Chapter 11
Buildings and Kac-Moody Groups

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Abstract. This survey paper provides an overview of some aspects of the theory buildings in connection with geometric and analytic group theory.

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1 Introduction

1. The general goal of this survey paper is to introduce a class of metric spaces with remarkable symmetry properties (*buildings*), and a class of finitely generated groups acting on some of them (*Kac-Moody groups*). Then – and mostly – we mention what the viewpoint of geometric group theory enabled one to prove in the very recent years. More precisely, we deal with the following topics – see the *structure of the paper* at the end of the introduction to find the exact places.

– *Simplicity:* Kac-Moody groups provide a wide class of infinite finitely generated, often finitely presented and Kazhdan, simple groups (Caprace–Rémy).
– *Rigidity:* these groups enjoy strong rigidity properties, e.g., of the type "higher-rank vs hyperbolic spaces" (Caprace–Rémy).
– *Amenability:* though these groups are not themselves amenable, they admit amenable actions on explicit compact spaces provided by boundaries of buildings (Caprace–Lécureux, Lécureux).
– *Quasi-morphisms:* the existence of some non-standard quasi-homomorphisms is understood in terms of the geometry of the buildings and the transitivity of the action (Caprace–Fujiwara).
– *Quasi-isometry:* Kac-Moody lattices provide infinitely many quasi-isometry classes of finitely presented simple groups (Caprace–Rémy).

This subject still has some motivating and fast developments; it follows a general trend according to which more and more analytic topics turn out to be relevant to geometric group theory. Therefore, in the core of the text, we mostly focus on analytic statements. Other very interesting results are alluded to below, in the second and third parts of the introduction; some of them are also described in more detail in [26].

2. We see the present paper as a kind of continuation of the previous survey paper on the topic, written slightly less than 10 years ago [58]. This is why this part of the introduction is dedicated to outline recent progress on some of the questions mentioned there (in the 5th section).

We will provide later some details on the way Kac-Moody theory provides finitely generated groups having a discrete action on the product of two (twinned) buildings. Let Λ be such a group, with associated buildings X_- and X_+. The buildings $X_\pm$ are locally finite, so the compact-open topology on the groups $\mathrm{Aut}(X_\pm)$ is locally compact. The Λ-action on a single factor $X_\pm$ has infinite stabilizers (therefore it is not discrete), and its kernel is the finite center $\mathrm{Z}(\Lambda)$. We denote by $\overline{\Lambda}^\pm$ the closure of the image of Λ in $\mathrm{Aut}(X_\pm)$. It was proved in [60] that the locally compact group $\overline{\Lambda}^\pm$ is locally pro-p, for p a well-defined prime number (equal to the characteristic of the finite ground field of Λ). By analogy with Lie groups over local fields of positive characteristic, some questions about $\overline{\Lambda}^\pm$ were addressed in [58, 5.5.2].

- *Decomposition into abstractly simple factors.* One of these questions is the decomposition of these groups into direct products of abstractly simple groups [58, Question 5.5.7] (the weaker result of a decomposition into topologically simple factors had been proved in [56]). It turns out that thanks to a clever combination of Tits' simplicity criterion for BN-pairs [11] and arguments from pro-p groups, Carbone et al. [21] could prove this result for a large class of Kac-Moody groups (those for which one, or equivalently any, cell-stabilizer is topologically finitely generated). This proves the decomposition for instance when the generalized Cartan matrix defining Λ is 2-*spherical* (i.e., any two canonical reflections in the Weyl group generate a finite group).
- *Non-linearity for compact open subgroups.* One other question was to decide whether some compact open subgroups in $\overline{\Lambda}^\pm$ are not linear (over any field) under suitable conditions on the geometry of the buildings [58, Question 5.5.6]. Strictly speaking, this question is not answered but it is related to a more interesting result on the Golod-Shafarevich property due to Ershov [31]; this result involves the pro-p completions of cell-stabilizers in Kac-Moody lattices Λ – see below.
- *Generalized arithmeticity.* Given the inclusion of a lattice Γ in a locally compact group G, one says that Γ is *arithmetic* in G if its commensurator in G
$$\mathrm{Comm}_G(\Gamma) = \{g \in G :\ \Gamma \cap g\Gamma g^{-1} \text{ has finite index both in } \Gamma \text{ and in } g\Gamma g^{-1}\}$$
is dense in G (recall that classically, according to a well-known criterion due to G. Margulis, a lattice in a semisimple Lie group is arithmetic if and only if its commensurator is dense in the ambient group [73, 6.2]). Facet stabilizers in Kac-Moody lattices provide interesting exotic examples of arithmetic *non-uniform* lattices in full tree automorphism groups [4], but it seems that the more general question asked in [58, Question 5.5.4] is still open.

