  $\mathcal{F}'$  of non-trivial linear subspaces. Pick a linear space  $L$  occurring in only one of them, say  $\mathcal{F}$ , and set  $i = \dim L$ . We embed  $\mathcal{F}'$  into a complete flag  $(L_0 \subset L_1 \subset \dots \subset L_{n-1})$  such that  $L_i \neq L$ .

We claim that this assumption guarantees the existence of two hyperplanes  $H$  and  $H'$  such that:

- $L \subset H$  and  $L_i \not\subset H$ ;
- $L_i \cap H = L_i \cap H'$  and  $L \not\subset H'$ .

To prove this claim, we argue with the corresponding linear quotient spaces of  $V$ . Let  $L = \mathbf{P}(V/W)$  and  $L_i = \mathbf{P}(V/W_i)$ , where  $W$  and  $W_i$  are different linear subspaces of  $V$  of dimension  $n - i$ . Choose a vector  $u \in W$  which is not contained in  $W_i$  and a vector  $u_i \in W_i$  which is not contained in  $W$ . We denote by  $U$  the line in  $V$  generated by  $u$  and by  $U'$  the line generated by  $u' = u + u_i$ . The corresponding hyperplanes  $H = \mathbf{P}(V/U)$  and  $H' = \mathbf{P}(V/U')$  have the desired properties.

In particular,  $H$  and  $H'$  are in the same position with respect to  $L_0, \dots, L_{n-1}$ . Given any equations  $v, v' \in V$  of  $H$  and  $H'$ , respectively, we thus obtain  $|v/v'| = 1$  on  $C_{Z'}$ . Let us now consider the flag  $\mathcal{F}$ . Any linear subspace  $M \in \mathcal{F}$  contained in  $H'$  is necessarily contained in  $L$  and hence in  $H$ ; therefore  $|v/v'| \leq 1$  on the ray  $\varepsilon_M(0, 1]$ . Since  $|v/v'| < 1$  on the interior of the ray  $\varepsilon_L(0, 1]$ , we deduce that  $|v/v'| < 1$  on the interior of the cone  $C_Z$ .  $\square$

*Proof of Proposition 2.1.* First, we observe that  $\mathfrak{S}(V)$  coincides with the set  $\Omega(V)_{\max}^{\text{an}}$  of maximal points of  $\Omega(V)^{\text{an}}$  for the following ordering:

$$x \preceq y \iff \forall f \in \mathcal{O}(\Omega(V)^{\text{an}}), |f(x)| \leq |f(y)|.$$

For any point  $x \in \Omega(V)^{\text{an}}$ , we have  $x \preceq \tau(x)$  because the fiber  $\tau^{-1}(\tau(x))$  is a  $k$ -affinoid domain with Shilov boundary  $\{\tau(x)\}$ . This implies the inclusion  $\Omega(V)_{\max}^{\text{an}} \subset \mathfrak{S}(V)$ .

We apply Lemma 2.2 to get the converse inclusion. If a point  $x \in \mathfrak{S}(V)$  is dominated by a point  $x' \in \Omega(V)^{\text{an}}$ , then it is also dominated by  $\tau(x')$ . However, for any two distinct points  $x$

and  $y$  in  $\mathfrak{S}(V)$ , there exists  $f \in \mathcal{O}(\Omega(V)^{\text{an}})^{\times}$  such that  $|f(x)| \neq |f(y)|$  and, hence, such that  $|f(x)| < |f(y)|$  and  $|(1/f)(x)| > |(1/f)(y)|$  or vice versa; therefore  $x$  and  $y$  are incomparable. In particular, we get  $x = \tau(x')$  and thus  $x$  is maximal.