- *Non-linearity for discrete subgroups.* Let Γ denote the stabilizer in Λ of some cell in X_ϵ (where $\epsilon = \pm$). Then Γ gives rise to a lattice for the building $X_{-\epsilon}$ of opposite sign. The subgroup Γ is much smaller than Λ; its closure in $\mathrm{Aut}(X_\epsilon)$ is an open virtually pro-p subgroup, contained as a finite index subgroup of a suitable maximal compact subgroup of $\overline{\Lambda}^{\epsilon}$. The question of deciding under which conditions Γ cannot be a group of matrices over any field [58, Question 5.5.5] is still open. Note that the only linearity to be disproved is over fields of characteristic p [53], but it may happen that Γ is linear since for suitable choices of generalized Cartan matrices and ground fields defining Λ, we can have $\Gamma \simeq \mathrm{SL}_n(\mathbf{F}_q[t])$ (for arbitrary integer $n \geqslant 2$ and prime power q). Note that according to Caprace, Gramlich and Mühlherr there is another class of very interesting lattices (for a single building) arising in this context, namely centralizers of suitable involutions of Λ; see for instance [33, 35] for an introduction to these lattices.

Note also that the (non-)linearity question for Λ is now solved by the answer to the simplicity question: according to Mal'cev, a linear finitely generated group – like Λ – is always residually finite [43], i.e. the intersection of its finite index (normal) subgroups is trivial; in particular, it cannot be simple if it is infinite. One difficulty, among others, for the linearity problem of the subgroups Γ is the fact that these groups are usually *not* finitely generated (see [3] for sufficient conditions, though).

3. Of course, most nice recent results on Kac-Moody groups (discrete and completed versions) were not proved after the questions mentioned in [58]! Here are some of them.

- *Harmonic analysis.* First, the analogy between the groups $\overline{\Lambda}^{\pm}$ and reductive Lie groups over local fields suggests the possibility of generalizing some results about non-commutative harmonic analysis on Lie groups. This leads to the question of proving the existence of a Gelfand pair with respect to a suitable maximal compact subgroup [62], the next step being the explicit computation of the corresponding spherical functions [42]. One striking result in this vein again reveals that the geometry of the buildings is crucial: Lécureux proved that whenever a building is *not* Euclidean, sufficiently transitive automorphism (i.e., sufficiently interesting) groups cannot have any Gelfand pair with respect to any maximal compact subgroup [41]. This means that convolution, restricted to any (Hecke) algebra of compactly supported functions bi-invariant under any compact subgroup, is never a commutative law. Actually, since maximal compact subgroups in this context are open, the Hecke algebras here are well-understood algebraic objects defined by relations closely related to infinite root systems; the resulting combinatorial problem is eventually solved by geometric arguments similar to those shortly described in Sect. 4.1.
- *The Golod–Shafarevich condition and property (T).* Roughly speaking, a pro-p group is said to have the *Golod–Shafarevich* property if it (admits a relation which) has few relations with respect to the number of generators. This condition can be made technically very precise, and was initially designed to

prove that some (Galois) groups are infinite. It turns out that this condition implies the largeness of the pro-p group under consideration in many more ways – we refer to [31, Introduction] for a nice and very efficient introduction to the subject. By passing to pro-p completions, the Golod–Shafarevich condition is relevant to the study of discrete groups. As already mentioned, to data needed to define a finitely generated Kac-Moody group are a generalized Cartan matrix and a finite field (see Sect. 2.3). If p denotes the characteristic of the latter ground field, then the rich combinatorial structure of Λ enables one to study the pro-p completion $\Gamma_{\widehat{p}}$ of a facet stabilizer Γ as above (note that such a Γ is residually finite: sufficiently many finite index subgroups are given for instance by pointwise stabilizers of combinatorial balls around the facet). After works by Lubotzky and Sarnak, some conjectures were made by Lubotzky and Zelmanov about the incompatibility between property (T) or (τ) and being Golod–Shafarevich [72]. The main result of [31] disproves some of these intuitions and says that for every sufficiently large prime number p, there exists a finitely generated group having property (T) and which is Golod–Shafarevich with respect to p. This is proved by using Kac-Moody theory, and in particular the pro-p completions $\Gamma_{\widehat{p}}$ of facet stabilizers.

- *Connection with the general theory of totally disconnected groups.* In a somewhat different direction, Willis started to develop a thorough study of arbitrary locally compact totally disconnected groups [70]. The main tool he uses to perform this study is the space of all compact open subgroups of the given group, which is non-empty (and big) by the assumption of being totally disconnected [12, III, Cor. 1, p.36]. The latter space is endowed with a natural metric which allows one to use some arguments of dynamical nature. This theory applies of course to groups defined as closed automorphism groups of locally finite cell complexes, and to algebraic groups over non-Archimedean local fields. One of its outcomes is to attach some invariants of the topological group structure to any totally disconnected locally compact group; one of them is a notion of rank, namely the *flat rank*, which coincides with the usual notion of rational rank when the group is obtained as the rational points of a reductive algebraic group over a non-archimedean local field. When the building is not Euclidean, which excludes any algebraic group consideration, the flat rank can still be related to another notion of rank in connection with maximal flat (i.e., Euclidean) subspaces in the building; see [16, 24]. Another connection between Kac-Moody groups and this theory is the study of contraction groups, which are so to speak dynamical generalizations of unipotent radicals of parabolic subgroups in the algebraic group case; it is proved in [15] that whenever the buildings are *not* Euclidean, contraction groups are *not* closed, which is another avatar, in addition to the dichotomy "simplicity vs linearity", of the dichotomy between Euclidean and non-Euclidean buildings.
- *Decomposition of abstract homomorphisms.* The last question that was solved and which can be mentioned in this introduction is purely of algebraic nature. The intuition leading to it is not the analogy with non-Archimedean Lie groups, but is the very starting point of the construction of Kac-Moody