The above characterization of  $\mathfrak{S}(V)$  as a closed subset of  $\Omega(V)^{\text{an}}$  implies that it is preserved by any  $k$ -automorphism  $\varphi$  of  $\Omega(V)$ . It remains to check that the homeomorphism of  $\mathfrak{S}(V)$  induced by  $\varphi$  also preserves the conical structure. Let  $\Phi$  denote the linear automorphism of  $\text{Hom}_{\mathbf{Ab}}(\mathcal{O}(\Omega(V)^{\text{an}})^{\times}, \mathbb{R}_{>0})$  deduced from  $\varphi$ . Given an  $n$ -dimensional cone  $C \subset \mathfrak{S}(V)$ , the image of its interior is disjoint from the  $(n - 1)$ -skeleton of  $\mathfrak{S}(V)$ ; otherwise, it would meet the interiors of two distinct  $n$ -dimensional cones  $C'$  and  $C''$ , and hence  $\langle \iota C' \rangle = \Phi(\langle \iota C \rangle) = \langle \iota C'' \rangle$ , contradicting Lemma 2.2. It follows that  $\varphi(C)$  is contained in some  $n$ -dimensional cone  $C'$ , and thus  $\varphi(C) = C'$  by considering  $\varphi^{-1}$ . The assertion for lower-dimensional cones follows at once by considering faces, since the automorphism  $\Phi$  is linear.

In particular, we see that  $\varphi$  preserves the 1-skeleton of  $\mathfrak{S}(V)$  and hence the set  $\Gamma(V)$  of discrete valuations on  $\kappa(V)$  associated with irreducible components of  $D = X - \Omega(V)$ , together with the simplicial structure reflecting the incidence relations between these components. By Proposition 1.4(i), this implies that  $\varphi$  extends to a  $k$ -automorphism of  $X$ .  $\square$

*Remark 2.3.* (i) Let  $D$  be a simple normal crossing divisor on a smooth and proper (connected) scheme  $X$  over  $k$ . Even if  $\Omega(V) = X - D$  is affine, Lemma 2.2 and its consequences may fail. For example, consider the case  $X = \mathbf{P}_k^n$ . If  $D$  is a hyperplane, then  $\mathfrak{S}_0(X, D)$  is a 1-dimensional cone whereas  $\Omega(V)_{\text{max}}^{\text{an}}$  is empty. If  $D$  is the union of the coordinate hyperplanes, then  $\Omega(V) = \mathbf{G}_m^n$  and  $\mathfrak{S}_0(X, D) = \Omega(V)_{\text{max}}^{\text{an}}$  is the toric fan; but the map  $\iota$  is bijective, and hence all maximal cones span the same linear space. In fact, the inversion  $(t_1, \dots, t_n) \mapsto (t_1^{-1}, \dots, t_n^{-1})$  on  $\mathbf{G}_m^n$  transforms the fan  $\mathfrak{S}_0(X, D)$  into its opposite, and thus does not preserve the conical structure. This reflects the fact that this automorphism of  $\mathbf{G}_m^n$  does not extend to  $\mathbf{P}^n$ .

(ii) The conical complex  $\mathfrak{S}(V)$  is also the *vectorial building* of  $\text{PGL}(V)$ , but this is somehow fortuitous and irrelevant from the viewpoint of automorphisms. In general, for any connected and split semi-simple  $k$ -group  $G$ , there exists a canonical embedding of the vectorial building  $\mathcal{V}(G, k)$  of  $G(k)$  into the analytification of an open affine subscheme  $\Omega$  in any flag variety  $Y$  of  $G$  (see [Ber90, §5.5]). However, this observation does not lead to a generalization of Theorem 1.1, at least not along the lines of the present proof. Indeed, while we have made crucial use of the fact that  $\mathfrak{S}(V)$  is the fan of a normal crossing divisor, we doubt that  $\mathcal{V}(G, k)$  can be realized as the fan of a toroidal compactification of  $\Omega(V)$  if  $(G', Y) \neq (\text{PGL}(V), \mathbf{P}(V)), (\text{PGL}(V), \mathbf{P}(V^{\vee}))$ .

(iii) It might be interesting to try to extend our method, based on the study of toroidal compactifications, to determine the automorphism groups of other period domains.

(iv) Whether the above proposition can be proved without analytic geometry is not clear.

### 3. Step 2: geometry of the blow-up

The second step in the proof of the theorem relies on elementary intersection theory on  $X$ , which we review in this section. The standard reference is [Ful97].

The Chow ring  $\text{CH}^*$  is a contravariant functor from the category of smooth  $k$ -schemes to the category of graded commutative rings. For any smooth  $k$ -scheme  $X$ , the abelian group underlying  $\text{CH}^*(X)$  is the free abelian group on integral subschemes of  $X$  modulo rational equivalence, and it is graded by codimension. Multiplication comes from the intersection product. We write  $[Z]$  for the class of a closed subscheme  $Z$  of  $X$ .

We are going to use the following two basic facts.

(a) Let  $Y$  be a regularly embedded closed subscheme of  $X$  of codimension at least 2, and let  $\pi : \tilde{X} \rightarrow X$  be the blow-up of  $X$  along  $Y$ , with exceptional divisor  $\tilde{Y}$ . The canonical map

$$\mathrm{CH}^1(X) \oplus \mathbb{Z}[\tilde{Y}] \rightarrow \mathrm{CH}^1(\tilde{X}), \quad (z, n[\tilde{Y}]) \mapsto \pi^*(z) + n[\tilde{Y}]$$

is an isomorphism [Ful97, Proposition 6.7].

(b) In the situation of (a), let  $V$  be an integral subscheme of  $X$  with strict transform  $\tilde{V}$ . If  $\mathrm{codim}(Y, X) \leq \mathrm{codim}(V \cap Y, V)$ , then

$$\pi^*[V] = [\tilde{V}]$$

in  $\mathrm{CH}^*(\tilde{X})$  (see [Ful97, Corollary 6.7.2]).

We now focus on the particular case where  $\pi : X \rightarrow \mathbf{P}(V)$  is the blow-up along the full hyperplane arrangement, with exceptional divisor  $D$ .

LEMMA 3.1. *We have*

$$\mathrm{CH}^1(X) = \mathbb{Z}h \oplus \bigoplus_{\mathbf{L}} \mathbb{Z}[E_{\mathbf{L}}],$$

where  $h = \pi^*[H]$  denotes the pullback of the hyperplane class  $[H]$  on  $\mathbf{P}(V)$  and  $\mathbf{L}$  runs over the set of non-trivial linear subspaces of  $\mathbf{P}(V)$  of codimension at least 2.

*Proof.* For any non-trivial linear subspace  $\mathbf{L}$  of  $\mathbf{P}(V)$  of dimension  $i \in \{0, \dots, n-1\}$ , let  $E_{\mathbf{L}}^{(i)} \subset X_i$  denote the blow-up of its strict transform in  $X_{i-1}$ ; this is a smooth irreducible hypersurface. Recall that we have  $\pi = \pi_0 \circ \pi_1 \circ \dots \circ \pi_{n-1}$ , where  $\pi_{n-1}$  is an isomorphism. Applying (a) iteratively to each blow-up  $\pi_0, \dots, \pi_{n-2}$ , we obtain that  $\mathrm{CH}^1(X)$  is the free abelian group on  $h$  and the classes  $(\pi_{i+1} \circ \dots \circ \pi_{n-1})^*[E_{\mathbf{L}}^{(i)}]$ , where  $i \in \{0, \dots, n-2\}$  and  $\mathbf{L}$  runs over the set of  $i$ -dimensional linear subspaces of  $\mathbf{P}(V)$ .

The conclusion follows from the additional fact that we have an equality

$$(\pi_{i+1} \circ \dots \circ \pi_{n-1})^*[E_{\mathbf{L}}^{(i)}] = [E_{\mathbf{L}}]$$

in  $\mathrm{CH}^1(X)$  for any linear subspace  $\mathbf{L}$  of dimension  $i \in \{0, \dots, n-2\}$ . This is an immediate consequence of (b), since the center of each blow-up  $\pi_j$ , with  $j \in \{i+1, \dots, n-1\}$ , is transversal to the strict transform of  $E_{\mathbf{L}}^{(i)}$  in  $X_{j-1}$ . □

For each integer  $d \geq 1$ , we define

$$\lambda(d) = \# \left\{ \begin{array}{l} \text{non-trivial linear subspaces of} \\ \text{codimension at least 2 in } \mathbf{P}_k^d \end{array} \right\}.$$

Additionally, we set  $\lambda(0) = 0$ .