groups by Tits, namely to introduce infinite-dimensional analogues of Chevalley group schemes [67]. Having this (initial) motivation in mind, one can ask whether there is, as in the finite-dimensional case, a well understood factorization for abstract group homomorphisms between groups of rational points over fields of Kac-Moody functors. The strongest results in this direction were obtained by Caprace [20].

1.1 Structure of the Paper

In Sect. 11.1, we recall some basic definitions in building theory and explain how the algebraic machinery of Kac-Moody theory provides examples endowed with interesting group actions. In Sect. 11.2, we present some recent results in connection with quotients and actions, namely we study the problems of simplicity and rigidity for Kac-Moody lattices. In Sect. 11.3, we investigate some more topics from analytic and metric group theory, namely amenability (for actions), existence of (exotic) quasi-characters and distortion.

1.2 Conventions

In this paper, letters like Λ and Γ denote discrete groups, letters like G and H denote non-discrete topological groups (assumed to be locally compact) and letters like X and Y denote metric spaces (most of the time assumed to be complete).

2 Buildings and Kac-Moody Groups

In this section, we briefly provide the material in Kac-Moody and building theory needed to understand the problems investigated in the sequel. The general reference for building theory is [1]; for Kac-Moody groups it is [54].

2.1 (Simplicial) Buildings

In what follows, we only deal with the simplicial point of view on buildings. This is because we are mostly interested here in new phenomena (with respect to the classical theory of Lie groups) which occur mainly while studying groups acting on *non-affine* buildings. Non-simplicial buildings are mostly interesting (so far) when they are of Euclidean type (via the Bruhat-Tits theory of reductive groups over non-discretely valued fields [59], or appearing as asymptotic cones of symmetric spaces or Bruhat–Tits buildings [39]).

First, let us recall the following preliminary notions [11].

- A *Coxeter group*, say W, is a group admitting a presentation: $W = \langle s \in S \mid (st)^{M_{s,t}} = 1 \rangle$ where $M = [M_{s,t}]s, t \in S$ is a *Coxeter matrix* (i.e., symmetric with 1's on the diagonal and other entries in $\mathbf{N}_{\geqslant 2} \cup \{\infty\}$).

– For any Coxeter system (W, S) there is a natural simplicial complex Σ on the maximal simplices of which W acts simply transitively: Σ is called the *Coxeter complex* of (W, S).

We can go the other way round; namely, let us use a theorem of Poincaré's [45, IV.H.11] starting with a suitable tesselation and providing a Coxeter group (the initial tesselation is eventually a geometric realization of the alluded to above Coxeter complex):

Example 2.1. Start with a Euclidean or hyperbolic periodic tiling whose dihedral angles are integral submultiples of π. Then the group generated by the reflections in the codimension 1 faces of the fundamental tile is a discrete subgroup of the full isometry group; in fact, it is a Coxeter group and the initial tiling realizes its Coxeter complex.

The reason why we introduced Coxeter complexes is that they are so to speak "slices" in buildings, as the following definition shows. We freely use the above notation W and Σ (Coxeter complex).

Definition 2.2. *A* building *of type* (W, S) *is a cellular complex, covered by subcomplexes all isomorphic to* Σ*, called the* apartments*, such that:*

(i) Any two simplices, called the facets*, are contained in a suitable apartment;*
(ii) Given any two apartments A *and* A'*, there is a cellular isomorphism* $A \simeq A'$ *fixing* $A \cap A'$*.*

The group W *is called the* Weyl group *of the building.*

When W is a Euclidean reflection group [11, V Sect. 3], one says that the building is *affine* or, equivalently, *Euclidean.*

Example 2.3. A tree with all vertices of valency $\geqslant 2$ (resp. a product of such trees) is a building with W equal to the infinite dihedral group D_∞ (resp. with W equal to a product of D_∞'s).

The above examples of trees are elementary, but they are the only ones with infinite Weyl group which can be reasonably drawn. Note that it is enough to consider these examples to see a difficulty in producing interesting group-theoretic situations from the theory of buildings: *it may very well happen that the automorphism group of a building be trivial* (take a tree in which any two distinct vertices have distinct valencies). It is precisely one feature of Kac-Moody theory to give rise to buildings automatically endowed with a highly transitive group action.

2.2 Analogies with Lie Groups and Exotic Examples

What are buildings good for? They were first designed to provide a uniform approach to simple algebraic groups whatever the ground field is, but it turned out that Chevalley's scheme-theoretic approach became more popular. Still, thanks to buildings, the idea to attach a suitable geometry to simple groups of Lie type was pushed far beyond the classical example of the interplay between simple real Lie groups and symmetric spaces – we refer to [26] for more details on these historical points.