LEMMA 3.2. *Let  $\mathbf{L} \subset \mathbf{P}(V)$  be a non-trivial linear subspace of dimension  $d$ ; note that  $d \in \{0, \dots, n-1\}$ .*

(i) *We have*

$$\mathrm{rk} \mathrm{CH}^1(E_{\mathbf{L}}) = \lambda(d) + \lambda(n-1-d) + \varepsilon(d),$$

where  $\varepsilon(d) = 1$  if  $d \in \{0, n-1\}$  and  $\varepsilon(d) = 2$  otherwise.

(ii) *For every linear subspace  $\mathbf{L}' \subset \mathbf{P}(V)$  of dimension  $d'$  satisfying  $d < d' < n-1-d$ , the following inequality holds:*

$$\mathrm{rk} \mathrm{CH}^1(E_{\mathbf{L}}) > \mathrm{rk} \mathrm{CH}^1(E_{\mathbf{L}'}).$$

*Proof.* (i) Let  $L_{d-1}$  (respectively,  $\tilde{L}$ ) denote the strict transform of  $L$  in  $X_{d-1}$  (respectively, in  $X_d$ ). The scheme  $E_L$  is the blow-up of  $\tilde{L}$  along the hypersurface arrangement induced by hyperplanes of  $\mathbf{P}(V)$  containing  $L$ . Applying (a), we obtain

$$\begin{aligned} \text{rk CH}^1(E_L) &= \text{rk CH}^1(\tilde{L}) + \# \left\{ \begin{array}{l} \text{linear spaces of codimension} \\ \text{at least 2 that strictly contain } L \end{array} \right\} \\ &= \text{rk CH}^1(\tilde{L}) + \lambda(n - d - 1). \end{aligned}$$

Since  $\tilde{L} = \mathbf{P}(\mathcal{N})$ , where  $\mathcal{N}$  is the conormal sheaf to  $L_{d-1}$  in  $X_{d-1}$ , of rank  $n - d$ , it follows from [Ful97, Theorem 3.3(b)] that

$$\text{rk CH}^1(\tilde{L}) = \text{rk CH}^0(L_{d-1}) + \text{rk CH}^1(L_{d-1}) = 1 + \text{rk CH}^1(L_{d-1})$$

if  $0 \leq d < n - 1$ , and that

$$\text{rk CH}^1(\tilde{L}) = \text{rk CH}^1(L_{d-1})$$

if  $d = n - 1$ .

Finally, since  $L_{d-1}$  is the blow-up of  $L$  along the full hyperplane arrangement,

$$\text{rk CH}^1(L_{d-1}) = \text{rk CH}^1(L) + \# \left\{ \begin{array}{l} \text{non-trivial linear subspaces} \\ \text{of codimension at least 2 in } L \end{array} \right\},$$

and hence

$$\text{rk CH}^1(L_{d-1}) = \begin{cases} 1 + \lambda(d) & \text{if } 0 < d \leq n - 1, \\ 0 & \text{if } d = 0. \end{cases}$$

(ii) In view of (i), it is enough to prove the inequality

$$\lambda(n - 1 - d) - \lambda(n - 1 - d') > \lambda(d') - \lambda(d) + 1 \tag{*}$$

for any  $d, d' \in \{0, \dots, n - 1\}$  such that  $d < d' < n - 1 - d$ .

Let us first show that this statement follows from the inequality

$$\lambda(t) - \lambda(t - 1) > \lambda(t - 1) + 1 \quad \text{for all } t \geq 2. \tag{\dagger}$$

Indeed, assuming  $(\dagger)$ , fix  $d \in \{0, \dots, n - 1\}$  and  $d'$  satisfying  $d < d' < n - 1 - d$ . Since  $d' \geq d + 1$ , we have  $n - 1 - d' \leq n - 1 - (d + 1)$ ; and since  $\lambda$  is an increasing function, this implies that

$$\lambda(n - 1 - d) - \lambda(n - 1 - d') \geq \lambda(n - 1 - d) - \lambda(n - 1 - (d + 1)).$$

Now put  $t = n - 1 - d$ . Note that  $t \geq 2$ . Then  $t - 1 = n - 1 - (d + 1)$ , and  $(\dagger)$  implies that

$$\lambda(n - 1 - d) - \lambda(n - 1 - (d + 1)) > \lambda(n - 1 - (d + 1)) + 1.$$

As  $d' < n - 1 - d$ , we have  $d' \leq n - 1 - d - 1 = n - 1 - (d + 1)$ . Once more, we use the fact that  $\lambda$  is a non-negative increasing function to deduce that

$$\lambda(n - 1 - (d + 1)) + 1 \geq \lambda(d') + 1 \geq \lambda(d') - \lambda(d) + 1.$$

Combining the previous inequalities gives  $(*)$ .

Therefore it remains to prove  $(\dagger)$ . If we fix a hyperplane  $H$  and count non-trivial linear subspaces of codimension 2 or greater in  $\mathbf{P}_k^t$ , taking into account their position with respect to  $H$  (transverse to  $H$ , or of codimension at least 2 or equal to 1 in  $H$ ), we obtain that for  $t \geq 2$ ,

$$\lambda(t) = \nu(t) + \lambda(t - 1) + \# \mathbf{P}^{t-1}(k) > \nu(t) + \lambda(t - 1) + 1,$$

where  $\nu(t)$  denotes the number of non-trivial linear subspaces of codimension at least 2 in  $\mathbf{P}_k^t$  which are not contained in  $H$ . Hence, it is enough to prove the inequality

$$\nu(t) \geq \lambda(t - 1)$$

for every integer  $t \geq 2$ . But this is obvious: given a hyperplane  $\mathbf{P}_k^{t-1} \subset \mathbf{P}_k^t$  and a rational point  $p$  in the complement of  $\mathbf{P}^{t-1}(k)$ , the map  $L \mapsto \langle L, p \rangle$  embeds the set of codimension- $d$  linear subspaces of  $\mathbf{P}_k^{t-1}$  into the set of codimension- $d$  linear subspaces of  $\mathbf{P}_k^t$  which are not contained in  $\mathbf{P}_k^{t-1}$ .  $\square$

We can now prove Theorem 1.1.

Let us first show part (i). Every  $k$ -automorphism  $\varphi$  of  $\Omega(V)$  extends to a  $k$ -automorphism  $\tilde{\varphi}$  of  $X$  by Proposition 2.1; hence it induces a permutation  $\hat{\varphi}$  of non-trivial linear subspaces of  $\mathbf{P}(V)$  defined by  $\tilde{\varphi}(E_L) = E_{\hat{\varphi}(L)}$ . Note that  $\hat{\varphi}$  preserves the simplicial structure of flags in  $\mathcal{L}(V)$ , because  $\tilde{\varphi}$  preserves the simplicial structure of strata of the boundary divisor. By Proposition 1.4(ii), it suffices to prove that  $\hat{\varphi}$  preserves *hyperplanes*.

We argue by induction on  $n = \dim V - 1 \geq 1$ . For  $n = 1$ , the result is obvious. For  $n = 2$ , it is enough to compare self-intersections of components of  $D$  to conclude that for a point  $p$  and a line  $\ell$ ,

$$\deg[E_p]^2 = -1 \quad \text{and} \quad \deg[E_\ell]^2 = \deg\left(h - \sum_{q \in \ell(k)} [E_q]\right)^2 = 1 - \#\ell(k) = -(\#k);$$

thus  $\hat{\varphi}$  maps a line to a line.