Eventually, as the examples below show, there is no rule in general on what comes first, the building or the group: spherical (resp. Euclidean) buildings are used to better understand (isotropic) algebraic groups over arbitrary (resp. non-Archimedean local) fields, but in some recent approaches, buildings were designed first in order to provide automorphism groups for which the analogy with reductive Lie group makes sense and is often very fruitful.

Example 2.4. Pick (the rational points of) a simple algebraic group over a non-Archimedean local field, e.g. $\mathrm{SL}_n(\mathbf{Q}_p)$. Then the group acts on a Euclidean building in a suitable way: the action is strongly transitive, that is transitive on the inclusions of a chamber (= maximal facet) in an apartment; this is one of the main results of Bruhat–Tits theory – see [18] and [19].

Which other buildings (with an interesting group action) can be exhibited? This question is natural. Actually, there exist lots of interesting examples which are not relevant to Borel–Tits or Bruhat–Tits theory. We will see that some of these buildings have apartments which are periodic tilings of real hyperbolic spaces; here is a list of selected results concerning such hyperbolic buildings.

- According to Haglund and Paulin, for some of these hyperbolic buildings the full isometry group is a locally compact, totally disconnected, uncountable *abstractly simple* group [38].
- According to Bourdon, some of them are characterized by any group acting discretely and cocompactly: this is a strong rigidity result "à la Mostow" [13].
- The latter result can be pushed further: according to Bourdon and Pajot, any quasi-isometry from a right-angled Fuchsian building to itself is at finite distance from a true isometry of the latter space – see [14, 71].

All of these statements strongly support the analogy between exotic buildings and symmetric spaces, between Lie groups and full automorphism groups of these new geometries.

2.3 Kac-Moody Groups and Kac-Moody Buildings

The previously mentioned examples were studied from a geometric (more precisely: metric) viewpoint. We turn now to Kac-Moody theory, which can be seen as an algebraic machinery to produce (as a first step) groups with good combinatorial properties – refinements of Tits systems –, and (as a second step) – by formal and now standard arguments – buildings automatically endowed with strongly transitive group actions. Let us briefly sum up the situation and refer to [54] for details and constructions of twisted variants.

- Kac-Moody groups (in the “minimal” version we are interested in, see [68]) were constructed by Tits [67] in order to generalize, as group functors, (split) reductive algebraic group schemes initally due to Chevalley and Demazure.
- They are defined by a presentation generalizing the generators and relations of SL_n (using elementary unipotent matrices) [64].
- The defining datum for a Kac-Moody group is: a field $\mathbf{K}$ and a *generalized Cartan matrix*, i.e., an integral matrix $A = [A_{s,t}]_{s,t\in S}$ such that $A_{s,s} = 2$ for all $s \in S$ and $A_{s,t} \leqslant 0$ for $s \neq t$, with $A_{s,t} = 0$ if and only if $A_{t,s} = 0$.
- Unfortunately, they have not yet been endowed with any structure from algebraic geometry, which would/will be, by the way, infinite-dimensional
- Nevertheless, they share many combinatorial properties with groups of points of Chevalley–Demazure group schemes: they have a (twin) BN-pair structure [67, Sects. 5 and 6].

Example 2.5. The standard example of such a group is $\Lambda = \mathbf{G}\big(\mathbf{K}[t, t^{-1}]\big)$ for $\mathbf{G}$ a simple (isotropic) matrix group over a field $\mathbf{K}$.

Still, we will see that the groups from this example are not the most interesting ones for the purposes mentioned in the first part of the introduction: they have an obvious matrix interpretation, and as such are residually finite groups. Note that the associated buildings are Bruhat–Tits buildings, hence are Euclidean.

The geometric counterpart to the above BN-pair combinatorics is the following.

Fact 2.6 *(i) Any Kac-Moody group Λ naturally acts on the product $X_- \times X_+$ of two isomorphic buildings $X_\pm$.*

(ii) The explicit rule for W deduces $[M_{s,t}]_{s,t\in S}$ from $A = [A_{s,t}]_{s,t\in S}$. More precisely, we have $M_{s,t} = 2,\ 3,\ 4,\ 6$ or ∞ according to whether $A_{s,t} \cdot A_{t,s}$ is 0, 1, 2, 3 or is $\geqslant 4$, respectively.

Reference. This is [67, Sects. 5 and 6] for the group-theoretic part, and the geometric side is explained in [54, Sects. 1 and 2]. □

Reading backwards $[A_{s,t}]_{s,t\in S} \mapsto [M_{s,t}]_{s,t\in S}$, we can produce buildings (with nice group actions) provided the Weyl group has its Coxeter exponents in $\{2; 3; 4; 6; \infty\}$. This is the only condition on the shape of apartments – see [55, Sect. 2] for the application to hyperbolic buildings

- The case of affine buildings corresponds exactly to the previous examples $\Lambda = \mathbf{G}\big(\mathbf{K}[t, t^{-1}]\big)$, with a concrete matrix interpretation. We say then that the generalized Cartan matrix A and the corresponding Kac-Moody group Λ are of *affine* type.
- When W is a Fuchsian group, e.g. when W is generated by a right-angled hyperbolic polygon or by a regular triangle of angle $\frac{\pi}{4}$ or $\frac{\pi}{6}$, then $X_\pm$ carries a negatively curved metric.

These are only examples; the general case is a mixture.

Let us finally mention the basic results saying that over a finite ground field, Kac-Moody groups are relevant to geometric group theory.

Fact 2.7 *(i) Any Kac-Moody group Λ over any finite field is finitely generated.*
(ii) The associated buildings $X_\pm$, for a suitable non-positively curved realization, are locally finite.

Reference More technically: according to Davis, any building admits a realization carrying a CAT(0) metric [29], and this realization is locally finite if and only if the ground field if finite. □

3 Simplicity and Rigidity

In this section, we are interested in the algebraic question of simplicity for the finitely generated groups obtained by taking Kac-Moody groups over finite fields. It turns out that in this case – where we have a nice geometric action – there is a simple answer to this basic question in group theory (it can be formulated in terms of the geometry of the buildings). The question of rigidity of group actions is less algebraic, but we review it here in order to explain the general idea to combine simplicity, non-positive curvature and representation-theoretic properties.

3.1 Covolume

From now on, we denote by Λ a Kac-Moody group defined by a generalized Cartan matrix, say $A = [A_{s,t}]_{s,t\in S}$, and a *finite* field $\mathbf{F}_q$; the associated buildings are denoted by $X_\pm$. The full automorphism groups $\mathrm{Aut}(X_\pm)$ are thus locally compact for the compact open topology; therefore $\mathrm{Aut}(X_\pm)$ admits Haar measures [10]. Using the Λ-actions on $X_\pm$ arising from the combinatorial structure of a twin BN-pair, we can see Λ as a (diagonally embedded) subgroup of $\mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)$. The starting point to combine Kac-Moody groups and geometric group theory is the following result, which establishes an analogy between Kac-Moody groups over finite fields and arithmetic groups in positive characteristics [53].

Theorem 3.1. *Assume the Weyl group W of Λ is infinite and denote by $W(t) = \sum_{w\in W} t^{\ell(w)}$ its growth series. If $W(\frac{1}{q}) < \infty$, then Λ is a lattice of $X_+ \times X_-$; it is never cocompact.*

Reference This is the result settled in [23] or in the note [52]. □

Remark 3.2.
1. When the Kac-Moody group is affine, i.e. when $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t,t^{-1}]\big)$, the condition $W(\frac{1}{q}) < \infty$ is empty since the virtually abelian group W has polynomial growth: it is a Euclidean reflection group. In this case, this is in fact a well-known consequence of a deeper result: Harder's reduction theory in positive characteristic [37].
2. The diagonal Λ-action on $X_+ \times X_-$ is always discrete and the real number $W(\frac{1}{q})$ is merely the covolume $\mu\Big(\dfrac{\mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)}{\Lambda}\Big)$ for a natural normalization μ of the Haar measure on $\mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)$.

Concerning computation of covolume, some natural questions are in order. For instance, one consists in fixing a building and checking whether the infimum of the normalized covolumes is > 0; one can also try to decide whether this infimum is reached. This is an adaptation, in a new situation of locally compact groups admitting lattices, of very classical questions and results going back to Siegel, one of the most important achievements being strong finiteness results due to Prasad [50]. In the new situation allowed by considering suitable non-affine buildings and their automorphism groups, some partial results are already available. For instance, Thomas exhibited, for right-angled Fuchsian buildings, an infinite increasing sequence of lattices [66]; she also showed that in some cases the sets of covolumes are unrestricted with respect to the classical "Lie" situation (in particular when super-rigidity holds) [65].

3.2 Simplicity

We keep the previous notation. The fact that a finitely generated Kac-Moody group such as Λ can be seen as a lattice of some reasonable geometry is the starting point to prove the following simplicity result.

Theorem 3.3. *Let Λ be a Kac-Moody group defined over the finite field $\mathbf{F}_q$. Assume that the Weyl group W is infinite and irreducible, and that $W(\frac{1}{q}) < \infty$. Then Λ is simple (modulo its finite center) whenever the buildings $X_\pm$ are not Euclidean and Λ is generated by its root subgroups.*

Reference This statement is contained in [27] – see in particular Sect. 4 of [loc. cit.]. □

What follows in this subsection is dedicated to explaining the general two-step strategy of the proof, but let us first make some comments on the statement itself and some extensions of it.

Remark 3.4. 1. Here is a rough but striking geometric reformulation of the above statement: whenever Λ (with irreducible Weyl group) has no obvious matrix interpretation – because it is not affine – it is a simple finitely generated group; moreover there is a geometric (Coxeter-theoretic) explanation for this.
2. There exist infinitely many generalized Cartan matrices A such that Λ is a *finitely presented, Kazhdan, simple group* for any $q \gg 1$.

Point 2 above is proved by combining the previous simplicity theorem together with:

(i) (Cohomological) finiteness results due to Abramenko and Mühlherr, e.g. [3] and [2].
(ii) Dymara and Januszkiewicz's criterion for property (T) for automorphism groups of buildings [30].