In general, for any rational hyperplane  $H$  of  $\mathbf{P}(V)$ , it follows from Lemma 3.2 that  $\hat{\varphi}(H)$  is either a hyperplane or a rational point. Let us now assume that  $n$  is at least 3 and that the theorem has been proved in lower dimensions. If  $\hat{\varphi}(H)$  is a rational point  $p$ , then  $\tilde{\varphi}$  induces a  $k$ -isomorphism  $\bar{\varphi}$  between  $E_H$  and  $E_p$  which maps the divisor  $D_H = \bigcup_{L \neq H} E_H \cap E_L$  onto the divisor  $D_p = \bigcup_{L \neq \{p\}} E_p \cap E_L$ .

Since  $E_H$  (respectively,  $E_p$ ) is the blow-up of  $H$  (respectively, of  $\mathbf{P}(T_p^\vee)$  where  $T_p$  denotes the tangent space of  $\mathbf{P}(V)$  at  $p$ ) along the full hyperplane arrangement, with exceptional divisor  $D_H$  (respectively,  $D_p$ ), the theorem in dimension  $n - 1$  implies that  $\bar{\varphi}$  is induced by a  $k$ -isomorphism between  $H$  and  $\mathbf{P}(T_p^\vee)$ , and hence maps the components of  $D_H$  defined by rational points of  $H$  to components of  $D_p$  defined by rational points of  $\mathbf{P}(T_p^\vee)$ , i.e. by (rational) lines in  $\mathbf{P}(V)$  containing  $p$ .

Let  $q$  be a rational point of  $H$ , and let  $\ell$  denote the line in  $\mathbf{P}(V)$  such that

$$\tilde{\varphi}(E_H \cap E_q) = E_p \cap E_\ell.$$

The two hypersurfaces  $E_\ell$  and  $\tilde{\varphi}(E_q)$  have the same non-empty intersection with  $\tilde{\varphi}(E_H) = E_p$ , so

$$\tilde{\varphi}(E_q) = E_\ell$$

since  $D$  is a normal crossing divisor. By Lemma 3.2, this implies  $n = 2$ , while we assumed  $n \geq 3$ .

Therefore,  $\hat{\varphi}$  preserves the set of hyperplanes.

*Remark.* Carlo Gasbarri suggested that it should be possible to prove that  $\hat{\varphi}$  preserves hyperplanes by looking at the canonical divisor on  $X$ , which is a fixed point of  $\tilde{\varphi}^*$  in  $\text{CH}^1(X)$ . We sketch a way to combine this idea with the results of § 3. Using the classical formula for the canonical divisor of a blow-up [Har77, Exercise II.8.5], we obtain

$$K_X = \pi^* K_{\mathbf{P}(V)} + \sum_{L \in \mathcal{L}(V)} (\text{codim } L - 1)[E_L] = -(n + 1)h + \sum_{i=0}^{n-2} (n - i - 1) \sum_{L \in \mathcal{L}^i(V)} [E_L]. \quad (1)$$

Let  $\Gamma$  denote the subgroup of  $\text{CH}^1(X)$  spanned by  $\{[E_L]\}_{\text{codim } L \geq 2}$ . For any hyperplane  $H$ , we have

$$\tilde{\varphi}^*h = \tilde{\varphi}^*\left([E_H] + \sum_{L \subsetneq H} [E_L]\right) = [E_{\tilde{\varphi}^{-1}(H)}] + \sum_{L \subsetneq H} [E_{\tilde{\varphi}^{-1}(L)}].$$

Since  $\tilde{\varphi}^{-1}$  preserves the simplicial structure of  $\mathcal{L}(V)$ , it maps the link of  $H$  to the link of  $W = \tilde{\varphi}^{-1}(H)$  and, hence, linear subspaces of  $H$  to linear subspaces of  $\mathbf{P}(V)$  contained in or containing  $W$ . Since there are  $\#\mathbf{P}(V/W)(k)$  hyperplanes containing  $W$ , we obtain

$$\tilde{\varphi}^*h \equiv \#\mathbf{P}(V/W)(k)h \pmod{\Gamma}.$$

In particular,  $\dim \tilde{\varphi}^{-1}(H)$  does not depend on the hyperplane  $H$ . Together with Lemma 3.2, this observation implies that  $\tilde{\varphi}$  either preserves hyperplanes or swaps hyperplanes and points.

Assume that  $\tilde{\varphi}$  swaps hyperplanes and points. Then

$$K_X \equiv \tilde{\varphi}^*K_X \equiv -(n+1)\tilde{\varphi}^*h + (n-1) \sum_{p \in \mathbf{P}(V)(k)} \tilde{\varphi}^*[E_p] \equiv -(n+1)\#\mathbf{P}^{n-1}(k) + (n-1)\#\mathbf{P}^n(k)h$$

modulo  $\Gamma$ , so equation (1) implies that

$$(n+1)(q^n - q) = (n-1)(q^{n+1} - 1)$$

with  $q = \#k$ . This identity cannot hold if  $n > 1$ , since it would imply  $q \mid n-1$  and hence that  $n \geq q+1 \geq 3$  and  $(n+1)/(n-1) \leq 2$ , whereas  $(q^{n+1} - 1)/(q^n - q) > q \geq 2$ . Therefore,  $\tilde{\varphi}$  has to preserve hyperplanes.

#### 4. Extension of the ground field

We now indicate how to prove the second part of Theorem 1.1. For every field extension  $K/k$ , the base change  $\Omega(V)_K$  of  $\Omega(V)$  coincides with the complement in  $\mathbf{P}(V)_K$  of all  $k$ -rational hyperplanes. Since blowing up commutes with base change, the  $K$ -scheme  $X_K = X \otimes_k K$  can be obtained by blowing up  $\mathbf{P}(V)_K$  along the arrangement of all  $k$ -rational hyperplanes. Moreover, every irreducible component  $E_L$  of  $D$  is geometrically irreducible, and its base change  $(E_L)_K$  is the irreducible component of  $X_K - \Omega(V)_K$  corresponding to the  $k$ -rational linear subspace  $L_K$  of  $\mathbf{P}(V)_K$ .

Let us consider a  $K$ -automorphism  $\varphi$  of  $\Omega(V)_K$ . One proves exactly as in Proposition 1.4 that  $\varphi$  extends to a  $K$ -automorphism of  $X_K$  (respectively, of  $\mathbf{P}(V)_K$ ) if and only if  $\varphi$  preserves the simplicial set  $\Gamma(V_K)$  of discrete valuations on  $\kappa(V_K)$  coming from irreducible components of  $D_K$  (respectively, preserves the subset of  $\Gamma(V_K)$  corresponding to hyperplanes). Once again, this condition is established via analytic geometry over the field  $K$  endowed with the trivial absolute value. The key point is Lemma 2.2, which holds for the fan  $\mathfrak{S}(V_K)$  of the normal crossing divisor  $D_K$  on  $X_K$ . The proof works verbatim, but one could also argue that  $\mathfrak{S}(V_K)$  coincides with the inverse image of  $\mathfrak{S}(V)$  under the projection map  $p : X_K^{\text{an}} \rightarrow X^{\text{an}}$ , so the statement holds for  $\mathfrak{S}(V_K)$  since it holds for  $\mathfrak{S}(V)$ . We then prove as above that  $\varphi$  extends to a  $K$ -automorphism of  $X_K$ .

Lemmas 3.1 and 3.2 also apply to  $X \otimes_k K$ , upon replacing ‘linear subspaces’ by ‘ $k$ -rational linear subspaces’. It follows that the permutation of  $k$ -rational linear subspaces induced by  $\tilde{\varphi}$  preserves the hyperplanes; hence  $\varphi$  induces a  $K$ -automorphism of  $\mathbf{P}(V)_K$ . This automorphism preserves the set of  $k$ -rational hyperplanes. Pick a  $k$ -basis of  $V$  and consider the corresponding coordinate hyperplanes; since they are mapped to  $k$ -rational hyperplanes,  $\varphi$  is induced by a  $k$ -automorphism of  $\mathbf{P}(V)$ .

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