We can now turn to roughly sketching the proof of the simplicity theorem. It owes a lot to Burger–Mozes' recent construction of finitely presented torsion-free simple groups. The latter groups appear as cocompact lattices in products of two trees (here, it is good to have in mind that trees are 1-dimensional buildings!).

The general idea in [8] is first to see the discrete groups under consideration as analogues of lattices in Lie groups in order to rule out infinite quotients, and finally to exploit decisive differences with the classical Lie group case in order to rule out finite quotients too.

- The *analogy part* is motivated by Margulis' normal subgroup theorem, which says that a normal subgroup in a higher-rank lattice must have finite index [73, Sect. 8]. The point is to obtain a generalization of this result without relying on any algebraic group structure on the ambient topological group given by the full automorphism group of the geometry (product of trees or, more generally, of buildings).
- The *difference part* is more specific to the situation. In the case of products of trees, it relies on the possibility of proving some *non*-residual finiteness criteria involving transitivity conditions on the local actions (around each vertex) for the projection of the lattice on each of the two trees. This part was eventually improved by the possibility to embed explicitly well-known non-residually finite groups into suitable cocompact lattices of products of trees. This enabled Rattaggi to reduce a lot the size of the presentation of some simple groups as constructed by Burger and Mozes [51].

Remark 3.5. A group all of whose quotients are finite is called just infinite; according to Wilson, such a finitely generated group either is residually finite (examples are given by linear groups) or contains a finite index subgroup which is a direct product of finitely many isomorphic simple groups.

We refer to [6, Th. 5.6 p. 59] for a more refined version of Wilson's alternative involving the so-called branch groups ("Grigorchuk's trichotomy").

Here is a slightly more precise summary of the proof of simplicity in the Kac-Moody case, with relevant references.

1. The analogy part follows Margulis' strategy for the normal subgroup property: any normal subgroup of a center-free Kac-Moody lattice has finite index or is finite and central (i.e. an irreducible center-free Kac-Moody lattice is just infinite). Actually, the crucial point to prove this is to use a criterion due to Shalom [63] (resp. Bader and Shalom [17]) to prove property (T) (resp. amenability) for the quotient group Λ/N, endowed with the discrete topology, where $N \triangleleft \Lambda$ is the normal subgroup under consideration (recall that a group which is both Kazhdan and amenable is compact). The paper [63] considers *cocompact* irreducible lattices in direct products, but Shalom notes himself that the cocompactness assumption can relaxed to a weaker integrability condition involving the induction cocycle; the latter integrability condition is checked in [57]

2. What can go wrong from being just infinite to being simple? Consider the affine (linear) example $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t,t^{-1}]\big)$; it has a lot of finite quotients (given for instance by the congruence kernels). This is where non-affineness of the Weyl group has to be used crucially. Indeed, a strengthening of Tits' alternative for Coxeter groups (due to Margulis–Noskov–Vinberg, see [48] and [49]) implies that non-affine Coxeter groups are "weakly hyperbolic" in the sense that there exist lots of triples of roots with empty pairwise intersections. This is what has to be combined together with a trick on infinite root systems and some defining relations for Kac-Moody groups, in order to rule out finite quotients for Λ [27, Sect. 4].

3.3 Rigidity

Why Care About Kazhdan's Property (T) for Simple Groups? A well-known general principle is that there is a deep connection between the representation theory of a locally compact group and its intrinsic topological group structure. Property (T) illustrates this perfectly: one of its features is that it has many equivalent formulations among which are definitions by representation-theoretic means. What we explain below may not be the strongest way to combine Kazhdan's property (T), simplicity and non-positive curvature for metric spaces acted upon, but it provides a reasonable motivation to investigate further rigidity questions in such situations.

Here are the main steps of the argument.

Residual Finiteness and Compactness First let us recall that a finitely generated group is *residually finite* (i.e. the intersection of its finite index subgroups is trivial) if, and only if, it embeds abstractly in a compact group. One implication is easy since residual finiteness amounts to saying that the natural homomorphism from the group to its profinite completion is injective. For the other direction, the trick is to use the Peter–Weyl theorem decomposing representations of compact groups – this is where the previous general principle is used (it is a variant of the way to use Peter–Weyl's theorem to understand general compact groups [69, Chap. V]). In our situation, where we are most interested in finitely generated groups, we can conclude that a finitely generated simple group Γ has trivial homomorphic image to any compact group.

Geometric Consequence for Cell Complexes Therefore, if such a Γ acts on a locally finite cell complex with a global fixed point, the action is actually trivial; indeed, by definition of the compact open topology on the automorphism group of a complex, the stabilizer of a point is (open and) compact.

Non-Positive Curvature The situation is better if we make some assumption on the curvature properties (in a singular sense) of the metric space acted upon. Indeed, Bruhat–Tits fixed point theorem says that a bounded subset in a complete non-positively curved space has a barycenter which is uniquely characterized by purely metric properties [18, Sect. 3.2]; this can be applied to bounded orbits of isometric group actions. In our situation, this implies that if Γ (simple) acts non-trivially on a CAT(0) locally finite space, then any of its orbits is unbounded.

Negative Curvature Recall that the CAT(0) *property* for a geodesic metric space is that all geodesic triangles be at least as thin as in the Euclidean plane [7, II.1]. The similar comparison with the real hyperbolic plane $\mathbb{H}^2_{\mathbf{R}}$ instead of $\mathbf{R}^2$ leads to the notion of a CAT(-1)*-space*. Now let Y be a proper CAT(-1)-space with Isom(Y) acting cocompactly (the properness assumption here means that closed metric balls are compact and the cocompactness assumption is a variant of requiring bounded geometry). Then according to Burger–Mozes, the stabilizer of any $\xi \in \partial_\infty Y$ is amenable, so any non-trivial action of a finitely generated *Kazhdan* simple group Γ on Y has no global fixed point in the compactification $Y \cup \partial_\infty Y$ (recall that “Kazhdan + amenable” implies “compact”).

Super-Rigidity By convention in this paper, the terminology *super-rigid* applies to group actions on non-positively curved spaces (or spaces derived from them). It means that if a discrete group has a “nice” action on a specific geometry, then it cannot have a non-degenerate action on a space which doesn’t look like the initial geometry (“nice” means for instance that the action of the discrete group enables one to see it as a lattice of the full isometry group of the geometry). A well-known result by Margulis [44, VII.5] says that higher-rank lattices in semisimple Lie groups are super-rigid; this means that the lattice actions on the symmetric spaces associated to the ambient Lie groups are super-rigid with respect to actions on symmetric spaces (or Bruhat–Tits buildings) of other type.

Example 3.6. There are many results disproving the existence of actions of higher-rank lattices, e.g. $\mathrm{SL}_n(\mathbf{Z})$ for $n \geqslant 3$, on the circle (which can be seen as the boundary at infinity of $\mathbb{H}^2_{\mathbf{R}} = \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}(2)$, the Poincaré plane).

We refer to Ghys’ paper [34] for a general report about group actions on the circle.

Let us go back to our specific topic: what makes a Kac-Moody lattice be of higher-rank? For non-affine buildings, Kazhdan’s property (T) and the existence of flats of dimension $\geqslant 2$ in the buildings are independent conditions (this is a difference with the classical case of non-Archimedean simple Lie groups). Once these two conditions are fulfilled, we can draw strong consequences on actions on some hyperbolic spaces in the sense of Gromov [7, Sect. III.H.1].

Theorem 3.7. *Let Λ be a simple Kac-Moody lattice and let Y be a proper* CAT(-1)*-space with cocompact isometry group. If the buildings $X_\pm$ of Λ contain flat subspaces of dimension $\geqslant 2$ and if Λ is Kazhdan, then the group Λ has no nontrivial action by isometries on Y.*

Reference This statement is contained in [27, Sect. 7]. □

The intuition behind this result is of course that the algebraic group structure of Λ encodes a substantial part of the geometry of the buildings, so that there is not “enough room” in a hyperbolic space to be compatible with existence of higher-dimensional flats in the buildings of Λ. One crucial ingredient for the proof is a general super-rigidity result due to Monod and Shalom [47].

4 Amenability, Quasi-Homomorphisms and Quasi-Isometry

This section is finally dedicated to reviewing some more advanced topics in geometric group theory, such as the existence of non-standard quasi-characters, or the beginnings of the (still widely open) quasi-isometric classification of Kac-Moody lattices. We start be mentioning another vein of very interesting results, relevant to an analytic approach to geometric group theory, namely the study of amenability properties for actions of building lattices.

4.1 Amenability

We first report on Caprace and Lécureux's work on the classification of amenable subgroups in automorphism groups of buildings, and then mention Lécureux's work on amenability of group actions on suitably compactified buildings.

Compactifications and Amenable Subgroups Once again, the beginning of the story is the situation of semisimple real Lie groups and their associated symmetric spaces. Given a simple real Lie group G, like $\mathrm{SL}_n(\mathbf{R})$, families of compactifications of the symmetric space $X = G/K$ (where K is a maximal compact subgroup) were defined in different ways – and for different puposes – by Furstenberg [32] and Satake [61] in the 1960s. These families depend on the choice of a conjugacy class of parabolic subgroups, and for the same choice of conjugacy class, they were eventually shown to be isomorphic. We are here interested in the fact that, given G, the maximal Satake–Furstenberg compactification of X provides a geometric (virtual) parametrization of maximal amenable subgroups in G: according to Moore, any point stabilizer in G is an amenable subgroup; conversely, any amenable subgroup of G has a finite index subgroup stabilizing a point in the maximal compactification $\overline{X}$ [46]. This result was extended later to the case of non-Archimedean simple Lie groups acting on their Bruhat–Tits buildings [36].

In view of the analogy described in Sect. 2.2, it is then natural (though not obvious at all!) to try to prove the following statement.

Theorem 4.1. *Any locally finite building X admits a compactification providing the same “classification” for amenable subgroups in $G = \mathrm{Isom}(X)$ as above.*

Reference This is one of the main results of [25]. □

The main difficulty is actually to define suitably the compactification, which is done by a clever use of projections onto residues in buildings; this idea enables

Caprace–Lécureux to obtain an embedding of the set of residues of a given building to a space of maps between sets of residues, the latter space being compact whenever the building is locally compact. Note that this compactification is not the one given by asymptotic classes of geodesic rays [7, Sects. II.8 and III.H.3]; it is related to the combinatorics of infinite root systems. Caprace and Lécureux also define a compactification by using the (compact) Chabauty topology [10] on the space of closed subgroups of a given locally compact group, and relate it to the previous one.

From Amenable Groups to Amenable Actions Roughly speaking, a G-action on a space S is called *amenable* if there is a sequence of maps $\{\mu_n : S \to \mathcal{M}^1(G)\}_{n\geqslant 0}$ (where $\mathcal{M}^1(G)$ denotes the space of the probability measures on G) such that

$$\lim_{n\to\infty} \| \mu_n(g.x) - g_*\mu_n(x) \| = 0$$

uniformly on compact subsets of $G \times S$. In other words, there exists a sequence of maps $S \to \mathcal{M}^1(G)$ which is "asymptotically" equivariant.

In the situation of the previous theorem, the group $G = \mathrm{Isom}(X)$ itself is not amenable in general, but we have the following.

Theorem 4.2. *For any building X, any proper action by a locally compact group on the above compactification $\overline{X}$ is amenable.*

Reference This is the main result of [40]. □

The main idea is to use the non-positive techniques already shown to be useful in the study of the strong Tits alternative and the weak hyperbolicity of Coxeter groups as mentioned at the end of Sect. 3.2. This enables Lécureux to embed an apartement (Coxeter complex) into a product of trees and then to define the desired family of measures.

Remark 4.3. Admitting an amenable action on a compact space is an important property in analytic group theory: for instance, it is related to the Novikov conjecture; it also provides theoretical resolutions in bounded cohomology and boundary maps in rigidity theory (which, by the way, might be useful in the study of the linearity of certain building lattices).

4.2 Quasi-Homomorphisms

In this subsection, we mention quickly some results by Caprace and Fujiwara about quasi-characters of automorphism groups of buildings (again seen as analogues of semisimple Lie groups).

(i) A *quasi-character* for a group G is by definition a map $f : G \to \mathbf{R}$ such that
$$\sup_{g,h\in G} | f(gh) - f(g) - f(h) | < \infty.$$
(ii) The set of all quasi-characters of G is denoted by $\mathrm{QH}(G)$.

(iii) The set of non-trivial quasi-characters is by definition

$$\widetilde{\mathrm{QH}(G)} = \frac{\mathrm{QH}(G)}{\mathrm{Hom}(G, \mathbf{R}) \oplus \ell^\infty(G)}.$$

Quasi-homomorphisms are related to rigidity questions; according to Burger and Monod, higher-rank lattices in Lie groups don't have non-trivial quasi-characters [9]. As we saw for rigidity questions in Sect. 3.3, it is not clear what to require to consider that a building is of higher rank. The next result shows that many buildings are *not* of higher-rank with respect to quasi-homomorphims.

Theorem 4.4. *Let (W, S) be an infinite, irreducible, non-affine Coxeter system and let X be a building of type (W, S). Let G be a group acting on X by automorphisms so that at least one of the following conditions is satisfied:*

(i) *The G-action on X is Weyl-transitive.*
(ii) *For some apartment $\mathbb{A} \subset X$, the stabilizer $\mathrm{Stab}_G(\mathbb{A})$ acts cocompactly on $\mathbb{A}$.*

Then $\widetilde{\mathrm{QH}(G)}$ is infinite-dimensional.

Reference This is the main result of [22]. □

The main idea is to combine Coxeter-theoretic ideas together with a criterion due to Bestvina and Fujiwara [5]; the key notion is that of a rank-one isometry in a CAT(0)-space (that is, an isometry with certain contraction properties).

Combined with Kac-Moody theory, this implies that, *up to isomorphism, there exist infinitely many finitely presented simple groups of strictly positive stable commutator length.*

4.3 Quasi-Isometry

Let G be a locally compact group admitting a finitely generated lattice Γ. This implies that G admits a compact generating subset, say $\widehat{\Sigma}$; we denote by $d_{\widehat{\Sigma}}$ the word metric associated with $\widehat{\Sigma}$. Similarly, we fix a finite generating set Σ for Γ and denote by d_Σ the associated word metric. The lattice Γ is called *undistorted* in G if d_Σ is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to Γ. This amounts to saying that the inclusion of Γ in G is a quasi-isometric embedding from (Γ, d_Σ) to $(G, d_{\widehat{\Sigma}})$.

Theorem 4.5. *Any Kac-Moody lattice $\Lambda < \mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$ is undistorted.*

Reference This is the main result of [28]. □

Again, combined with simplicity results from Kac-Moody theory, this implies that *there exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.*

The above result uses the metric and combinatorial shape of the apartments in a rough way; in particular, it is far from solving the question of quasi-isometric classification of Kac-Moody lattices.

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